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INFINITE ELEMENTARY DIVISOR STRUCTURE-PRESERVING TRANSFORMATIONS FOR POLYNOMIAL MATRICES

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The main purpose of this work is to propose new notions of equivalence between polynomial matrices that preserve both the finite and infinite elementary divisor structures. The approach we use is twofold: (a) the 'homogeneous polynomial matrix approach', where in place of the polynomial matrices we study their homogeneous polynomial matrix forms and use 2-D equivalence transformations in order to preserve their elementary divisor structure, and (b) the 'polynomial matrix approach', where some conditions between the 1-D polynomial matrices and their transforming matrices are proposed.

Keywords: equivalence, transformation, infinite elementary divisors, autoregressive representations

1. Introduction

Consider a linear homogeneous matrix difference equation of the form

$$A(\sigma)\beta(k) = 0, \quad k \in [0, N], \tag{1}$$

$$A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + \cdots + A_0 \in \mathbb{R}[s]^{r\times r}, \tag{2}$$

where σ denotes the forward shift operator. From (Antoniou *et al.*, 1998) it is known that (1) exhibits forward behavior due to the finite elementary divisors of $A(\sigma)$ and backward behavior due to the infinite elementary divisors of $A(\sigma)$ (and not due to its infinite zeros, as in the continuous time case). Actually, if $A(s)$ is nonsquare or square with zero determinant, then, additionally, the right minimal indices play a crucial role in both the forward and backward behavior of the AR-representation (Karampetakis, 2002b). Therefore it seems quite natural to search for relations that preserve both the finite and infinite elementary divisor structures of polynomial matrices. Pugh and Shelton (1978) proposed the extended unimodular equivalence relation (e.u.e.) which has the nice property of preserving only the finite elementary divisors. However, the e.u.e. preserves the i.e.d. only if additional conditions are added.

In the present work we use two different ways to approach and solve this problem of polynomial matrix equivalence. Specifically, in Section 3 we notice that the finite elementary divisor structure of the homogeneous polynomial matrix form $A_q\sigma^q + A_{q-1}\sigma^{q-1}w + \cdots + A_0w^q$ corresponding to (2) gives us complete information on both the finite and infinite elementary divisor structures of (2). Based on this line of thought, we reduce the problem of the equivalence between 1-D polynomial matrices to an equivalence between 2-D polynomial matrices. A more direct and transparent approach is given in Section 4, where we propose additional conditions to the e.u.e. It is shown that both relations provide necessary conditions for two polynomial matrices to possess the same elementary divisor structure. However, in the special set of square and nonsingular polynomial matrices: (a) the provided conditions are necessary and sufficient, and (b) the proposed relations are equivalent relations and define the same equivalence class.

2. Discrete-Time Autoregressive Representations and Elementary Divisor Structure

In what follows, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the fields of real and complex numbers, and $\mathbb{Z}$ and $\mathbb{Z}^+$ denote respectively the integers and non-negative integers. By $\mathbb{R}[s]$ and $\mathbb{R}[s]^{p\times m}$ we denote the sets of polynomials and $p\times m$ polynomial matrices, respectively, with real coefficients and indeterminate $s \in \mathbb{C}$. Consider the polynomial matrix

$$A(s) = A_qs^q + A_{q-1}s^{q-1} + \cdots + A_0 \in \mathbb{R}[s]^{p\times m}, \tag{3}$$

where $A_j \in \mathbb{R}^{p\times m}$, $j = 0, 1, \ldots, q \geq 1$, $A_q \neq 0$.

Definition 1. Let $A(s) \in \mathbb{R}[s]^{p\times m}$ with $\operatorname{rank}_{R(s)} A(s) = r \leq \min(p, m)$. The values $\lambda_i \in \mathbb{C}$ that satisfy the condition $r_i = \operatorname{rank}_{\mathbb{C}} A(\lambda_i) < r$ are called the finite zeros of $A(s)$. Assume that $A(s)$ has l distinct zeros $\lambda_1, \lambda_2, \ldots, \lambda_l \in \mathbb{C}$, and let

$$S^{\mathbb{C}}_{A(s)}(s) = \begin{bmatrix} Q(s) & 0_{r,m-r} \\ 0_{p-r,r} & 0_{p-r,m-r} \end{bmatrix},$$

$$Q(s) = \operatorname{diag}\{\underbrace{1, 1, \ldots, 1}_{k-1}, \varepsilon_k(s), \varepsilon_{k+1}(s), \ldots, \varepsilon_r(s)\},$$

$1 \le k \le r$, be the Smith form of $A(s)$ (in $\mathbb{C}$), where $\varepsilon_i(s) \in \mathbb{R}[s]$ are the invariant polynomials of $A(s)$ and $\varepsilon_j(s)|\varepsilon_{j+1}(s)$, $j = k, k+1, \ldots, r-1$. Assume that each invariant polynomial $\varepsilon_j(s)$ is decomposed into irreducible elementary divisors over $\mathbb{R}$, i.e., let

$$\varepsilon_j(s) = \prod_{i=1}^{l} (s - \lambda_i)^{m_{ij}},$$

where $m_{ij} \in \mathbb{Z}^+$ and $0 \le m_{i1} \le m_{i2} \le \cdots \le m_{ir}$ are the partial multiplicities of the eigenvalue $\lambda_i, i \in l$. The terms $(s-\lambda_i)^{m_{ij}}$ are called the *finite elementary divisors (f.e.d.)* of $A(s)$ at $s = \lambda_i$. We also set

$$S^{\lambda_i}_{A(s)}(s) = \begin{bmatrix} Q_i(s) & 0_{r,m-r} \\ 0_{p-r,r} & 0_{p-r,m-r} \end{bmatrix},$$

$$Q_i(s) = \operatorname{diag}\{\underbrace{1, 1, \ldots, 1}_{k_i-1}, (s-\lambda_i)^{m_{ik_i}}, \ldots, (s-\lambda_i)^{m_{i,r}}\}$$

as the local Smith form of $A(s)$ at $s = \lambda_i$. Finally, define $n := \sum_{i=1}^{l} \sum_{j=1}^{r} m_{ij}$.

Definition 2. If $A_0 \neq 0$, the *dual* matrix $\tilde{A}(s)$ of $A(s)$ is defined as $\tilde{A}(s) := A_0 s^q + A_1 s^{q-1} + \cdots + A_q$. Since $\operatorname{rank} \tilde{A}(0) = \operatorname{rank} A_q$, the dual matrix $\tilde{A}(s)$ of $A(s)$ has zeros at $s = 0$ iff $\operatorname{rank} A_q < r$. Let $\operatorname{rank} A_q < r$ and let

$$S^0_{\tilde{A}(s)}(s) = \begin{bmatrix} \operatorname{diag}\{\underbrace{1, 1, \ldots, 1}_{g-1} s^{\mu_{g+1}}, \ldots, s^{\mu_r}\} & 0_{r,m-r} \\ 0_{p-r,r} & 0_{p-r,m-r} \end{bmatrix} \tag{4}$$

be the local Smith form of $\tilde{A}(s)$ at $s = 0$, where $\mu_j \in \mathbb{Z}^+$ and $0 \le \mu_{g+1} \le \mu_{g+2} \le \cdots \le \mu_r$. The *infinite elementary divisors (i.e.d.)* of $A(s)$ are defined as the finite elementary divisors s^{μ_j} of its dual $\tilde{A}(s)$ at $s = 0$. Also, define $\mu = \sum_{i=g+1}^{r} \mu_i$.

An interesting consequence of the above definition is that in order to prove that the polynomial matrix (3) has no infinite elementary divisors, it is enough to prove that $\operatorname{rank} A_q = r$. It is also easily seen that the finite elementary divisors of $A(s)$ describe the finite zero structure of the matrix polynomial. In contrast, the infinite elementary divisors give a complete description of the total structure at infinity (the pole and zero structure) and not simply that associated with the zeros (Hayton *et al.*, 1988; Vardulakis, 1991).

The structured indices of a polynomial matrix (finite-infinite elementary divisors and right-left minimal indices) are connected with the rank and the degree of the matrix as follows:

Proposition 1. (Antoniou *et al.*, 1998; Praagman, 1991)

(a) If $A(s) = A_0 + A_1 s + \cdots + A_q s^q \in \mathbb{R}[s]^{r\times r}$ and $\det A(s) \neq 0$, then the total number of elementary divisors (finite and infinite ones and multiplicities accounted for) is equal to the product rq, i.e., $n + \mu = rq$.

(b) If $A(s) = A_0 + A_1 s + \cdots + A_q s^q$ is nonsquare or square with zero determinant, then the total number of elementary divisors plus the left and right minimal indices of $A(s)$ (order accounted for) is equal to rq, where r denotes now the rank of the polynomial matrix $A(s)$.

Example 1. Consider the polynomial matrix

$$A(s) = \begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A_0} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A_1} s + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A_2} s^2$$

and its dual

$$\tilde{A}(s) = \begin{bmatrix} s^2 & 1 \\ 0 & s+s^2 \end{bmatrix} = A_2 + A_1 s + A_0 s^2.$$

Then

$$S^{\mathbb{C}}_{A(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix}, \quad S^0_{\tilde{A}(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s^3 \end{bmatrix}.$$

Therefore $A(s)$ has one finite elementary divisor $(s+1)$ and one infinite elementary divisor s^3, i.e., $n + \mu = 1 + 3 = 2 \times 2 = r \times q$. ♦

The elementary divisor structure of a polynomial matrix plays a crucial role in the study of the behavior of discrete-time AR-representations over a closed time interval. Consider, e.g., the q-th order discrete time autoregressive representation

$$A_q \xi_{k+q} + A_{q-1} \xi_{k+q-1} + \cdots + A_0 \xi_k = 0. \tag{5}$$

If σ denotes the forward shift operator $\sigma^i \xi_k = \xi_{k+i}$, then (5) can be written as

$$A(\sigma)\, \xi_k = 0, \quad k = 0, 1, \ldots, N-q, \tag{6}$$

where $A(\sigma)$ is as in (3) and $\xi_k \in \mathbb{R}^r$, $k = 0, 1, \ldots, N$ is a vector sequence. The solution space or behavior $B^N_{A(\sigma)}$ of the AR-representation (6) over the finite time interval $[0, N]$ is defined as

$$B^N_{A(\sigma)} := \left\{ \begin{array}{c} (\xi_k)_{k=0,1,\ldots,N} \subseteq \mathbb{R}^r | \\ \xi_k \text{ satisfies (6) for } k \in [0, N] \end{array} \right\} \subseteq (\mathbb{R}^r)^{N+1} \tag{7}$$

and we have the following result:

Theorem 1. *(a) (Regular case, (Antoniou et al., 1998)) If $A(s) \in \mathbb{R}[s]^{r\times r}$, $\det A(s) \neq 0$, then the dimension of the behavior of (6) $B^N_{A(\sigma)}$ over the finite time interval $k = 0, 1, \ldots, N \geq q$ is given by*

$$\dim B^N_{A(\sigma)} = rq = n + \mu.$$

(b) (Non-regular case, (Karampetakis, 2002a)) If $A(s)$ is nonsquare or square with zero determinant, then by

$$R\big(\xi_1(k), \xi_2(k)\big) = \left\{ \begin{array}{l} (\xi_1(k), \xi_2(k)) \in B^N_{A(\sigma)} \times B^N_{A(\sigma)} : \xi_1(i) = \xi_2(i) \\ \text{for } i = 0, 1, \ldots, q-1 \text{ (same initial conditions)} \\ \text{and } i = N-q+1, \ldots, N \text{ (same final conditions)} \end{array} \right\},$$

we define the equivalence relation that divides the space $B^N_{A(\sigma)}$ into equivalence classes and creates the space $\hat{B}^N_{A(\sigma)} := B^N_{A(\sigma)}/R$. The dimension of $\hat{B}^N_{A(\sigma)}$ is $n + \mu + 2\varepsilon$, where ε denotes the total number of right minimal indices (since the left minimal indices play no role in the construction of the right solution space).

$B^N_{A(\sigma)}$ consists of two subspaces (Karampetakis, 2002a): the one corresponding to the finite elementary divisors of $A(\sigma)$ (and right minimal indices for the non-regular case), which gives rise to solutions moving in the forward direction of time, and the other corresponding to the infinite elementary divisors of $A(\sigma)$ (and right minimal indices for the nonregular case), which gives rise to solutions moving in the backward direction of time.

Example 2. Consider the AR-representation

$$\underbrace{\begin{bmatrix} 1 & \sigma^2 \\ 0 & \sigma+1 \end{bmatrix}}_{A(\sigma)} \underbrace{\begin{bmatrix} \xi^1_k \\ \xi^2_k \end{bmatrix}}_{\xi_k} = 0_{2\times 1}.$$

Then

$$S^{\mathbb{C}}_{A(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix}, \quad S^0_{\tilde{A}(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s^3 \end{bmatrix},$$

$$B^N_{A(\sigma)} = \Bigg\langle \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix} (-1)^k,}_{\text{due to } S^{\mathbb{C}}_{A(s)}(s)}$$

$$\underbrace{\begin{pmatrix} \delta_{N-k} \\ 0 \end{pmatrix}, \begin{pmatrix} \delta_{N-k+1} \\ 0 \end{pmatrix}, \begin{pmatrix} \delta_{N-k+2} \\ -\delta_{N-k} \end{pmatrix}}_{\text{due to } S^0_{\tilde{A}(s)}(s)} \Bigg\rangle,$$

where $\langle \cdot \rangle$ denotes the space spanned by the included discrete-time vectors, and the discrete-time impulse δ_i is defined by

$$\delta_i = \begin{cases} 1 & \text{if} \quad i = 0, \\ 0 & \text{if} \quad i \neq 0. \end{cases}$$

It is easily seen that

$$\dim B^N_{A(\sigma)} = rq = 2 \times 2 = 1 + 3 = n + \mu,$$

where r is the dimension of the square polynomial matrix $A(\sigma)$ and q is the highest degree among the coefficients of the matrix $A(s)$. ♦

Example 3. Consider the polynomial matrix description

$$\begin{aligned} (\sigma^2)\, \xi_k &= -u_k, \\ y_k &= (\sigma+1)\, \xi_k. \end{aligned}$$

In order to find the state-input pair which gives rise to the zero output (the output zeroing problem), we have to solve the following system of difference equations:

$$\underbrace{\begin{bmatrix} \sigma^2 & 1 \\ \sigma+1 & 0 \end{bmatrix}}_{P(\sigma)} \underbrace{\begin{bmatrix} \xi_k \\ u_k \end{bmatrix}}_{x_k} = 0_{2\times 1}.$$

It is easily seen that the above discrete time AR-representation is the one we have already studied in the previous example and therefore the state-input pair which gives rise to the zero output is given by a simple transformation of the space $B^N_{A(\sigma)}$ defined in the previous example, i.e.,

$$\begin{pmatrix} \xi_k \\ u_k \end{pmatrix} = \begin{pmatrix} -l_1(-1)^k - l_4\delta_{N-k} \\ l_1(-1)^k + l_2\delta_{N-k} + l_3\delta_{N-k+1} + l_4\delta_{N-k+2} \end{pmatrix}.$$

♦

Since the elementary divisor structure of a polynomial matrix plays a crucial role in the study of discrete time AR-representations and/or polynomial matrix descriptions over a closed time interval, we are interested in finding relations that leave invariant the elementary divisor structure of polynomial matrices.

3. Homogeneous Polynomial Matrix Approach

We present two different approaches to study the infinite elementary divisor structure of a polynomial matrix. The first approach is to apply a suitably chosen conformal mapping to bring the infinity point to some finite point, while the second approach is to use homogeneous polynomials to study the infinity point.

Let $P(m,l)$ be the class of $(r+m)\times(r+l)$ polynomial matrices in any number of variables, where l and m are fixed integers and r ranges over all integers which are greater than $\max(-m,-l)$.

Definition 3. (Pugh and Shelton, 1978) $A_1(s),\ A_2(s)\in P(m,l)$ are said to be *extended unimodular equivalent* (e.u.e.) if there exist polynomial matrices $M(s)$ and $N(s)$ such that

$$\begin{bmatrix} M(s) & A_2(s)\end{bmatrix}\begin{bmatrix} A_1(s)\\ -N(s)\end{bmatrix}=0, \tag{8}$$

where the compound matrices

$$\begin{bmatrix} M(s) & A_2(s)\end{bmatrix},\quad \begin{bmatrix} A_1(s)\\ -N(s)\end{bmatrix} \tag{9}$$

have full rank $\forall s\in\mathbb{C}$.

The e.u.e. relates matrices of different dimensions and preserves the f.e.d. of the polynomial matrices involved (Pugh and Shelton, 1978). In the case in which we are interested to preserve only the elementary divisors at a specific point s_0, we introduce the $\{s_0\}$-equivalence relation.

Definition 4. (Karampetakis *et al.*, 1994) $A_1(s), A_2(s)\in P(m,l)$ are said to be $\{\mathrm{s}_0\}$*-equivalent* if there exist rational matrices $M(s)$ and $N(s)$, having no poles at $s=s_0$, such that (8) is satisfied and where the compound matrices in (9) have full rank at $s=s_0$.

The $\{s_0\}$-equivalence preserves only the f.e.d. of $A_1(s),\ A_2(s)\in P(m,l)$ of the form $(s-s_0)^i,\ i>0$ (Karampetakis *et al.*, 1994).

Based on the above polynomial matrix relations, we can easily define the following polynomial matrix relation:

Definition 5. $A_1(s),A_2(s)\in P(m,l)$ are said to be *strongly equivalent* if there exist

(i) *polynomial matrices* $M_1(s),\ N_1(s)$, such that (8) is satisfied and where the compound matrices in (9) have full rank $\forall s\in\mathbb{C}$,

(ii) *rational matrices* $M_2(s),\ N_2(s)$, having no poles at $s=0$, such that (8) between the dual polynomial matrices $\tilde{A}_1(s),\ \tilde{A}_2(s)$ is satisfied and where the respective compound matrices in (9) have full rank at $s=0$.

Theorem 2. *(a) The strong equivalence is an equivalence relation on* $P(m,l)$.

(b) $A_1(s),\ A_2(s)\in P(m,l)$ *are strongly equivalent iff* $S^{\mathbb{C}}_{A_1(s)}(s)$ *is a trivial expansion of* $S^{\mathbb{C}}_{A_2(s)}(s)$ *and* $S^{0}_{\tilde{A}_1(s)}(s)$ *is a trivial expansion of* $S^{0}_{\tilde{A}_2(s)}(s)$*, i.e., the s.e. leaves the finite and infinite elementary divisors invariant.*

Proof.

(a) The e.u.e. and the $\{s_0\}$-equivalence are equivalence relations on $P(m,l)$ (Karampetakis *et al.*, 1994; Pugh and Shelton, 1978), and thus the strong equivalence is an equivalence relation on $P(m,l)$ since it is an intersection of the e.u.e. and the $\{s_0\}$-equivalence.

(b) The strong equivalence is an intersection of the e.u.e. and $\{s_0\}$-equivalence relations. However, $A_1(s),\ A_2(s)\in P(m,l)$ are (i) e.u.e. iff $S^{\mathbb{C}}_{A_1(s)}(s)$ is a trivial expansion of $S^{\mathbb{C}}_{A_2(s)}(s)$, and (ii) $\{s_0\}$-equivalent iff $S^{0}_{\tilde{A}_1(s)}(s)$ is a trivial expansion of $S^{0}_{\tilde{A}_2(s)}(s)$. ■

Based on the properties of the e.u.e. and the $\{s_0\}$-equivalence of preserving respectively the f.e.d. and the i.e.d. at $s=s_0$, we can easily observe that the above relation has the nice property of preserving both the finite and infinite elementary divisors of $A_i(s)$.

Example 4. Consider the polynomial matrices

$$A_1(s)=\begin{bmatrix}1 & s^2\\ 0 & s+1\end{bmatrix},\quad A_2(s)=\begin{bmatrix}s&0&-1&0\\0&s&0&-1\\1&0&0&s\\0&1&0&1\end{bmatrix},$$

and their dual polynomial matrices

$$\tilde{A}_1(s)=\begin{bmatrix}s^2 & 1\\ 0 & s+s^2\end{bmatrix},\quad \tilde{A}_2(s)=\begin{bmatrix}1&0&-s&0\\0&1&0&-s\\s&0&0&1\\0&s&0&s\end{bmatrix}.$$

Then we can find polynomial matrices $M(s),\ N(s)$ and $\tilde{M}(s),\tilde{N}(s)$ such that

$$\underbrace{\begin{bmatrix}0&0\\0&0\\1&0\\0&1\end{bmatrix}}_{M(s)}\underbrace{\begin{bmatrix}1 & s^2\\ 0 & s+1\end{bmatrix}}_{A_1(s)}=\underbrace{\begin{bmatrix}s&0&-1&0\\0&s&0&-1\\1&0&0&s\\0&1&0&1\end{bmatrix}}_{A_2(s)}\underbrace{\begin{bmatrix}1&0\\0&1\\s&0\\0&s\end{bmatrix}}_{N(s)}$$

is an e.u.e. and

$$\underbrace{\begin{bmatrix}0&0\\0&0\\1&0\\0&1\end{bmatrix}}_{\tilde{M}(s)}\underbrace{\begin{bmatrix}s^2 & 1\\ 0 & s+s^2\end{bmatrix}}_{\tilde{A}_1(s)}=\underbrace{\begin{bmatrix}1&0&-s&0\\0&1&0&-s\\s&0&0&1\\0&s&0&s\end{bmatrix}}_{\tilde{A}_2(s)}\underbrace{\begin{bmatrix}s&0\\0&s\\1&0\\0&1\end{bmatrix}}_{\tilde{N}(s)}$$

is a $\{0\}$-equivalence relation. Therefore $A_1(s)$ and $A_2(s)$ are s.e. and thus, according to Theorem 2, they possess the same finite and infinite elementary divisors, i.e.,

$$S^C_{A(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix}, \quad S^C_{A_2(s)}(s) = \begin{bmatrix} I_3 & 0 \\ 0 & s+1 \end{bmatrix},$$
$$S^0_{\tilde{A}_1(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s^3 \end{bmatrix}, \quad S^0_{\tilde{A}_2(s)}(s) = \begin{bmatrix} I_3 & 0 \\ 0 & s^3 \end{bmatrix}.$$

♦

The strong equivalence relation has the disadvantage of consisting of two separate relations. In order to overcome this difficulty, we use the homogeneous variable to represent the infinity.

As the *homogeneous form* of $A(s)$ define the matrix

$$A^H(s,w) = A_0 s^n + A_1 s^{n-1} w + \cdots + A_n w^n. \quad (10)$$

It is easily seen that the finite elementary divisors of $A(s)$ are actually the finite elementary divisors of $A^H(s,1)$. Similarly to the previous definition, we can easily see that the infinite elementary divisors of $A(s)$ are actually the finite elementary divisors of $A^H(1,w)$ at $w=0$. An alternative definition of the finite and infinite elementary divisors in terms of the homogeneous polynomial matrix (10) is given below.

Definition 6. (Praagman, 1991) Let D_i be the greatest common divisor of $i \times i$ minors of A^H, and define $D_0 = 1$. Then $D_i | D_{i+1}$, and let $D_i | D_{i-1} =: c_i \prod (as - bw)^{\ell_i(b/a)}$, where the product is taken over all pairs $(1,b)$ and $(0,1)$, and $1/0$ is denoted by ∞. The factors $(as-bw)^{\ell_i(b/a)}$ with $\ell_i(b/a) \neq 0$ are called the elementary divisors of $A(s)$, and the integers ℓ_i are said to be the elementary exponents of $A(s)$.

It is easily seen that the pairs $(0,1)$ correspond to the i.e.d. while the remaining pairs to the f.e.d.

Example 5. Consider the polynomial matrix $A(s)$ defined in Example 4, i.e.,

$$A(s) = \begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix}.$$

Also, define the homogeneous polynomial matrix

$$A^H(s,w) = \begin{bmatrix} w^2 & s^2 \\ 0 & sw+w^2 \end{bmatrix}.$$

Then

$$D_0 = 1, \quad D_1 = 1, \quad D_2 = w^3(s+w),$$

and therefore the Smith form of $A^H(s,w)$ over $\mathbb{R}[s,w]$ is given by

$$S^{\mathbb{C}}_{A^H(s,w)}(s,w) = \begin{bmatrix} 1 & 0 \\ 0 & w^3(s+w) \end{bmatrix},$$

where we have the following pairs of (a,b): $(1,1)$ with the exponent 1 and $(0,1)$ with the exponent 3. The first pair corresponds to the f.e.d. $(s+1)^1$, while the second pair corresponds to the i.e.d. w^3. ♦

An extension of the e.u.e. to the 2-D setting is given by two relations, factor and zero coprime equivalences (Johnson, 1993). While both of them preserve the invariant polynomials of the equivalent matrices (Johnson, 1993), the latter has the additional property of preserving the ideals of a polynomial matrix (Pugh and El-Nabrawy, 2003) and therefore is more restrictive. Since we are interested only in the invariant polynomials of the homogeneous polynomial matrices and not in their corresponding ideals, we present and use only the former relation.

Definition 7. $A_1(s,w),\ A_2(s,w) \in P(m,l)$ are said to be *factor coprime equivalent* (f.c.e.) if there exist polynomial matrices $M(s,w)$ and $N(s,w)$ such that

$$\begin{bmatrix} M(s,w) & A_2(s,w) \end{bmatrix} \begin{bmatrix} A_1(s,w) \\ -N(s,w) \end{bmatrix} = 0, \quad (11)$$

where the compound matrices

$$\begin{bmatrix} M(s,w) & A_2(s,w) \end{bmatrix}, \quad \begin{bmatrix} A_1(s,w) \\ -N(s,w) \end{bmatrix} \quad (12)$$

are factor coprime, i.e., if all $(r+m)\times(r+m)$ (resp. $(r+l)\times(r+l)$) minors of $\begin{bmatrix} M(s,w) & A_2(s,w) \end{bmatrix}$ (resp. $\begin{bmatrix} A_1(s,w) \\ -N(s,w) \end{bmatrix}$) have no polynomial factor.

Theorem 3. (Johnson, 1993; Levy, 1981)

1. The f.c.e. is only reflexive and transitive and therefore it is not an equivalence relation. The f.c.e. is an equivalence relation on the set of square and nonsingular polynomial matrices.

2. If $A_1(s),\ A_2(s) \in P(m,l)$ are f.c.e., then they have the same invariant polynomial.

Since (a) the above relation leaves the invariant polynomials of the equivalent polynomial matrices invariant and (b) the elementary divisor structure of a polynomial matrix is completely characterized by the invariant polynomials of its homogeneous polynomial matrix, it seems quite natural to reduce the problem of the equivalence between two 1-D polynomial matrices to the problem of the equivalence between its respective homogeneous polynomial matrices.

Definition 8. $A_1(s),\ A_2(s) \in P(m,l)$ are defined to be *factor equivalent* if their respective homogeneous polynomial matrices $A_1^H(s,w),\ A_2^H(s,w)$ are factor coprime equivalent.

Due to the properties of the factor coprime equivalence, it is easy to prove the following result:

Corollary 1. *(i) The f.e. is reflexive and transitive. It is an equivalence relation on the set of square and nonsingular polynomial matrices.*

(ii) If $A_1(s),\ A_2(s) \in P(m,l)$ are f.e., then they have the same finite and infinite elementary divisors.

Although the factor coprime equivalence does not satisfy the symmetry property in the general class of two variable polynomial matrices, this does not necessary imply that the symmetry property is not met for the special class of homogeneous polynomial matrices, either. However, as we can see from the following counter-example, the symmetry property is not satisfied for the class of homogeneous polynomial matrices either and therefore the f.e. is not an equivalence relation.

Example 6. Consider the polynomial matrices

$$A_1(s) = \begin{bmatrix} 1 & s \end{bmatrix}, \quad A_2(s) = \begin{bmatrix} 1 & s & 0 \\ 0 & 1 & s \end{bmatrix}$$

and their respective homogeneous polynomial matrices

$$A_1^H(s,w) = \begin{bmatrix} w & s \end{bmatrix}, \quad A_2^H(s,w) = \begin{bmatrix} w & s & 0 \\ 0 & w & s \end{bmatrix}.$$

Then we can find polynomial matrices $M(s,w)$ and $N(s,w)$ such that

$$\underbrace{\begin{bmatrix} q_{31}w \\ q_{41}w \end{bmatrix}}_{M(s,w)} \underbrace{\begin{bmatrix} w & s \end{bmatrix}}_{A_1^H(s,w)}$$

$$= \underbrace{\begin{bmatrix} w & s & 0 \\ 0 & w & s \end{bmatrix}}_{A_2^H(s,w)} \underbrace{\begin{bmatrix} q_{31}w - q_{41}s & q_{31}s \\ q_{41}w & 0 \\ 0 & q_{41}w \end{bmatrix}}_{N(s,w)},$$

where

$$S^{\mathbb{C}}_{\left[M \ A_2^H \right]}(s,w) = \begin{bmatrix} I_2 & 0_{2\times 2} \end{bmatrix},$$

$$S^{\mathbb{C}}_{\left[\begin{smallmatrix} A_1^H \\ -N \end{smallmatrix} \right]}(s,w) = \begin{bmatrix} I_2 \\ 0_{2\times 2} \end{bmatrix}.$$

Therefore, $A_1(s)$ and $A_2(s)$ are f.e. and thus, according to Corollary 1, they possess the same f.e.d. and i.e.d. (none). However, it is easily seen that the only symmetry transformation is of the form

$$\underbrace{\begin{bmatrix} m_1 & n_6 + m_{21}w \end{bmatrix}}_{M(s,w)} \underbrace{\begin{bmatrix} w & s & 0 \\ 0 & w & s \end{bmatrix}}_{A_2^H(s,w)}$$

$$= \underbrace{\begin{bmatrix} w & s \end{bmatrix}}_{A_1^H(s,w)} \underbrace{\begin{bmatrix} m_1 - n_{41}s & m_{21}w + n_6 - n_{51}s & m_{21}s \\ n_{41}w & m_1 + n_{51}w & n_6 \end{bmatrix}}_{N(s,w)},$$

where the compound matrix

$$\begin{bmatrix} A_2^H(s,w) \\ -N(s,w) \end{bmatrix}$$

has the common factor $q(s) = n_{41}s^2 - m_1 s - n_{51}ws + n_6 w$. Therefore, the symmetric transformation is not factor coprime or otherwise the symmetry property of f.e. is not satisfied. ♦

In the sequel, we give another example where the finite and infinite elementary divisors of the polynomial matrices $A_1(s),\ A_2(s)$ do not reduce to the empty set.

Example 7. Consider the polynomial matrices $A_1(s)$ and $A_2(s)$ defined in Example 4, and their respective homogeneous polynomial matrices

$$A_1^H(s,w) = \begin{bmatrix} w^2 & s^2 \\ 0 & sw + w^2 \end{bmatrix},$$

$$A_2^H(s,w) = \begin{bmatrix} s & 0 & -w & 0 \\ 0 & s & 0 & -w \\ w & 0 & 0 & s \\ 0 & w & 0 & w \end{bmatrix}.$$

Then we can find polynomial matrices $M(s,w)$ and $N(s,w)$ such that

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{M(s,w)} \underbrace{\begin{bmatrix} w^2 & s^2 \\ 0 & sw + w^2 \end{bmatrix}}_{A_1^H(s,w)}$$

$$= \underbrace{\begin{bmatrix} s & 0 & -w & 0 \\ 0 & s & 0 & -w \\ w & 0 & 0 & s \\ 0 & w & 0 & w \end{bmatrix}}_{A_2^H(s,w)} \underbrace{\begin{bmatrix} w & 0 \\ 0 & w \\ s & 0 \\ 0 & s \end{bmatrix}}_{N(s,w)},$$

where

$$S^{\mathbb{C}}_{\left[\begin{smallmatrix} M & A_2^H \end{smallmatrix}\right]}(s,w) = \begin{bmatrix} I_4 & 0_{4\times 2} \end{bmatrix},$$

$$S^{\mathbb{C}}_{\left[\begin{smallmatrix} A_1^H \\ -N \end{smallmatrix}\right]}(s,w) = \begin{bmatrix} I_2 \\ 0_{4\times 2} \end{bmatrix}.$$

Therefore $A_1(s)$ and $A_2(s)$ are f.e. and thus, according to Corollary 1, they possess the same f.e.d. and i.e.d. However, it is easily seen that the compound matrix $\begin{bmatrix} M & A_1 \end{bmatrix}$ has singularities at $s = w = 0$ for any matrix $M(s,w)$ and, therefore, $A_1^H(s,w)$ and $A_2^H(s,w)$ are not zero coprime equivalent (Johnson, 1993; Levy, 1981), although they possess the same invariant polynomials. Therefore, it is seen that the zero coprime equivalence would be quite restrictive for our purpose. This is easily checked out in the case where $A_1(s)$, $A_2(s)$ are of different dimensions. Then there is no zero coprime equivalence relation between $A_1^H(s,w)$, $A_2^H(s,w)$. ♦

For 1-D systems, (Gohberg *et al.*, 1982) presented an algorithm that reduces a general arbitrary polynomial matrix $A(s)$ to an equivalent matrix pencil. More specifically, given the polynomial matrix $A(s)$ in (3) and the matrix pencil

$$sE - A := \begin{bmatrix} sI_m & -I_m & 0 & \cdots & 0 \\ 0 & sI_m & -I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -I_m \\ A_0 & A_1 & A_2 & \cdots & A_q s + A_{q-1} \end{bmatrix}, \tag{13}$$

the following holds:

Theorem 4. *The polynomial matrix $A(s)$ defined in (3) and the matrix pencil $sE - A$ defined in (13) are f.e.*

Proof. Consider the relation

$$\underbrace{\begin{bmatrix} 0_{(q-1)m,p} \\ I_p \end{bmatrix}}_{M(s,w)} A^H(s,w) = [sE - wA] \underbrace{\begin{bmatrix} w^{q-1} I_m \\ w^{q-2} s I_m \\ \vdots \\ w s^{q-2} I_m \\ s^{q-1} I_m \end{bmatrix}}_{N(s,w)}.$$

Then the compound matrix $\begin{bmatrix} M(s,w) & sE - wA \end{bmatrix}$ has two $qm \times qm$ minors equal to $s^{(q-1)m}$ and $(-w)^{(q-1)m}$, respectively, and thus the matrices are factor coprime. These minors are

$$\det \begin{bmatrix} sI_m & -wI_m & 0 & \cdots & 0 & 0 \\ 0 & sI_m & -wI_m & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_m & 0 \\ A_0 w & A_1 w & A_2 w & \cdots & A_{q-2} w & I_p \end{bmatrix},$$

$$\det \begin{bmatrix} -wI_m & 0 & \cdots & 0 & 0 \\ sI_m & -wI_m & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -wI_m & 0 \\ A_1 w & A_2 w & \cdots & A_q s + A_{q-1} w & I_m \end{bmatrix},$$

and they are equal to $s^{(q-1)m}$ and $(-w)^{(q-1)m}$, respectively. Similarly, the compound matrix $\left[\begin{smallmatrix} A^H(s,w) \\ -N(s,w) \end{smallmatrix}\right]$ has two coprime $m \times m$ minors, $s^{(q-1)m}$ and $w^{(q-1)m}$, and thus it is factor coprime, i.e.,

$$\det \left[w^{q-1} I_m \right] = w^{(q-1)m}, \quad \det \left[s^{q-1} I_m \right] = s^{(q-1)m}.$$

Therefore, the matrices $\begin{bmatrix} M(s,w) & sE - wA \end{bmatrix}$ and $\left[\begin{smallmatrix} A^H(s,w) \\ -N(s,w) \end{smallmatrix}\right]$ are factor coprime, $A^H(s,w)$ and $sE - wA$ are factor coprime equivalent, whereas $A(s)$ and $sE - A$ are factor equivalent. ■

An illustrative example of the above theorem has already been given as Example 7. A direct consequence of the above theorem is given by the following result:

Corollary 2. *$A(s)$ and $sE - A$ possess the same finite and infinite elementary divisor structures.*

Proof. $A(s)$ and $sE - A$ are f.e. from Theorem 4 and thus, according to Corollary 1, they possess the same finite and infinite elementary divisor structures. ■

A completely different and more transparent approach to the problem of the equivalence between 1-D polynomial matrices, without using the theory of 2-D polynomial matrices, is given in the next section.

4. Polynomial Matrix Approach

Although the e.u.e. preserves the finite elementary divisors, it does not preserve the infinite elementary divisors, as we can see in the following.

Example 8. Consider the following e.u.e. relation:

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{M(s)} \underbrace{\begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix}}_{A_1(s)} = \underbrace{\begin{bmatrix} 1 & s^3 \\ 0 & s+1 \end{bmatrix}}_{A_2(s)} \underbrace{\begin{bmatrix} 1 & s^2 - s^3 \\ 0 & 1 \end{bmatrix}}_{N(s)}.$$

Although $A_1(s)$ and $A_2(s)$ have the same finite elementary divisors, i.e.,

$$S^{\mathbb{C}}_{A_1(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix} = S^{\mathbb{C}}_{A_2(s)}(s),$$

they have different infinite elementary divisors, i.e.,

$$S^{0}_{\tilde{A}_1(s)}(s) = S^{0}_{\left[\begin{smallmatrix} s^2 & 1 \\ 0 & s+s^2 \end{smallmatrix}\right]} = \begin{bmatrix} 1 & 0 \\ 0 & s^3 \end{bmatrix},$$

$$S^{0}_{\tilde{A}_2(s)}(s) = S^{0}_{\left[\begin{smallmatrix} s^3 & 1 \\ 0 & s^2+s^3 \end{smallmatrix}\right]} = \begin{bmatrix} 1 & 0 \\ 0 & s^5 \end{bmatrix}.$$

♦

The above example indicates that further restrictions must be placed on the compound matrices (9) in order to ensure that the associated relation will leave both the finite and infinite elementary divisors invariant. A new relation between polynomial matrices of the same set $P(m,l)$ is given in the following definition:

Definition 9. Two matrices $A_1(s),\ A_2(s) \in P(m,l)$ are said to be *divisor equivalent* (d.e.) if there exist polynomial matrices $M(s),\ N(s)$ of appropriate dimensions such that (8) is satisfied, where

(i) the compound matrices in (9) are left prime and right prime matrices, respectively,

(ii) the compound matrices in (9) have no infinite elementary divisors,

(iii) the following degree conditions are satisfied:

$$d\begin{bmatrix} M(s) & A_2(s) \end{bmatrix} = d\left[A_2(s)\right] \quad \text{or } d\left[M(s)\right] \le d\left[A_2(s)\right],$$

$$d\begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} = d\left[A_1(s)\right] \quad \text{or } d\left[N(s)\right] \le d\left[A_1(s)\right], \tag{14}$$

where $d[P]$ denotes the degree of $P(s)$ seen as a polynomial with nonzero matrix coefficients.

Theorem 5. *If $A_1(s),\ A_2(s) \in P(m,l)$ are divisor equivalent, then they have the same finite and infinite elementary divisors.*

Proof. According to condition (i) of "divisor equivalence", $A_1(s)$ and $A_2(s)$ are also e.u.e. and thus they have the same finite elementary divisors.

By setting $s = 1/w$ (8) may be rewritten as the equation

$$\begin{bmatrix} M\left(\frac{1}{w}\right) & A_2\left(\frac{1}{w}\right) \end{bmatrix} \begin{bmatrix} A_1\left(\frac{1}{w}\right) \\ -N\left(\frac{1}{w}\right) \end{bmatrix} = 0.$$

Then premultiplying and postmultiplying it by

$$w^{d[\, M(s) \ A_2(s)\,]} \quad \text{and} \quad w^{d\left[\begin{smallmatrix} A_1(s) \\ -N(s) \end{smallmatrix}\right]},$$

respectively, gives

$$w^{d[\, M(s) \ A_2(s)\,]} \begin{bmatrix} M\left(\frac{1}{w}\right) & A_2\left(\frac{1}{w}\right) \end{bmatrix} \times \begin{bmatrix} A_1\left(\frac{1}{w}\right) \\ -N\left(\frac{1}{w}\right) \end{bmatrix} w^{d\left[\begin{smallmatrix} A_1(s) \\ -N(s) \end{smallmatrix}\right]} = 0$$

$$\Longleftrightarrow \begin{bmatrix} \widetilde{M(w) \quad A_2(w)} \end{bmatrix} \begin{bmatrix} \widetilde{A_1(w)} \\ -N(w) \end{bmatrix} = 0, \tag{15}$$

where '~' denotes the dual matrix. Now, since

$$d\begin{bmatrix} M(s) & A_2(s) \end{bmatrix} = d\left[A_2(s)\right]$$

and

$$d\begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} = d\left[A_1(s)\right],$$

(15) may be rewritten as

$$\begin{bmatrix} M'(w) & \tilde{A}_2(w) \end{bmatrix} \begin{bmatrix} \tilde{A}_1(w) \\ -N'(w) \end{bmatrix} = 0. \tag{16}$$

The compound matrix $\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}$ (resp. $\left[\begin{smallmatrix} A_1(s) \\ -N(s) \end{smallmatrix}\right]$) has no infinite elementary divisors and therefore its dual $\begin{bmatrix} M'(w) & \tilde{A}_2(w) \end{bmatrix}$ (resp. $\left[\begin{smallmatrix} \tilde{A}_1(w) \\ -N'(w) \end{smallmatrix}\right]$) has no finite zeros at $w=0$. Therefore, the relation (15) is a $\{0\}$-equivalence relation which preserves the finite elementary divisors of $\tilde{A}_1(w)$ and $\tilde{A}_2(w)$ at $w=0$ or otherwise the infinite elementary divisors of $A_1(s)$ and $A_2(s)$. ■

Example 9. Consider the polynomial matrices $A_1(s)$ and $A_2(s)$ defined in Example 4. Then we can find polynomial matrices $M(s)$ and $N(s)$ such that

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ s-2 & 0 \\ 0 & s-2 \end{bmatrix}}_{M(s)} \underbrace{\begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix}}_{A_2(s)}$$

$$= \underbrace{\begin{bmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 1 & 0 & 0 & s \\ 0 & 1 & 0 & 1 \end{bmatrix}}_{A_1(s)} \underbrace{\begin{bmatrix} s-2 & 0 \\ 0 & s-2 \\ s(s-2) & 0 \\ 0 & s(s-2) \end{bmatrix}}_{N(s)}$$

is a divisor equivalence relation, i.e.,

$$S^{\mathbb{C}}_{[\,M \;\; A_1\,]}(s) = S^{0}_{[\,\tilde{M} \;\; \tilde{A}_1\,]}(s) = \begin{bmatrix} I_4 & 0_{4\times 2} \end{bmatrix},$$

$$S^{\mathbb{C}}_{\left[\begin{smallmatrix} A_2 \\ -N \end{smallmatrix}\right]}(s) = S^{0}_{\left[\begin{smallmatrix} \tilde{A}_2 \\ -\tilde{N} \end{smallmatrix}\right]}(s) = \begin{bmatrix} I_2 \\ 0_{4\times 2} \end{bmatrix},$$

and

$$d\begin{bmatrix} M & A_1 \end{bmatrix} = 1 = d\,[A_1]\,,$$

$$d\begin{bmatrix} A_2 \\ -N \end{bmatrix} = 2 = d\,[A_2]\,.$$

Therefore $A_1(s)$ and $A_2(s)$ are divisor equivalent and thus, according to Theorem 5, they possess the same finite and infinite elementary divisors. ♦

Although the d.e. preserves both the f.e.d. and i.e.d., it is not known if the d.e. is an equivalence relation on $P(p,m)$ and it provides necessary and sufficient conditions for two polynomial matrices to possess the same f.e.d. and i.e.d. Also the exact geometrical meaning of the degree conditions appearing in the definition of the d.e. is under research. Now, consider the following set of polynomial matrices:

$$\mathbb{R}_c[s] := \left\{ \begin{matrix} A(s) = A_0 + A_1 s + \cdots + A_q s^q \in \mathbb{R}[s]^{r\times r} \\ \det A(s) \neq 0 \text{ and } c = rq,\ r \geq 2 \end{matrix} \right\}. \tag{17}$$

Example 10. The polynomial matrices $A_1(s)$ and $A_2(s)$ defined in Example 4 belong to $\mathbb{R}_4[s]$ since $r_1 q_1 = 2 \times 2 = 1 \times 4 = r_2 q_2$. ♦

The degree conditions of the d.e. in $\mathbb{R}_c[s]$ are redundant as we can see in the following result:

Lemma 1. (Karampetakis *et al.*, 2002)

(a) Let $A_1(s)$ *and* $A_2(s) \in \mathbb{R}_c[s]$ *with dimensions* $m\times m$ *and* $(m+r)\times(m+r)$*, respectively, where* $r \neq 0$*. Then the first two conditions of the d.e. imply the degree conditions of the d.e., i.e.,* $\deg M(s) \leq \deg A_2(s)$ *and* $\deg N(s) \leq \deg A_1(s)$.

(b) Let $A_1(s)$ *and* $A_2(s) \in \mathbb{R}_c[s]$ *have the same dimensions* $m \times m$ *and therefore the same degree* d*. If* $A_1(s)$ *and* $A_2(s)$ *satisfy (8) and the first two conditions of the d.e., then* $\deg M(s) = \deg N(s)$.

Therefore, in this special case we are able to reformulate the definition of the d.e. on $\mathbb{R}_c[s]$ with only two conditions.

Definition 10. Two matrices $A_1(s),\ A_2(s) \in \mathbb{R}_c[s]$ are called *divisor equivalent* (d.e.) if there exist polynomial matrices $M(s)$ and $N(s)$ of appropriate dimensions, such that (8) is satisfied where the compound matrices in (9) have full rank and neither f.e.d. nor i.e.d.

Some properties of the d.e. are given in the following result:

Theorem 6. (Karampetakis *et al.*, 2002)

(a) $A_1(s),\ A_2(s) \in \mathbb{R}_c[s]$ *are d.e. iff they have the same f.e.d. and i.e.d.*

(b) The d.e. is an equivalence relation on $\mathbb{R}_c[s]$.

A different approach concerning the equivalence between two polynomial matrices on $\mathbb{R}_c[s]$ is presented in (Vardulakis and Antoniou, 2001).

Definition 11. (Vardulakis and Antoniou, 2001) $A_1(s)$ and $A_2(s) \in \mathbb{R}_c[s]$ are called *strictly equivalent* iff their equivalent matrix pencils $sE_1 - A_1 \in \mathbb{R}^{c\times c}$ and $sE_2 - A_2 \in \mathbb{R}^{c\times c}$ proposed in (13) are strictly equivalent in the sense of (Gantmacher, 1959).

D.e. and s.e. define the same equivalence class on $\mathbb{R}_c[s]$.

Theorem 7. (Karampetakis *et al.*, 2002) *Strict equivalence (cf. Definition 11) gives the same equivalence class as d.e.*

A geometrical meaning of d.e. is given in the sequel.

Definition 12. (Vardulakis and Antoniou, 2001) Two AR-representations

$$A_i(\sigma)\,\xi^i_k = 0, \quad k = 0,1,2,\ldots,N,$$

where σ is the shift operator, $A_i(\sigma) \in \mathbb{R}_c[\sigma]^{r_i\times r_i}$, $i = 1,2$ will be *called fundamentally equivalent (f.e.)* over the finite time interval $k = 0,1,2,\ldots,N$ iff there exists a bijective polynomial map between their respective behavior $\mathcal{B}^N_{A_1(\sigma)}$ and $\mathcal{B}^N_{A_2(\sigma)}$.

S.e. and f.e. define the same equivalence class on $\mathbb{R}_c[s]$.

Theorem 8. (Antoniou and Vardulakis, 2003) *Strict equivalence (cf. Definition 11) gives the same equivalence class as fundamental equivalence.*

A direct consequence of Theorems 7 and 8 is given by the following result:

Theorem 9. $A_1(s),\ A_2(s) \in \mathbb{R}_c[s]$ *are d.e. iff they are f.e.*

Proof. Although the proof is a direct consequence of Theorems 7 and 8, here we give an alternative proof of the "if" part.

From (8) we have

$$M(\sigma)A_1(\sigma) = A_2(\sigma)N(\sigma). \tag{18}$$

Multiplying (18) on the right by ξ_k^1, we get

$$\begin{aligned} M(\sigma)A_1(\sigma)\xi_k^1 &= A_2(\sigma)N(\sigma)\xi_k^1 \\ &\Longrightarrow 0 = A_2(\sigma)N(\sigma)\xi_k^1 \\ &\Longrightarrow \exists \xi_k^2 \in \mathcal{B}_{A_2(\sigma)} \text{ s.t. } \xi_k^2 = N(\sigma)\xi_k^1. \qquad (19) \end{aligned}$$

The map defined by the polynomial matrix $N(\sigma)$: $\mathcal{B}^N_{A_1(s)} \to \mathcal{B}^N_{A_2(s)} \mid \xi_k^1 \mapsto \xi_k^2$ is injective iff $N(\sigma)\xi_k^1 = 0$ implies $\xi_k^1 = 0$. Since $\xi_k^1 \in \mathcal{B}^N_{A_1(s)}$, we have additionally that $A_1(\sigma)\xi_k^1 = 0$. Therefore we get

$$\begin{bmatrix} A_1(\sigma) \\ N(\sigma) \end{bmatrix} \xi_k^1 = 0.$$

However, the above system has only the zero solution (Karampetakis, 2002a), i.e., $\xi_k^1 = 0$, iff the compound matrix $[\ A_1(\sigma)^T \quad -N(\sigma)^T\ ]^T$ has full rank and neither f.e.d. nor i.e.d., the condition that is satisfied by the conditions of d.e. Therefore the map defined by the polynomial matrix $N(\sigma) : \mathcal{B}^N_{A_1(s)} \to \mathcal{B}^N_{A_2(s)} \mid \xi_k^1 \mapsto \xi_k^2$ is injective. Furthermore, $\dim \mathcal{B}^N_{A_1(s)} = c = \dim \mathcal{B}^N_{A_2(s)}$, since $A_i(\sigma) \in \mathbb{R}_c[\sigma]^{r_i \times r_i}$, and thus $N(\sigma)$ is a bijection between $\mathcal{B}^N_{A_1(\sigma)}$ and $\mathcal{B}^N_{A_2(\sigma)}$. ■

5. Conclusions

The forward and backward behaviour of a discrete time AR-representation over a closed time interval is connected with the finite and infinite elementary divisor structures of the polynomial matrix involved in the AR-representation. Furthermore, it is known that a polynomial matrix description can always be written as an AR-representation, and many problems arising from the Rosenbrock system theory can be reduced to problems based on AR-representation theory. This was the motivation of the present work, which presents three new polynomial matrix relations, namely, strong equivalence, factor equivalence and divisor equivalence, which preserve both the finite and infinite elementary divisor structures of polynomial matrices. More specifically, it was shown that strong equivalence is an equivalence relation and provides necessary and sufficient conditions for two polynomial matrices to possess the same elementary divisor structure. However, its main disadvantage is that it consists of two separate relations. We showed that we can overcome this problem using the homogeneous polynomial matrix form of univariate polynomial matrices and then using known relations from 2-D systems theory. Following this reasoning, we introduced the factor equivalence relation. Although factor equivalence is simpler in the sense that it uses only one pair of transformation matrices instead of two (strong equivalence), it suffers from an extra step (homogenization) that is needed. A solution to this problem is given by adding extra conditions to the extended unimodular equivalence relation giving rise to divisor equivalence. We showed that both factor and divisor equivalences provide necessary conditions for two polynomial matrices to possess the same elementary divisor structure. The conditions become necessary and sufficient in the case of square and nonsingular matrices. In this special set of matrices, both relations are equivalence relations sharing the same equivalence class. A geometrical interpretation of the d.e. in terms of maps between the solution spaces of AR-representations is given in the special case of square and nonsingular polynomial matrices.

Finally, certain questions remain open concerning the sufficiency of the divisor equivalence for nonsquare polynomial matrices, or square polynomial matrices with zero determinant. The work (Vardulakis and Antoniou, 2001) proposed a new notion of equivalence, called the fundamental equivalence, in terms of mappings between discrete time AR-representations described by square and nonsingular polynomial matrices. Further research is now focused on the following problems: (a) How can fundamental equivalence be extended to nonsquare polynomial matrices? (b) What are its invariants? (c) What is the connection between the relations presented in this work and the f.e. relation? An extension of these results to the Rosenbrock system matrix theory is also under research.

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