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DECENTRALIZED STATIC OUTPUT FEEDBACK CONTROLLER DESIGN FOR LINEAR INTERCONNECTED SYSTEMS

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Many interconnected systems in the real world, such as power systems and chemical processes, are often composed of subsystems. A decentralized controller is suitable for an interconnected system because of its more practical and accessible implementation. We use the homotopy method to compute a decentralized controller. Since the centralized controller constitutes the starting point for the method, its existence becomes very important. This paper introduces a non-singular matrix and a design parameter to generate a centralized controller. If the initial centralized controller fails, we can change the value of the design parameter to generate a new centralized controller. A sufficient condition for a decentralized controller is given as a bilinear matrix inequality with three matrix variables: a controller gain matrix and a pair of other matrix variables. Finally, we present numerical examples to validate the proposed decentralized controller design method.

Keywords: output feedback, decentralized controller, homotopy method, interconnected system, matrix inequality.

1. Introduction

Many systems in real phenomena have been modeled as interconnected systems, such as power systems, chemical processes, and communication networks. General interconnected systems consist of some subsystems which exchange information. In centralized control, the construction of each local controller is based on the information (state or output) of all subsystems. Thus, if the information of some subsystems cannot be obtained timely due to physical problems, it is difficult to implement the local controller. In contrast, the concept of decentralized control is that each local controller only uses the information of neighboring subsystems and itself. Thus, the computation and implementation problem is more accessible and more practical. This benefit is crucial for many large-scale systems that must continue to run even when some individual components fail. The system dimension information structure constraints, and delays in the accuracy of the transmitted information are three main reasons for using the decentralized controller (Siljak, 1991). These reasons reveal that decentralized controllers are considered for large-scale systems.

In practical applications, full-state measurements are not usually possible. Therefore, we use the output feedback control to allow flexibility and simplicity of decentralized controller implementation. Straka and

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Punčochář (2020) design and analyze the performance of active fault diagnosis for large scale systems with decentralized and distributed architectures. A decentralized adaptive controller is proposed for a class of uncertain interconnected systems with unknown modeling errors and interactions (Cai *et al.*, 2022), where the proposed controller can ensure that all closed-loop system signals are bounded by using backstepping methods. Harno and Petersen (2014) demonstrate that the decentralized controller can exploit known nonlinearities and interconnections between subsystems without treating them as uncertainties. A decentralized state estimator is proposed by Liu and Yu (2018) for spatially interconnected systems with arbitrary connection relations, and an optimization problem based on linear matrix inequalities (LMIs) is constructed for the computations of improved subsystem parameter matrices. The decentralized event-triggered controller is proposed for a nonlinear interconnected system (Huo *et al.*, 2021).

Zhai *et al.* (2013) construct a decentralized output feedback controller for linear interconnected feedback systems with quantized measurement outputs. The interconnection involves output signals from its own subsystems only. A two-level decentralized controller is generated by Lavaei (2009), where the top level represents the centralized controller. A new procedure for the design of decentralized static and output-feedback tracking controllers is presented for a class of interconnected and disturbed Takagi–Sugeno systems. The decentralized controller's design conditions are given in terms of LMIs via extended quadratic Lyapunov functions (Jabri *et al.*, 2020). Ben Amor and Elloumi (2018) proposed a method to obtain a decentralized controller for an interconnected system via resolving an LMI problem.

In some references (Zhai *et al.*, 2001; Chen *et al.*, 2005; Benlatreche *et al.*, 2008; Veselỳ and Thuan, 2011; Qu *et al.*, 2014) BMIs are used to get a decentralized controller. BMI solution techniques are constantly being investigated since BMI formulations have some advantages. When considering spectral abscissa optimization, BMIs formulations can avoid a nonsmooth objective function that is hard to handle (Burke *et al.*, 2002). BMI can yield less conservative designs than LMI formulations (Chiu, 2017) and may outperform LMI approaches that can fail to compute the stability of Takagi–Sugeno (T–S) fuzzy systems (Kiriakidis, 2001). A systematic way to solve the BMI problem for dynamic output feedback controllers is proposed by Javanmardi *et al.* (2021). The authors decompose the original BMIs to a BMI problem and an LMI problem, reducing the dimension and complexity of the iterative algorithm. Ojaghi and Rahmani (2017) present a robust controller to translate the underlying matrix inequality to an LMI. Based on these results, the linear nature of the LMI formulation makes it tractable for finding the best solution.

Nonetheless, the transformation employs conservative approaches that reduce the overall performance by constricting the design area and settling for a sub-optimal solution. A path-following approach for calculating BMIs is set forth by Hassibi *et al.* (1999). It is assumed that the open- and closed-loop systems are different. However, the findings only apply to unknown variables with a specific structure. To overcome the BMI problem, a novel branch-and-bound approach is proposed by Tuan and Apkarian (2000), in which the bilinear terms are substituted with new variables, and the BMIs turn into LMIs. According to these results, the computational cost and conservatism of the optimization issue are both high. Wang *et al.* (2018) use convex-concave decomposition techniques to convert BMIs to LMIs; a strictly feasible initial value is necessary, and the conservatism of the optimization issue, in general, is high.

In contrast to LMIs, there are crossing terms between two matrix variables in BMIs. As such, computations over BMI constraints are more complicated than LMI constraints. Accordingly, we need a reasonably efficient method to obtain the solution. A homotopy method is proposed by Zhai *et al.* (2001) to get a feasible solution of BMI for the decentralized controller problem. The homotopy method works by fixing one group of variables and solving for the other variables from the LMI. This method is similar to alternate minimization (AM), which has more simplicity and effectiveness than other BMI methods (Javanmardi *et al.*, 2022). The initial point in that method is the value of the centralized controller. Therefore, the existence of a suitable centralized controller is essential. In the work of Zhai *et al.* (2001), the initial value of the centralized controller is obtained by considering only constant free parameters whose singular values are less than γ. Chen *et al.* (2005), assumed that no uncertainty exists and eliminated the bilinear term so that BMI became an LMI. The initial centralized controller for the double homotopy approach was obtained with some existing method (Gahinet and Apkarian, 1994) by Qu *et al.* (2014).

In this paper, we formulate the decentralized output feedback controller using a BMI. The homotopy algorithm is used to solve the BMI. To find a suitable initial centralized controller K for which the algorithm converges, we introduce a nonsingular H matrix and a design parameter β. The introduced H matrix inspired by De Oliveira *et al.* (1999) as well as Chang and Yang (2014) can help us to obtain K using LMIs. In our method, the first step is generating a random value of the design parameter β and solving the LMI of the centralized controller. Then with the homotopy algorithm, a decentralized controller is computed. In this case, the procedure is repeated using a different β if the algorithm does not find a decentralized controller. This method will be applied to two cases: a numerical example and a storey building system.

Notation. The notation of $\mathbb{R}^n$ generally regards $n \times 1$ column vectors or $1 \times n$ row vectors. A zero matrix or a null matrix is a matrix in which all entries are zero, and is denoted by $\mathbf{0}$. A block diagonal matrix whose diagonal blocks are $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$ in order, is represented by the notation $\mathrm{diag}\{\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n\}$. We use $\mathcal{X} < 0$ ($\mathcal{X} > 0$) notation to represent a symmetric negative (resp., positive) definite matrix. The $\star$ notation in a symmetric matrix denotes the entry implied by symmetry.

The $\mathcal{L}_2$ norm of a vector $v(k)$ is defined as $\sqrt{\sum_{j=0}^{\infty} v^T(j)v(j)}$, and $\mathcal{L}_2[0,\infty)$ denotes the set of all time-varying vectors that have finite $\mathcal{L}_2$ norms.

Let $\mathcal{S}$ be the transfer function of a discrete-time system. The H_∞ norm of $\mathcal{S}$ is defined as the supremum (minimal upper bound) of the largest singular number of its transfer function over the unit circle:

$$\| \mathcal{S} \|_\infty = \sup_{\omega \in [-\pi,\pi]} \bar{\sigma}\left(\mathcal{S}(e^{j\omega})\right),$$

where the $\bar{\sigma}$ notation represents the maximum singular value. The H_∞ norm for a dynamic system captures how a measurable signal in $\mathcal{L}_2$ with finite energy is amplified at the monitored output of the system.

An element-wise multiplication of two matrices with the same dimension is called Hadamard product. In addition, the Hadamard product is commutative, associative, and distributive,

$$\begin{aligned}
\mathcal{T} \odot \mathcal{U} &= \mathcal{U} \odot \mathcal{T},\\
\mathcal{T} \odot (\mathcal{U} \odot \mathcal{V}) &= (\mathcal{T} \odot \mathcal{U}) \odot \mathcal{V},\\
\mathcal{T} \odot (\mathcal{U} + \mathcal{V}) &= \mathcal{T} \odot \mathcal{U} + \mathcal{T} \odot \mathcal{V},\\
\mathcal{T} \odot \mathbf{0} &= \mathbf{0} \odot \mathcal{T} = \mathbf{0},\\
(k\mathcal{T}) \odot \mathcal{U} &= \mathcal{T} \odot (k\mathcal{U}) = k(\mathcal{T} \odot \mathcal{U}),
\end{aligned}$$

where the $\odot$ notation is the Hadamard product operator and $\mathcal{T}, \mathcal{U}, \mathcal{V}, \mathbf{0}$ are matrices of the same dimension, $k \in \mathbb{R}$.

2. System description and problem formulation

Consider a linear interconnected system consisting of $\mathcal{N}$ subsystems. For the i-th subsystem, we define

$$\begin{aligned}
x_i(k+1) &= \sum_{j=1}^{\mathcal{N}} A_{ij}x_j(k) + B_{1i}w_i(k) + B_{2i}u_i(k),\\
z_i(k) &= C_{1i}x_i(k) + D_{11i}w_i(k) + D_{12i}u_i(k),\\
y_i(k) &= C_{2i}x_i(k) + D_{21i}w_i(k),
\end{aligned} \tag{1}$$

where $i = 1, 2, \dots, \mathcal{N}$ is the index number of subsystems, $x_i(k) \in \mathbb{R}^{n_i}$ is the state variable of the i-th subsystem,

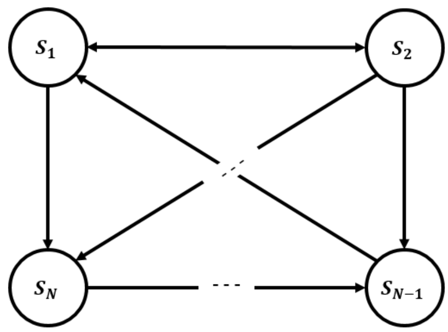

Fig. 1. Diagram of an interconnected system.

$w_i(k) \in \mathbb{R}^{r_i}$ is the disturbance of the i-th subsystem, $u_i(k) \in \mathbb{R}^{m_i}$ is the control input of the i-th subsystem, $z_i(k) \in \mathbb{R}^{p_i}$ is the controlled output variable of the i-th subsystem, $y_i(k) \in \mathbb{R}^{q_i}$ is the measurement output of the i-th subsystem. The matrices A_{ij}, B_{1i}, B_{2i}, C_{1i}, C_{2i}, D_{11i}, D_{12i}, and D_{21i} are constant with appropriate sizes.

As shown in Fig. 1, $S_1, S_2, \dots, S_{\mathcal{N}}$ represent the subsystems. A node in the graph represents a subsystem.

The interconnection between subsystems in system (1) is defined by a directed graph and represented as an adjacency matrix $L = [\ell_{ij}]_{\mathcal{N}\times\mathcal{N}}$ defined by

$$\ell_{ij} = \begin{cases} 1 & \text{if there is an edge from } S_j \text{ to } S_i,\\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Consider the decentralized static output feedback controller for the linear interconnected system. The decentralized controller of the i-th subsystem is described as

$$u_i(k) = \sum_{j=1}^{\mathcal{N}} K_{ij}\ell_{ij}y_j(k), \tag{3}$$

where $K_{ij} \in \mathbb{R}^{m_i \times q_j}$ are decentralized controller gains to be determined. By applying the decentralized controller (3) to the linear interconnected system (1), we have

$$\begin{aligned}
x_i(k+1) &= \sum_{j=1}^{\mathcal{N}} A_{ij}x_j(k) + B_{1i}w_i(k)\\
&\quad + B_{2i}\sum_{j=1}^{\mathcal{N}} K_{ij}\ell_{ij}y_j(k)\\
&= \sum_{j=1}^{\mathcal{N}} A_{ij}x_j(k) + B_{1i}w_i(k)\\
&\quad + B_{2i}\sum_{j=1}^{\mathcal{N}} K_{ij}\ell_{ij}\left(C_{2j}x_j(k) + D_{21j}w_j(k)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{\mathcal{N}} A_{ij}x_j(k) + B_{2i}\sum_{j=1}^{\mathcal{N}} K_{ij}\ell_{ij}C_{2j}x_j(k) \\
&\quad + B_{1i}w_i(k) + B_{2i}\sum_{j=1}^{\mathcal{N}} K_{ij}\ell_{ij}D_{21j}w_j(k), \\
z_i(k) &= C_{1i}\ x_i(k) + D_{11i}w_i(k) \\
&\quad + D_{12i}\sum_{j=1}^{\mathcal{N}} K_{ij}\ell_{ij}y_j(k) \\
&= C_{1i}\ x_i(k) + D_{11i}w_i(k) \\
&\quad + D_{12i}\sum_{j=1}^{\mathcal{N}} K_{ij}\ell_{ij}\left(C_{2j}x_j(k) + D_{21j}w_j(k)\right) \\
&= C_{1i}x_i(k) + D_{12i}\sum_{j=1}^{\mathcal{N}} K_{ij}\ell_{ij}C_{2j}x_j(k) \\
&\quad + D_{11i}w_i(k) + D_{12i}\sum_{j=1}^{\mathcal{N}} K_{ij}\ell_{ij}D_{21j}w_j(k).
\end{aligned} \tag{4}
$$

Equation (4) is the closed-loop system of the i-th subsystem. To describe the closed-loop system in a compact form, we collect the state x_i, the disturbance w_i, and the controlled output z_i as

$$
\begin{aligned}
x &= \begin{bmatrix} x_1^T & \cdots & x_{\mathcal{N}}^T \end{bmatrix}^T \in \mathbb{R}^n, \\
&\quad n = n_1 + \cdots + n_{\mathcal{N}}, \\
w &= \begin{bmatrix} w_1^T & \cdots & w_{\mathcal{N}}^T \end{bmatrix}^T \in \mathbb{R}^r, \\
&\quad r = r_1 + \cdots + r_{\mathcal{N}}, \\
z &= \begin{bmatrix} z_1^T & \cdots & z_{\mathcal{N}}^T \end{bmatrix}^T \in \mathbb{R}^p, \\
&\quad p = p_1 + \cdots + p_{\mathcal{N}}.
\end{aligned} \tag{5}
$$

For the coefficient matrices of each subsystem, we collect them as

$$
\begin{aligned}
A &= \begin{bmatrix} A_{11} & \cdots & A_{1\mathcal{N}} \\ \vdots & \vdots & \ddots \\ A_{\mathcal{N}1} & \cdots & A_{\mathcal{N}\mathcal{N}} \end{bmatrix} \in \mathbb{R}^{n\times n}, \\
B_1 &= \operatorname{diag}\{B_{11}, \cdots, B_{1\mathcal{N}}\} \in \mathbb{R}^{n\times r}, \\
B_2 &= \operatorname{diag}\{B_{21}, \cdots, B_{2\mathcal{N}}\} \in \mathbb{R}^{n\times m}, \\
&\quad m = m_1 + \cdots + m_{\mathcal{N}}, \\
C_1 &= \operatorname{diag}\{C_{11}, \cdots, C_{1\mathcal{N}}\} \in \mathbb{R}^{p\times n}, \\
C_2 &= \operatorname{diag}\{C_{21}, \cdots, C_{2\mathcal{N}}\} \in \mathbb{R}^{q\times n}, \\
&\quad q = q_1 + \cdots + q_{\mathcal{N}}, \\
D_{11} &= \operatorname{diag}\{D_{111}, \cdots, D_{11\mathcal{N}}\} \in \mathbb{R}^{p\times r}, \\
D_{12} &= \operatorname{diag}\{D_{121}, \cdots, D_{12\mathcal{N}}\} \in \mathbb{R}^{p\times m}, \\
D_{21} &= \operatorname{diag}\{D_{211}, \cdots, D_{21\mathcal{N}}\} \in \mathbb{R}^{q\times r},
\end{aligned} \tag{6}
$$

and for the interconnection matrix, we collect the interconnection coefficient ℓ_{ij} as

$$
L_D = \begin{bmatrix} \ell_{11}J_{m_1,q_1} & \cdots & \ell_{1\mathcal{N}}J_{m_1,q_{\mathcal{N}}} \\ \vdots & \ddots & \vdots \\ \ell_{\mathcal{N}1}J_{m_{\mathcal{N}},q_1} & \cdots & \ell_{\mathcal{N}\mathcal{N}}J_{m_{\mathcal{N}},q_{\mathcal{N}}} \end{bmatrix}, \tag{7}
$$

where $L_D \in \mathbb{R}^{m\times q}$ is the interconnection matrix of the decentralized controller and $J_{m,q}$ is a matrix of size $m \times q$ where the entries are one.

By substituting (5)–(7) to the closed-loop system (4), we get the compact form of the closed-loop system as

$$
\begin{aligned}
x(k+1) &= \left(A + B_2\left(L_D \odot K\right)C_2\right)x(k) \\
&\quad + \left(B_1 + B_2\left(L_D \odot K\right)D_{21}\right)w(k) \\
z(k) &= \left(C_1 + D_{12}\left(L_D \odot K\right)C_2\right)x(k) \\
&\quad + \left(D_{11} + D_{12}\left(L_D \odot K\right)D_{21}\right)w(k).
\end{aligned} \tag{8}
$$

The unknown quantity in the formulation (8) is matrix K, whereas the other matrices are obtained from (1). In this paper, the control problem is to design a decentralized controller such that the following two conditions are satisfied:

1. The closed-loop system (8) is asymptotically stable when $w(k) = 0$.
2. The closed-loop system has a prescribed level γ of H_∞ noise attenuation, i.e., under the zero initial condition,

$$
\sum_{k=0}^{\infty} z^T(k)z(k) < \gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k),
$$

is satisfied for any nonzero $w(k) \in \mathcal{L}_2[0,\infty)$.

If the above statements are satisfied, we say that the system (8) is asymptotically stable with the H_∞ performance γ.

3. Centralized controller

In this section, we discuss a method to design a centralized output feedback controller for linear interconnected systems. Each subsystem receives the information from itself and all other subsystems. In the centralized controller, the controller gain K is a full matrix. We can replace L_D in (8) with L_F, where L_F is the interconnection of the centralized controller, represented by the $m \times q$ matrix of ones which leads to

$$
L_F \odot K = K. \tag{9}
$$

The closed-loop system for the centralized controller is

$$
\begin{aligned}
x(k+1) &= (A + B_2KC_2)x(k) \\
&\quad + (B_1 + B_2KD_{21})w(k) \\
z(k) &= (C_1 + D_{12}KC_2)x(k) \\
&\quad + (D_{11} + D_{12}KD_{21})w(k).
\end{aligned} \tag{10}
$$

Our centralized controller result is obtained by using a matrix decoupling technique with the introduction of a non-singular H matrix and a design parameter β, which enables us to derive new LMI conditions to guarantee that the system (10) is asymptotically stable with the H_∞ performance γ. De Oliveira *et al.* (1999) expand the discrete Lyapunov condition for stability analysis by introducing a new matrix variable. As a result, a linear matrix inequality is obtained in which the Lyapunov matrix is not involved in any product with the dynamic system matrix. Based on simulations, it is claimed that such a method does not result in conservativeness due to the presence of the extra degree of freedom provided by the introduction of the matrix. Chang and Yang (2014) use an extra non-singular matrix to eliminate the coupling term. Simulations suggest that for static output feedback controllers this approach significantly relaxes the conservativeness of the existing ones.

The proposed nonsingular matrix is intended to give some more flexibility for improving the iterative computation of the algorithm (Li *et al.*, 2011). The nonsingular matrix H also maintains two identical conditions of consensuality and nonnegativity. In addition, the system matrices, controller gain, and Lyapunov matrices are separated in the conditions, which makes parameterization easier (Liu *et al.*, 2021). The following preliminary lemma is used to deal with the output feedback controller design.

Lemma 1. (Chang, 2014) *The system* (10) *is asymptotically stable with the H_∞ performance γ if there exist a symmetric matrix $P > 0$ and matrix H such that*

$$\begin{bmatrix} \mathcal{P}_{11} & \star \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} < 0, \tag{11}$$

where

$$\begin{aligned} \mathcal{P}_{11} &= \begin{bmatrix} -P & \mathbf{0} \\ \mathbf{0} & -\gamma^2 I_r \end{bmatrix}, \\ \mathcal{P}_{21} &= H \begin{bmatrix} A + B_2 K C_2 & B_1 + B_2 K D_{21} \\ C_1 + D_{12} K C_2 & D_{11} + D_{12} K D_{21} \end{bmatrix}, \\ \mathcal{P}_{22} &= -H - H^T + \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & I_p \end{bmatrix}. \end{aligned}$$

From (11), $\mathcal{P}_{22} < 0$ guarantees that H is nonsingular. We can rewrite $\mathcal{P}_{21}$ in (11) to move the centralized controller gain K outside as follows

$$\begin{aligned} &H \begin{bmatrix} A + B_2 K C_2 & B_1 + B_2 K D_{21} \\ C_1 + D_{12} K C_2 & D_{11} + D_{12} K D_{21} \end{bmatrix} \\ &= H \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} + H \begin{bmatrix} B_2 K C_2 & B_2 K D_{21} \\ D_{12} K C_2 & D_{12} K D_{21} \end{bmatrix} \\ &= H \tilde{\tilde{A}} + H \tilde{\tilde{B}} K \tilde{\tilde{C}}, \end{aligned} \tag{12}$$

where

$$\begin{aligned} &\tilde{\tilde{A}} = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}, \quad \tilde{\tilde{B}} = \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}, \\ &\tilde{\tilde{C}} = \begin{bmatrix} C_2 & D_{21} \end{bmatrix}. \end{aligned} \tag{13}$$

Next, we define $V = UK$, where U is a nonsingular matrix. Then we can rewrite the matrix inequality (11) as

$$\begin{aligned} &\begin{bmatrix} \mathcal{P}_{11} & \star \\ H\tilde{\tilde{A}} + \tilde{\tilde{B}} V \tilde{\tilde{C}} & \mathcal{P}_{22} \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{0} \\ I \end{bmatrix} (H\tilde{\tilde{B}} - \tilde{\tilde{B}}U) U^{-1} V \tilde{\tilde{C}} \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \\ &+ \begin{bmatrix} I & \mathbf{0} \end{bmatrix}^T \tilde{\tilde{C}}^T V^T U^{-T} (H\tilde{\tilde{B}} - \tilde{\tilde{B}}U)^T \begin{bmatrix} \mathbf{0} \\ I \end{bmatrix}^T < 0. \end{aligned} \tag{14}$$

The following results are needed to deal with the above matrix inequality.

Lemma 2. (Zhou and Khargonekar, 1988) *The following inequality holds for matrices $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{J} > 0$ with appropriate sizes:*

$$\mathcal{X}\mathcal{Y} + \mathcal{Y}^T \mathcal{X}^T \le \mathcal{X}\mathcal{J}\mathcal{X}^T + \mathcal{Y}^T \mathcal{J}^{-1} \mathcal{Y}.$$

Lemma 3. (Chang and Yang, 2014) *The following condition is satisfied for scalar β and matrices $\mathcal{T}$, $\mathcal{Q}$, $\mathcal{Y}$, and $\mathcal{A}$ with appropriate sizes:*

$$\begin{bmatrix} \mathcal{T} & \star \\ \mathcal{Y}\mathcal{A} & -\beta\mathcal{Y} - \beta\mathcal{Y}^T + \beta^2 \mathcal{Q} \end{bmatrix} < 0 \tag{15}$$

if and only if

$$\mathcal{T} + \mathcal{A}^T \mathcal{Q} \mathcal{A} < 0. \tag{16}$$

According to Lemma 2, the matrix inequality (14) holds if the following condition is satisfied for a positive definite matrix Z:

$$\begin{aligned} &\begin{bmatrix} \mathcal{P}_{11} & \star \\ H\tilde{\tilde{A}} + \tilde{\tilde{B}} V \tilde{\tilde{C}} & \mathcal{P}_{22} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ I \end{bmatrix} Z \begin{bmatrix} \mathbf{0} \\ I \end{bmatrix}^T \\ &\quad + \begin{bmatrix} I & \mathbf{0} \end{bmatrix}^T \tilde{\tilde{C}}^T V^T U^{-T} (H\tilde{\tilde{B}} - \tilde{\tilde{B}}U)^T \\ &\quad \times Z^{-1} (H\tilde{\tilde{B}} - \tilde{\tilde{B}}U) U^{-1} V \tilde{\tilde{C}} \begin{bmatrix} I & \mathbf{0} \end{bmatrix} < 0. \end{aligned} \tag{17}$$

For the matrix inequality (17), by using Lemma 3 with

$$\begin{aligned} \mathcal{T} &= \begin{bmatrix} \mathcal{P}_{11} & \star \\ H\tilde{\tilde{A}} + \tilde{\tilde{B}} V \tilde{\tilde{C}} & \mathcal{P}_{22} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ I \end{bmatrix} Z \begin{bmatrix} \mathbf{0} \\ I \end{bmatrix}^T, \\ \mathcal{A} &= U^{-1} V \tilde{\tilde{C}} \begin{bmatrix} I & \mathbf{0} \end{bmatrix}, \\ \mathcal{Q} &= (H\tilde{\tilde{B}} - \tilde{\tilde{B}}U)^T Z^{-1} (H\tilde{\tilde{B}} - \tilde{\tilde{B}}U), \\ \mathcal{Y} &= U, \end{aligned}$$

and by substituting $\mathcal{T}$, $\mathcal{A}$, $\mathcal{Q}$ and $\mathcal{Y}$ into (15), we obtain the following matrix condition:

$$\begin{bmatrix} \mathcal{P}_{11} & \star & \star \\ H\tilde{\tilde{A}} + \tilde{\tilde{B}}V\tilde{\tilde{C}} & \mathcal{P}_{22} + Z & \star \\ V\tilde{\tilde{C}} & 0 & \Pi \end{bmatrix} < 0, \tag{18}$$

where $\Pi = -\beta U - \beta U^T + \beta^2 (H\tilde{\tilde{B}} - \tilde{\tilde{B}}U)^T Z^{-1} (H\tilde{\tilde{B}} - \tilde{\tilde{B}}U)$ and β is the design parameter. Next, the Schur complement is applied to (18), which results in

$$\begin{bmatrix} \mathcal{P}_{11} & \star & \star & \star \\ H\tilde{\tilde{A}} + \tilde{\tilde{B}}V\tilde{\tilde{C}} & \mathcal{P}_{22} + Z & \star & \star \\ V\tilde{\tilde{C}} & \mathbf{0} & -\beta U - \beta U^T & \star \\ \mathbf{0} & \mathbf{0} & \beta H\tilde{\tilde{B}} - \beta\tilde{\tilde{B}}U & -Z \end{bmatrix} < 0. \tag{19}$$

Scalar β, $P > 0$, $Z > 0$, V, and nonsingular matrices U, H are to be determined. The centralized controller K is obtained by

$$K = U^{-1}V. \tag{20}$$

We suppose that a centralized controller always exists, which means that we can adjust β to obtain the decentralized controller. Moreover, the solution of (20) will be used as the initial value to compute the decentralized controller for linear interconnected systems.

4. Decentralized controller via the homotopy method

In this section, we will design a decentralized static output feedback controller for linear interconnected systems so that the closed-loop system is asymptotically stable. To design conditions of decentralized H_∞ controllers for the closed-loop system (8), we need the following result.

Lemma 4. *The system* (8) *is asymptotically stable with the H_∞ performance γ if there exist a symmetric matrix $P > 0$ and a matrix H such that*

$$\begin{bmatrix} \mathcal{P}_{11} & \star \\ \mathcal{P}_{\lceil 21} & \mathcal{P}_{22} \end{bmatrix} < 0, \tag{21}$$

where $\mathcal{P}_{\lceil 21} = H\left(\tilde{\tilde{A}} + \tilde{\tilde{B}}(L_D \odot K)\tilde{\tilde{C}}\right)$ *and* $\mathcal{P}_{11}$, $\mathcal{P}_{22}$ *are defined in* (11).

For the proof, see Appendix.

Using the same discussion as in the previous section, we can rewrite (21) in Lemma 4 defining

$$\begin{aligned} &F(L_D, K, P, H) \\ &\quad = \begin{bmatrix} \begin{bmatrix} -P & \mathbf{0} \\ \mathbf{0} & -\gamma^2 I_r \end{bmatrix} & \star \\ H\tilde{\tilde{A}} & -H - H^T + \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & I_p \end{bmatrix} \end{bmatrix} \\ &\qquad + \begin{bmatrix} \mathbf{0} \\ I \end{bmatrix} H\tilde{\tilde{B}}(L_D \odot K)\tilde{\tilde{C}} \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \\ &\qquad + \begin{bmatrix} I & \mathbf{0} \end{bmatrix}^T \tilde{\tilde{C}}^T (L_D \odot K)^T \tilde{\tilde{B}}^T H^T \begin{bmatrix} \mathbf{0} \\ I \end{bmatrix}^T < 0, \end{aligned} \tag{22}$$

where $\tilde{\tilde{A}}$, $\tilde{\tilde{B}}$, and $\tilde{\tilde{C}}$ are in (13). Then we obtain the following result.

Theorem 1. *System* (8) *is asymptotically stable with the H_∞ performance γ if there exist a matrix K, a symmetric positive-definite matrix P and a nonsingular matrix H such that*

$$F(L_D, K, P, H) < 0. \tag{23}$$

Because there are multiplication terms among more than two unknown matrices in (23), we can see that it is a BMI problem. We employ the homotopy method suggested by Zhai *et al*. (2001) to solve (23). The initial step of this method consists in computing the centralized controller, as described in Section 3. At the next step, we solve the BMI by fixing one of the variables alternately. Consider the homotopy function as follows:

$$\begin{aligned} &H_{om}(K, P, H, \sigma) \\ &\qquad = F((1-\sigma)L_F + \sigma L_D, K, P, H), \end{aligned} \tag{24}$$

$\sigma \in [0, 1]$ being a real number. Depending on σ, the matrix function (24) can be written down in the following manner:

$$H_{om}(K, P, H, \sigma) = \begin{cases} F(L_F, K, P, H), & \sigma = 0, \\ F(L_D, K, P, H), & \sigma = 1. \end{cases} \tag{25}$$

We can rewrite the decentralized controller design problem in (25) as an inequality as follows:

$$H_{om}(K, P, H, \sigma_k) < 0. \tag{26}$$

Based on (26), we need to set the initial value of K, or P and H to reduce the BMI to an LMI. Since there is no multiplication between P and H in (22), we decide that P and H are always given as an initial value and solved simultaneously. For example, when $\sigma_k = 0$, we can set the initial value of K by using the centralized controller (20) to solve for P and H. Next, for $\sigma_k = 1$, we can use P and H that we computed in the previous step to determine the decentralized controller $L_D \odot K$. Because decentralized controllers are a subset of centralized controllers we suppose that the initial centralized controller always exists for the same control performance γ. We note that nonconvergence of the algorithm for some centralized controller does not generally indicate that the decentralized controller problem has no solution (Zhai *et al*., 2001).

For more details, the homotopy method to solve the decentralized control problem will be written as Algorithm 1. A step can be considered feasible as Algorithm 1 if a solution to the matrix inequality (26) exists.

Algorithm 1. Homotopy algorithm for the decentralized controller.

1: Generate β randomly.
2: Compute matrix variables P, H and centralized controller gain K.
3: Set $k := 0$, $K_0 := K$, $P_0 := P$, and $H_0 := H$
4: Initialize a positive integer M where the upper bound is $M_{\max}$, e.g., $M := 2$ and $M_{\max} := 2^{10}$.
5: **while** $k < M$ and $M \leq M_{\max}$ **do**
6:   set $\sigma_{k+1} := (k+1)/M$
7:   Compute a solution K of LMI $H_{om}(K, P_k, H_k, \sigma_{k+1}) < 0$
8:   **if** Step 7 is feasible **then**
9:     Set $K_{k+1} := K$
10:     Compute a solution P and H of LMI $H_{om}(K_{k+1}, P, H, \sigma_{k+1}) < 0$, then set $P_{k+1} := P$ and $H_{k+1} := H$
11:     Set $k := k + 1$
12:   **else**
13:     Compute a solution P and H of LMI $H_{om}(K_k, P, H, \sigma_{k+1}) < 0$
14:     **if** Step 13 is feasible **then**
15:       Set $P_{k+1} := P$ and $H_{k+1} := H$
16:       Compute a solution K of LMI $H_{om}(K, P_{k+1}, H_{k+1}, \sigma_{k+1}) < 0$, then set $K_{k+1} := K$
17:       Set $k := k + 1$
18:     **else**
19:       Set $M := 2M$, $P_{2k} := P_k$, $H_{2k} := H_k$, $K_{2k} := K_k$
20:       Set $k := 2k$
21:     **end if**
22:   **end if**
23: **end while**
24: **if** $k = M$ **then**
25:   Matrices K_M, P_M, and H_M are a solution of (23).
26: **else**
27:   Return to Step 1.
28: **end if**

Remark 1. The original problem is a BMI with P, H, and K. There is no coupling between γ and these variables. In consequence, it is possible to consider the optimization problem to minimize γ subject to (23).

Remark 2. Because we solve at most three LMIs in each iteration in Algorithm 1, the total computed LMIs is $3M$ for a given division number M. If the original problem is feasible, we can expect the algorithm will succeed for M large enough. It is commonly recognized that LMIs can be solved very efficiently by using the MATLAB LMI toolbox.

Remark 3. Notice that the algorithm executes the homotopy method again by using different β if there is no solution by using the current value of β.

5. Numerical examples

Example 1. In this part, we shall apply the homotopy algorithm to compute the decentralized static output feedback controller. Consider a linear discrete-time system with four subsystems and the interconnection between subsystems shown in Fig. 2 by a directed graph.

In this case, S_1 has three state variables, while S_2, S_3, and S_4 have two state variables. The control inputs and measurement outputs in each subsystem have the same dimension equal to 2. Based on Fig. 2, the connection between four subsystems is represented as

$$L = \begin{bmatrix} 1 & 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 & 1 \\ 1 & \mathbf{0} & \mathbf{0} & 1 \end{bmatrix}. \tag{27}$$

Due to the self loop in every node, we can fill the diagonal of the interconnection matrix with ones. Based on (27), we know that S_1 can use the information from S_2, S_2 can use the information from S_3, S_3 can use the information from S_4, and S_4 can use the information from S_1. For the coefficient matrices of each subsystem, we have the original interconnected system in (28)–(35).

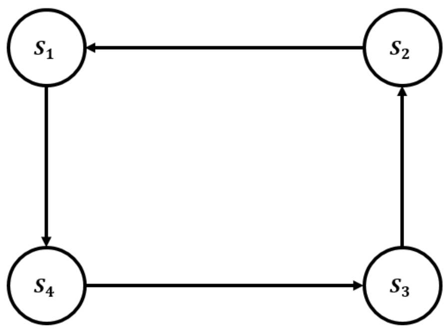

Fig. 2. Interconnection of four subsystems.

$$A = \left[\begin{array}{ccc:cc:cc:cc} 1.00 & 1.00 & 0.57 & 1.00 & 0 & 0 & 0.01 & -0.75 & 0 \\ 0.30 & 0.60 & 0.30 & 0.81 & 0 & 0.50 & 0 & 0.08 & 0.70 \\ 0.40 & -0.40 & 0.47 & 1.00 & 0.01 & 0 & 0.08 & 1.00 & -0.23 \\ \hdashline 0.01 & 0.05 & -0.07 & 0.51 & 1.00 & 0.10 & 0 & 0 & 0.01 \\ 0.77 & 0 & 0 & -0.21 & 0.34 & 0 & 0.20 & 0.80 & 0 \\ \hdashline 0 & 0 & 0.23 & 1.00 & 0 & -0.60 & 0.50 & 0 & 0.03 \\ 0.01 & 0.68 & -1.00 & 1.00 & 0.96 & 0.30 & 0.60 & 0 & 1.00 \\ \hdashline 0.01 & 0 & 0 & -0.06 & 0 & 0.70 & 0 & 1.00 & 0.06 \\ 1.00 & 0.01 & 0.04 & -0.02 & 0.54 & 0 & 0.10 & 0.45 & -0.10 \end{array}\right], \tag{28}$$

$$B_1 = \operatorname{diag}\left\{ \begin{bmatrix} -0.11 & 0.14 \\ 0.03 & 0.03 \\ -0.014 & -0.05 \end{bmatrix}, \begin{bmatrix} -0.02 & 0.14 \\ -0.05 & -0.05 \end{bmatrix}, \begin{bmatrix} -0.16 & 0.11 \\ -0.02 & 0.07 \end{bmatrix}, \begin{bmatrix} 0.02 & -0.06 \\ 0.18 & 0.03 \end{bmatrix} \right\}, \tag{29}$$

$$B_2 = \operatorname{diag}\left\{ \begin{bmatrix} -0.87 & 0.64 \\ -0.78 & 2.05 \\ -0.34 & 0.79 \end{bmatrix}, \begin{bmatrix} -1.07 & -0.55 \\ -0.20 & -0.29 \end{bmatrix}, \begin{bmatrix} 1.18 & 0.99 \\ 0.37 & 0.60 \end{bmatrix}, \begin{bmatrix} -0.79 & -1.35 \\ 0.93 & 0.79 \end{bmatrix} \right\}, \tag{30}$$

$$C_1 = \operatorname{diag}\left\{ \begin{bmatrix} 0.08 & 0.10 \\ 0.10 & -0.10 \\ 0.06 & 0.20 \end{bmatrix}^T, \begin{bmatrix} -0.80 & 0.11 \\ 0.10 & 0.20 \end{bmatrix}^T, \begin{bmatrix} 0 & -0.10 \\ 0.10 & -0.20 \end{bmatrix}^T, \begin{bmatrix} -0.14 & 0.05 \\ 0.10 & -0.20 \end{bmatrix}^T \right\}, \tag{31}$$

$$C_2 = \operatorname{diag}\left\{ \begin{bmatrix} 1.00 & -2.00 \\ 1.00 & 0.50 \\ 0 & 1.00 \end{bmatrix}^T, \begin{bmatrix} -0.20 & 0 \\ 0 & 1.00 \end{bmatrix}^T, \begin{bmatrix} 1.00 & 0.50 \\ 0 & 2.00 \end{bmatrix}^T, \begin{bmatrix} 1.00 & -3.00 \\ 0 & 1.00 \end{bmatrix}^T \right\}, \tag{32}$$

$$D_{11} = \operatorname{diag}\left\{ \begin{bmatrix} 0.05 & 0.45 \\ 0.05 & 0.32 \end{bmatrix}, \begin{bmatrix} 0.27 & 0.07 \\ 0.05 & -0.14 \end{bmatrix}, \begin{bmatrix} 0.05 & -0.05 \\ -0.27 & 0.32 \end{bmatrix}, \begin{bmatrix} -0.05 & -0.32 \\ 0.09 & 0.36 \end{bmatrix} \right\}, \tag{33}$$

$$D_{12} = \operatorname{diag}\left\{ \begin{bmatrix} -0.01 & 0.08 \\ -0.10 & -0.01 \end{bmatrix}, \begin{bmatrix} -0.01 & -0.01 \\ -0.09 & 0.02 \end{bmatrix}, \begin{bmatrix} 0.02 & 0.02 \\ 0.10 & -0.01 \end{bmatrix}, \begin{bmatrix} 0.01 & 0.01 \\ 0.11 & -0.01 \end{bmatrix} \right\}, \tag{34}$$

$$D_{21} = \operatorname{diag}\left\{ \begin{bmatrix} -0.20 & 0.10 \\ -0.20 & -0.10 \end{bmatrix}, \begin{bmatrix} -0.20 & -0.01 \\ -0.10 & 0.30 \end{bmatrix}, \begin{bmatrix} 0.01 & 0.01 \\ 0.02 & -0.01 \end{bmatrix}, \begin{bmatrix} 0.01 & 0.10 \\ 0.02 & -0.01 \end{bmatrix} \right\}, \tag{35}$$

$$K = \begin{bmatrix} 0.6411 & 0.0910 & 0.0039 & -0.0882 & 0.0855 & 0.1052 & -0.2008 & 0.0553 \\ -0.1483 & -0.0303 & 0.0836 & -0.1122 & -0.2555 & -0.0308 & -0.6377 & -0.1625 \\ 0.0393 & -0.0062 & -0.2086 & 0.4213 & 0.1252 & 0.0274 & 0.0777 & -0.0158 \\ 0.1108 & -0.0454 & 0.0664 & 0.3300 & -0.1288 & 0.0301 & 0.2842 & 0.0534 \\ -0.1349 & -0.0228 & -0.0730 & 0.1402 & 0.2085 & -0.1161 & -0.2731 & -0.0487 \\ -0.1485 & 0.0267 & 0.1969 & -0.3391 & 0.1266 & -0.1977 & -0.5512 & -0.2408 \\ -0.1195 & 0.0949 & 0.1331 & -0.2358 & -0.1152 & 0.0175 & -0.5067 & -0.0907 \\ 0.0158 & 0.0027 & -0.0830 & 0.1136 & 0.3543 & -0.0042 & 0.1358 & -0.1242 \end{bmatrix}. \tag{36}$$

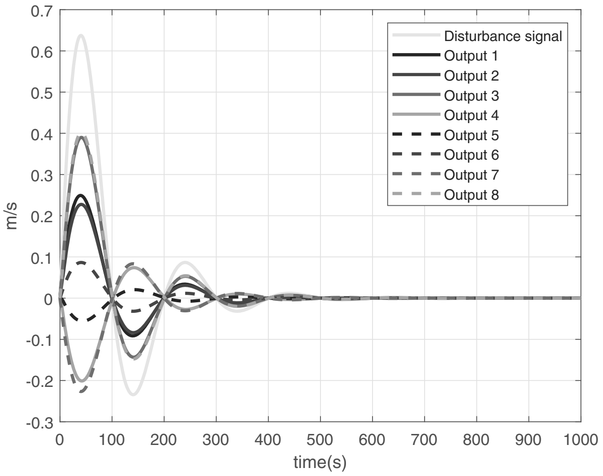

Fig. 3. Response of the discrete-time system.

The eigenvalues of the state matrix A in (28) can be used to check the stability of the original system.

The state matrix A is unstable because the absolute value of some eigenvalues is greater than 1. Therefore, to stabilize the linear interconnected system (28)–(35), a decentralized controller will be designed using the homotopy algorithm. To get the initial value for the algorithm, we need to compute a full matrix for the centralized controller K firstly. We compute K by using (20). Based on the simulation, we obtain the centralized controller K with given $\beta = 1.1$ and $\gamma = 1.5$ as in (36).

We get the asymptotic stability of the closed-loop system since the absolute value of all eigenvalues is smaller than 1 and the H_∞ disturbance attenuation level of a centralized static output feedback controller (36) is $0.5845 < 1.5$.

Next, we use the homotopy method to compute the decentralized static output feedback controller. The decentralized controller K_D for the linear interconnected system (28)–(35) with a specified structure is represented in (37). We know that the specified structure of decentralized controller K_D in (37) is similar to the structure of the Laplacian graph in (27). The H_∞ disturbance attenuation level of decentralized controller (37) is $0.6034 < 1.5$.

Figure 3 displays the plots of the simulated time response of the closed-loop system to the disturbance input $w(k) = e^{-0.01k}\sin(0.01\pi k)$, $k = 0, 1, 2, \ldots, 1000$ with duration of 1000 seconds (1000 iterations). By giving a disturbance input to the closed-loop system, we can observe that the plots of the outputs converge to zero quickly enough. ⧫

Example 2. In recent years, there has been a lot of interest in the technology for managing the construction of multi-storey structures. However, as we all know, the storey construction system has several issues, such as lowering structural reaction, speed, displacement, acceleration, and force in the face of disturbances like earthquakes, high winds, and other disasters.

The matrix equation of motion of the structural system is represented by

$$M\ddot{q}(t) + D\dot{q}(t) + Cq(t) = Ew(t) + Gu(t), \tag{38}$$

where $q(t) \in \mathbb{R}^n$ is the vector of displacements relative to the ground, the mass, spring, and damper matrix coefficients of the storey building system are M, C, and D, respectively, whereas $u(t) \in \mathbb{R}^m$ is a force vector, and $w(t) \in \mathbb{R}^r$ is an external disturbance. The matrices $E \in \mathbb{R}^{n\times r}$ and $G \in \mathbb{R}^{n\times m}$ represent external disturbance and control force coefficients, respectively.

A disturbance like an earthquake ground acceleration $w(t) \in \mathbb{R}^r$ and active control force vector $u(t) \in \mathbb{R}^m$ are applied to the system (38). In this case, the external disturbance $w(t)$ is assumed to be one-dimensional, and the control forces between adjoining floors are specified as $u(t)$. A positive control force is described as moving the floor above the device to the right and moving the floor below the device to the left.

The external disturbance and the control force location matrix for a five-storey building are defined as (Wang *et al.*, 2009)

$$G = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad E = -M \begin{bmatrix} 10^3 \\ 10^3 \\ 10^3 \\ 10^3 \\ 10^3 \end{bmatrix}.$$

We can rewrite (38) in the state space realization as

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t), \tag{39}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, A is a $2n \times 2n$ system matrix, B_1 is a $2n \times r$ matrix; in this case we use a one-dimensional disturbance $r = 1$, and B_2 is a $2n \times m$ matrix,

$$x(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix},$$

$$A = \begin{bmatrix} \mathbf{0} & I \\ -M^{-1}C & -M^{-1}D \end{bmatrix},$$

$$B_1 = \begin{bmatrix} \mathbf{0} \\ M^{-1}E \end{bmatrix},$$

$$B_2 = \begin{bmatrix} \mathbf{0} \\ M^{-1}G \end{bmatrix}.$$

To reflect that the control force and the disturbance are constant during the sampling period T, we define

$$u(t) = u(kT), \tag{40}$$

$$w(t) = w(kT), \tag{41}$$

$$L_D \odot K = \begin{bmatrix} 1.5086 & 0.0837 & -1.5509 & -0.1594 & 0 & 0 & 0 & 0 \\ 0.1795 & -0.0794 & -0.2367 & -0.1887 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.8726 & 0.5190 & -0.1425 & -0.1944 & 0 & 0 \\ 0 & 0 & 0.2950 & 0.5954 & 0.3767 & 0.4037 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.2056 & 0.7499 & 4.2245 & 1.6906 \\ 0 & 0 & 0 & 0 & 0.7455 & -1.1822 & -5.7127 & -2.1745 \\ -0.9827 & 0.0790 & 0 & 0 & 0 & 0 & -3.3340 & -0.9380 \\ 0.6910 & 0.0157 & 0 & 0 & 0 & 0 & 2.6377 & 0.6702 \end{bmatrix}. \quad (37)$$

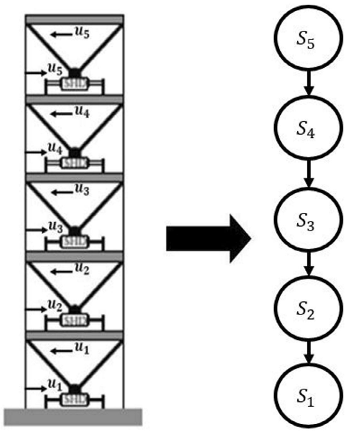

Fig. 4. Illustration of connection of a five-storey building system.

where $kT \le t < (k+1)T$. The solution of the state equation of motion (39) can be given as

$$\begin{aligned} x(t) =& e^{A(t-t_0)}x(t_0) \\ &+ \int_{t_0}^{t} e^{A(t-\theta)} B_2 u(\theta)\,\mathrm{d}\theta \\ &+ \int_{t_0}^{t} e^{A(t-\theta)} B_1 w(\theta)\,\mathrm{d}\theta. \end{aligned}$$

Let $t_0 = kT$, $t = (k+1)T$, and $\tau = (k+1)T - \theta$. One can obtain the following discrete-time system equation of motion:

$$\begin{aligned} x((k+1)T) =& \mathcal{A}(T)x(kT) \\ &+ \mathcal{B}_1(T)w(kT) + \mathcal{B}_2(T)u(kT), \end{aligned}$$

where

$$\begin{aligned} \mathcal{A}(T) &= e^{AT}, \\ \mathcal{B}_1(T) &= \int_0^T e^{A\tau}\,\mathrm{d}\tau B_1, \\ \mathcal{B}_2(T) &= \int_0^T e^{A\tau}\,\mathrm{d}\tau B_2. \end{aligned}$$

For convenience, the discrete-time system equation of motion can be represented as in system (1).

In this case, the homotopy algorithm is used for five-storey building systems. Each storey building is characterized as a subsystem, represented in Fig. 4 by a node. The edges in the directed graph describe the connections within a five-storey building.

For the particular values of the matrices, we have

$$M = 10^3 \times \begin{bmatrix} 215.2 & 0 & 0 & 0 & 0 \\ 0 & 209.2 & 0 & 0 & 0 \\ 0 & 0 & 207.0 & 0 & 0 \\ 0 & 0 & 0 & 204.8 & 0 \\ 0 & 0 & 0 & 0 & 266.1 \end{bmatrix},$$

$$D = 10^3 \times \begin{bmatrix} 650.4 & -231.1 & 0 & 0 & 0 \\ -231.1 & 548.9 & -202.5 & 0 & 0 \\ 0 & -202.5 & 498.6 & -182.0 & 0 \\ 0 & 0 & -182.0 & 466.7 & -171.8 \\ 0 & 0 & 0 & -171.8 & 318.5 \end{bmatrix},$$

$$C = \begin{bmatrix} 260 & -113 & 0 & 0 & 0 \\ -113 & 212 & -99 & 0 & 0 \\ 0 & -99 & 188 & -89 & 0 \\ 0 & 0 & -89 & 173 & -84 \\ 0 & 0 & 0 & -84 & 84 \end{bmatrix} \times 10^6,$$

where M is expressed in kg, D in N/(m/s), and C in N/m. The original system of the five-storey building is asymptotically stable. To show the interconnection between storeys we write the matrix A in (43). The relationship between storeys is represented as the interconnection matrix

$$L = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (42)$$

The centralized and decentralized H_∞ controllers are obtained in (44)–(45), respectively.

$$A = \left[\begin{array}{cc:cc:cc:cc:cc} -0.03 & -0.00 & 0.12 & 0.14 & -0.06 & -0.00 & -0.00 & -0.00 & -0.00 & 0.00 \\ -0.00 & 0.08 & 0.19 & 0.16 & 0.00 & -0.00 & -0.01 & -0.00 & 0.00 & 0.01 \\ \hdashline 0.13 & 0.19 & 0.13 & 0.07 & 0.22 & -0.00 & -0.00 & -0.00 & 0.00 & 0.00 \\ 0.15 & 0.17 & 0.07 & 0.12 & 0.37 & -0.00 & 0.00 & 0.00 & -0.00 & 0.00 \\ \hdashline -0.05 & 0.00 & 0.17 & 0.28 & 0.36 & 0.00 & 0.00 & 0.00 & 0.00 & -0.00 \\ 4.48 & -0.64 & 0.94 & 1.75 & -2.80 & -0.02 & -0.00 & 0.13 & 0.15 & -0.07 \\ \hdashline -0.66 & 5.10 & 1.41 & 0.30 & -3.39 & -0.00 & 0.09 & 0.19 & 0.16 & -0.00 \\ 0.98 & 1.42 & 4.61 & -2.87 & -1.54 & 0.13 & 0.20 & 0.14 & 0.06 & 0.21 \\ \hdashline 1.84 & 0.31 & -2.90 & 2.71 & -1.09 & 0.15 & 0.17 & 0.06 & 0.13 & 0.37 \\ -2.26 & -2.67 & -1.20 & -0.84 & 2.94 & -0.05 & -0.00 & 0.17 & 0.28 & 0.37 \end{array}\right], \tag{43}$$

$$K = 10^{-3} \times \begin{bmatrix} -0.1620 & 0.0040 & -0.0022 & -0.0010 & -0.0040 \\ 0.0030 & -0.1541 & -0.0010 & -0.0017 & -0.0016 \\ 0.0022 & -0.0060 & -0.0707 & 0.0062 & -0.0074 \\ -0.0018 & 0.0018 & 0.0009 & -0.0790 & 0.0003 \\ -0.0008 & -0.0059 & -0.0385 & -0.0092 & -0.0948 \end{bmatrix}, \tag{44}$$

$$L_D \odot K = \begin{bmatrix} -0.1300 & -0.0125 & 0 & 0 & 0 \\ 0 & -0.1420 & 0.0295 & 0 & 0 \\ 0 & 0 & -0.3627 & -0.0027 & 0 \\ 0 & 0 & 0 & -0.3261 & 0.1549 \\ 0 & 0 & 0 & 0 & -0.1643 \end{bmatrix}. \tag{45}$$

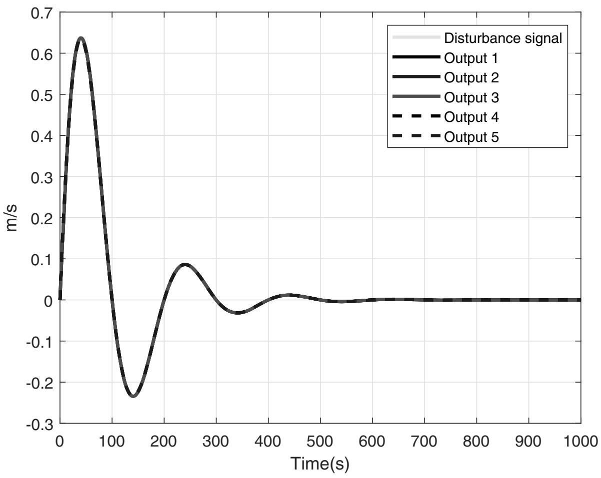

Fig. 5. Response of the five-storey building system.

The H_∞ norm of the decentralized H_∞ controller is 1.0026, where $\gamma = 1.5$. In this simulation, we simulate the five-storey building system responses to $w(k) = e^{-0.01k}\sin(0.01\pi k)$, $k = 0, 1, 2, \ldots, 1000$, cf. Fig. 5.

Each trajectory in Fig. 5 represents the response of one of the five system outputs to the $w(k)$ signal applied to all inputs. Based on the simulation results, there is no different path between the outputs. This result happens because the original system is asymptotically stable. We can observe that the system state converges to zero very quickly. ♦

6. Conclusions

In this paper, we introduce a method to generate a new centralized controller. The new centralized controller is used when the homotopy method does not solve the BMI. Based on the simulations, the proposed method can synthesize a decentralized controller for some interconnected systems. Although the simulation shows good results, the algorithm's initial values use the centralized controller solution. However, the solution of the centralized controller may not exist. Therefore, in our future research we will address this issue. We are also interested in applying this technique to other problems.

Acknowledgment

It is a tribute to Helisyah Nur Fadhilah's supervisor, Prof. Erna Apriliani, who passed away on July 6, 2021, during the writing of this paper. This work is supported by the National Research and Innovation Agency Republic of Indonesia under doctoral research (main contract no. 008/E5/PG.02.00.PT/2022 and derivative contract no. 1559/PKS/ITS/2022).

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Appendix

Proof of Lemma 4

In this case, we use the following quadratic Lyapunov function to assure the internal stability of (8):

$$\mathbf{V}(k) = x^T(k)Px(k), \quad P > 0. \tag{A1}$$

Define the following function:

$$\begin{aligned} g(z(k), w(k)) &= \|z(k)_2^2 - \gamma^2\|\|w(k)\|_2^2 \\ &= z^T(k)z(k) - \gamma^2 w^T(k)w(k), \end{aligned} \tag{A2}$$

and the result of the total difference is

$$\begin{aligned} &\mathbf{V}(k+1) - \mathbf{V}(k) + g(z(k), w(k)) \\ &= \mathbf{V}(k+1) - \mathbf{V}(k) \\ &\quad + z^T(k)z(k) - \gamma^2 w^T(k)w(k) \\ &= x^T(k+1)Px(k+1) - x^T(k)Px(k) \\ &\quad + z^T(k)z(k) - \gamma^2 w^T(k)w(k) \\ &= ((A + B_2(L_D \odot K)C_2)x(k) \\ &\quad + (B_1 + B_2(L_D \odot K)D_{21})w(k))^T P \\ &\quad \times ((A + B_2(L_D \odot K)C_2)x(k) \\ &\quad + (B_1 + B_2(L_D \odot K)D_{21})w(k)) \\ &\quad - x^T(k)Px(k) \\ &\quad + ((C_1 + D_{12}(L_D \odot K)C_2)x(k) \\ &\quad + (D_{11} + D_{12}(L_D \odot K)D_{21})w(k))^T \\ &\qquad ((C_1 + D_{12}(L_D \odot K)C_2)x(k) \\ &\quad + (D_{11} + D_{12}(L_D \odot K)D_{21})w(k)) \\ &\quad - \gamma^2 w^T(k)w(k) \end{aligned}$$

$$
\begin{aligned}
&=\eta^T(k)\,([A+B_2(L_D\odot K)C_2\ B_1\\
&\quad +B_2(L_D\odot K)D_{21}]^T\\
&\quad P[A+B_2(L_D\odot K)C_2B_1\\
&\quad +B_2(L_D\odot K)D_{21}]\\
&\quad +[C_1+D_{12}(L_D\odot K)C_2\ D_{11}\\
&\quad +D_{12}(L_D\odot K)D_{21}]^T\\
&\quad [C_1+D_{12}(L_D\odot K)C_2\ D_{11}\\
&\quad +D_{12}(L_D\odot K)D_{21}]+\mathcal{P}_{11})\,\eta(k),
\end{aligned}
$$

where

$$\eta(k)=\begin{bmatrix} x(k)\\ w(k)\end{bmatrix}.$$

Next, we obtain the following inequality:

$$\sum_{k=0}^{\infty}(\mathbf{V}(k+1)-\mathbf{V}(k)+g(z(k),w(k)))<0, \quad \text{(A3)}$$

for any $\eta(k)\neq 0$, if (A4) holds. We have

$$
\begin{aligned}
&\big[A+B_2(L_D\odot K)C_2 \quad B_1+B_2(L_D\odot K)D_{21}\big]^T\\
&P\big[A+B_2(L_D\odot K)C_2 \quad B_1+B_2(L_D\odot K)D_{21}\big]\\
&+\big[C_1+D_{12}(L_D\odot K)C_2 \quad D_{11}+D_{12}(L_D\odot K)D_{21}\big]^T\\
&\big[C_1+D_{12}(L_D\odot K)C_2 \quad D_{11}+D_{12}(L_D\odot K)D_{21}\big]\\
&\qquad +\mathcal{P}_{11}<0.
\end{aligned} \quad \text{(A4)}
$$

By substituting

$$\mathcal{P}_{11}=\begin{bmatrix}-P & \mathbf{0}\\ \mathbf{0} & -\gamma^2 I_r\end{bmatrix}$$

and applying the Schur complement (Zhang, 2006) to (A4), we get

$$\begin{bmatrix}\begin{bmatrix}-P & \mathbf{0}\\ \mathbf{0} & -\gamma^2 I_r\end{bmatrix} & \star\\ \tilde{\tilde{A}}+\tilde{\tilde{B}}(L_D\odot K)\tilde{\tilde{C}} & -\begin{bmatrix}P^{-1} & \mathbf{0}\\ \mathbf{0} & I_p\end{bmatrix}\end{bmatrix}<0. \quad \text{(A5)}$$

Suppose that

$$-G=-\begin{bmatrix}P & \mathbf{0}\\ \mathbf{0} & I_p\end{bmatrix}.$$

We see that $-G^{-1}$ is the (2,2)-th element of (A5). To eliminate the inverse terms in (A5), we pre- and post-multiply the matrix inequality by $\mathrm{diag}\{I_{n+r},H\}$ and its transpose, respectively. We get

$$\begin{bmatrix}\begin{bmatrix}-P & \mathbf{0}\\ \mathbf{0} & -\gamma^2 I_r\end{bmatrix} & \star\\ H\left(\tilde{\tilde{A}}+\tilde{\tilde{B}}(L_D\odot K)\tilde{\tilde{C}}\right) & -HG^{-1}H^T\end{bmatrix}<0. \quad \text{(A6)}$$

Note that $(H-G)G^{-1}(H-G)^T\geq 0$, $G>0$ leads to $HG^{-1}H^T+G\geq H+H^T$, which implies that $-HG^{-1}H^T\leq -H-H^T+G$ that is equal to $\mathcal{P}_{22}$ in (21).

The above discussion implies that, if (21) is true, then we obtain (A3). When $w(k)=0$, from (A3) it follows that

$$\mathbf{V}(k+1)<\mathbf{V}(k) \quad \text{(A7)}$$

for any positive k and nonzero $x(k)$, which means that $\mathbf{V}(k)$ is monotone decreasing. Since the lower bound of $\mathbf{V}(k)$ is zero, this implies that $\mathbf{V}(k)\to 0$; thus $x(k)\to 0$. This result shows that the closed-loop system (8) is asymptotically stable.

Next, for $w(k)\neq 0$ we can rewrite (A3) as

$$
\begin{aligned}
&\sum_{k=0}^{\infty}(\mathbf{V}(k+1)-\mathbf{V}(k))\\
&+\sum_{k=0}^{\infty}z^T(k)z(k)-\gamma^2\sum_{k=0}^{\infty}w^T(k)w(k)<0,
\end{aligned} \quad \text{(A8)}
$$

which yields

$$\mathbf{V}(\infty)-\mathbf{V}(0)+\parallel z\parallel_2^2-\gamma^2\parallel w\parallel_2^2<0.$$

Since $\mathbf{V}(0)=0$ and $\mathbf{V}(\infty)\to 0$, for any nonzero $w(k)\in\mathcal{L}_2[0,\infty)$ we obtain $\|z\|_2^2<\gamma^2\|w\|_2^2$. This completes the proof.

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