*Appl. Math. and Comp. Sci., 1998, Vol.8, No.2, 253–268*

# $\mu$-ANALYSIS AND ROBUST STABILITY OF POSITIVE LINEAR SYSTEMS

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In this paper, we develop a $\mu$-analysis for nonnegative matrices and apply the results to analyse robust stability of positive linear continuous-time systems under arbitrary affine parameter perturbations. It is shown that real and complex stability radii of positive systems coincide for arbitrary affine perturbation structures, in particular for block-diagonal disturbances. Estimates and computable formulae are derived for these stability radii. The results hold for arbitrary perturbation norms induced by absolute vector norms (e.g. $p$-norms, $1 \leq p \leq \infty$).

## 1. Introduction

The notion of structured singular value (or $\mu$-value) introduced in (Doyle, 1982) is an important linear-algebra tool to study robust stability of uncertain linear systems. Parameter uncertainties in control can often be represented by block-diagonal perturbations (Packard and Doyle, 1993) and the study of this class of perturbations is the subject of $\mu$-analysis. In general, $\mu$-values are difficult to determine, but there exist algorithms for computing upper and lower bounds in the complex case (Packard and Doyle, 1993) Very little is known about the real case (where only real perturbations are considered). Because of these difficulties it is of interest to look for system classes of practical importance for which $\mu$-analysis can be carried out more easily and effectively.

The aim of the present paper is to develop a $\mu$-analysis for nonnegative matrices and to examine robust stability of positive linear systems under arbitrary affine parameter perturbations. An $n$-dimensional linear system

$$\Sigma : \qquad \dot{x}(t) = Ax(t), \qquad t \geq 0$$

is said to be *positive* if it leaves the positive orthant $\mathbb{R}^n_+ = [0, \infty)^n$ invariant in the sense that $e^{At}x_0 \in \mathbb{R}^n_+$ for all $t \geq 0$ whenever $x_0 \in \mathbb{R}^n_+$. It is well-known that a system $\Sigma$ is positive if and only if $A$ is a Metzler matrix, i.e. all off-diagonal elements of $A$ are nonnegative. The mathematical theory of Metzler matrices has a close relationship to the theory of nonnegative matrices founded by Perron and Frobenius. As references we mention (Berman and Plemmons, 1979; Gantmacher, 1959; Luenberger, 1979).

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Suppose that the system $\Sigma$ is asymptotically stable and that the matrix $A$ is subjected to parameter perturbations of the form

$$A \rightsquigarrow A + D\Delta E, \quad \Delta \in \mathcal{D}$$

Here $D$ and $E$ are given matrices specifying the structure of the pertubations, $\mathcal{D}$ is a given class of perturbations and $\Delta \in \mathcal{D}$ is an unknown disturbance matrix whose size is measured by some operator norm $||\Delta||$. If complex perturbations are allowed, the maximal number $\rho$ for which all the perturbed matrices in the set

$$\left\{A + D\Delta E, \Delta \in \mathcal{D} ||\Delta|| < \rho\right\}$$

are asymptotically stable is called the *complex stability radius* (Hinrichsen and Pritchard, 1986). If only real perturbations are considered, the *real stability radius* is obtained. These two stability radii are in general distinct. The complex stability radius is known to be more easily analysed and computed than the real one.

Up till now the theory of stability radii has mainly dealt with single block perturbations where $\mathcal{D} = \mathbb{K}^{l\times q}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$. A detailed analysis of the complex stability radius can be found in (Hinrichsen and Pritchard, 1990). A general formula for the real stability radius (for single block perturbations) was given in (Qiu *et al.*, 1995). The computation of the real stability radius requires the solution of a complicated global optimization problem. It was shown in (Hinrichsen and Pritchard, 1992) that the real stability radius may be an overoptimistic indicator of robustness in the presence of time-varying, nonlinear or dynamic perturbations of the system. Therefore it is of interest to know under which conditions the real and the complex stability radii are equal. It was shown in (Son and Hinrichsen, 1996) that if $A$ is a Metzler matrix, $D$, $E$ are nonnegative and $A$ is subjected to single block perturbations,

$$A \rightsquigarrow A + D\Delta E, \quad \Delta \in \mathbb{K}^{l\times q} \tag{1}$$

the real and complex stability radii coincide and can be determined via a simple computable formula.

In the present paper, we shall extend the results of (Son and Hinrichsen, 1996) to arbitrary affine parameter perturbations of the system $\Sigma$. We consider multi-perturbations of the form

$$A \rightsquigarrow A + \sum_{i=1}^{N} D_i \Delta_i E_i \tag{2}$$

and arbitrary affine perturbations of $A$:

$$A \rightsquigarrow A + \sum_{i=1}^{N} \delta_i A_i \tag{3}$$

where the matrices $A_i$, $D_i$ and $E_i$ are given nonnegative matrices defining the structure of the perturbations and $\Delta_i$ (resp. $\delta_i$) are unknown matrices (resp. scalars). We will prove that real and complex stability radii coincide and can be computed by formulae which extend the formula of (Son and Hinrichsen, 1996) to affine perturbations (3). For multi-perturbations (2), a lower bound for the real stability radius will be given. Throughout the paper the size of the disturbances $\Delta_i$ is measured by arbitrary operator norms induced by ***absolute norms.*** This extends the applicability of the formulae beyond the limits of usual $\mu$-analysis where only spectral norms are considered.

The paper is organized as follows. In the next section, we present some preliminary results concerning Metzler matrices. The equality of real and complex stability radii is established in Section 3. Computable formulae and estimates are derived in Section 4.

## 2. Preliminaries

In this section, we introduce some notations and present some preliminary results which will be of use in the later sections. For more details and proofs we refer to (Son and Hinrichsen, 1996).

Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and $n$, $l$, $q$ be positive integers. The set of all nonnegative $l \times q$-matrices is denoted by $\mathbb{R}_+^{l\times q}$. Inequalities between real matrices or vectors will be understood componentwise, i.e. for two real $l \times q$-matrices $A = (a_{ij})$ and $B = (b_{ij})$, the inequality $A \geq B$ means $a_{ij} \geq b_{ij}$ for $i = 1, \dots, l$, $j = 1, \dots, q$. If $x \in \mathbb{K}^n$ and $P \in \mathbb{K}^{l\times q}$ we define

$$|x| = (|x_i|)\,, \quad |P| = (|p_{ij}|)$$

For any matrix $A \in \mathbb{K}^{n\times n}$ the *spectral radius* and *spectral abscissa* of $A$ are respectively denoted by

$$\rho(A) = \max\Big\{|\lambda|;\ \lambda \in \sigma(A)\Big\}, \quad \gamma(A) = \max\Big\{\operatorname{Re}\lambda;\ \lambda \in \sigma(A)\Big\}$$

where $\sigma(A) \subset \mathbb{C}$ is the spectrum of $A$. The spectral radius has the following monotonicity properties: If $P, Q \in \mathbb{K}^{q\times q}$ and $|P| \leq Q$ then

$$\rho(P) \leq \rho(|P|) \leq \rho(Q) \tag{4}$$

By the Perron-Frobenius theorem

$$\rho(B) = \gamma(B) \in \sigma(B), \quad B \in \mathbb{R}_+^{n\times n}$$

Hence, if $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix, then $\gamma(A) = \rho(tI_n + A) - t$ if $tI_n + A \geq 0$. As a consequence, the spectral abscissa of a Metzler matrix has similar properties as the spectral radius of a nonnegative matrix. The following proposition follows from the Perron-Frobenius Theorem (Berman and Plemmons, 1979; Gantmacher, 1959; Luenberger, 1979).

**Lemma 1.** *Suppose that $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix. Then*

*(i) $\gamma(A)$ is an eigenvalue of $A$ and there exists a nonnegative eigenvector $x \geq 0$, $x \neq 0$ such that $Ax = \gamma(A)x$.*

(ii) *Given* $\alpha \in \mathbb{R}$, *there exists a nonzero vector* $x \geq 0$ *such that* $Ax \geq \alpha x$ *if and only if* $\gamma(A) \geq \alpha$.

(iii) $(tI_n - A)^{-1}$ *exists and is nonnegative if and only if* $t > \gamma(A)$.

With every matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ we associate the Metzler matrix

$$M(A) = A_d + |A - A_d| \tag{5}$$

where $A_d = \operatorname{diag}(a_{11}, \dots, a_{nn})$. The following lemma will be an important tool in the later analysis. For a proof, see (Son and Hinrichsen, 1996).

**Lemma 2.** *Suppose* $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}_+^{n \times n}$, $C \in \mathbb{C}^{n \times n}$. *Then*

$$|C| \leq B \quad \Longrightarrow \quad \gamma(A + C) \leq \gamma(M(A) + B) \tag{6}$$

*In particular,*

$$\gamma(A) \leq \gamma(M(A)) \tag{7}$$

*and if* $A$ *is a Metzler matrix*

$$\gamma(A) \leq \gamma(A + B) \tag{8}$$

A norm $\|\cdot\|$ on $\mathbb{K}^n$ is said to be *absolute* if $\|x\| = \|\,|x|\,\|$ for all $x \in \mathbb{K}^n$. Every $p$-norm on $\mathbb{K}^n$, $1 \leq p \leq \infty$, is absolute. The operator norm of a matrix $M \in \mathbb{K}^{m \times n}$ is defined by

$$\|M\| = \max_{\|y\|=1} \|My\|$$

Suppose that $\mathbb{K}^m$ and $\mathbb{K}^n$ are provided with absolute norms. Then the corresponding operator norm $\|\cdot\|$ will in general *not* be an absolute norm on $\mathbb{K}^{m \times n}$. However, one has the following monotonicity properties (Son and Hinrichsen, 1996):

**Lemma 3.** *Suppose that* $\mathbb{K}^m$, $\mathbb{K}^n$ *are provided with absolute norms and* $\|\cdot\|$ *denotes the corresponding operator norm on* $\mathbb{K}^{m \times n}$. *Then*

(i) *If* $P \in \mathbb{K}^{m \times n}$, $Q \in \mathbb{R}_+^{m \times n}$ *and* $|P| \leq Q$, *then* $\|P\| \leq \|\,|P|\,\| \leq \|Q\|$.

(ii) *If* $P \in \mathbb{K}^{m \times n}$ *is of rank one, then* $\|P\| = \|\,|P|\,\|$.

## 3. $\mu$-Values and Stability Radii

In this section, we study the structured singular values of nonnegative matrices and apply the results in order to study stability radii of positive systems with respect to arbitrary block-diagonal perturbations.

Structured singular values, or $\mu$-values, were introduced by Doyle (1982) as a tool to analyse structured perturbations of control systems for which the availabl singular-value-based techniques gave only conservative results. He considered complex block-diagonal perturbations of the form

$$\Delta \in \mathcal{D} := \Big\{ \operatorname{diag}[\delta_1 I_{r_1}, \dots, \delta_s I_{r_s}, \Delta_1, \dots, \Delta_f] \in \mathbb{C}^{l \times q};$$
$$\delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{l_j \times q_j}, 1 \le i \le s, 1 \le j \le f \Big\}$$

where $I_r$ denotes the $r \times r$ identity matrix and $r_i, l_j, q_j, s, f \ge 1$ are given natural numbers. The associated *$\mu$-value* of a matrix $M \in \mathbb{C}^{q \times l}$ is defined as

$$\mu_{\mathcal{D}}(M) = \Big[\inf \Big\{ \bar{\sigma}(\Delta);\ \Delta \in \mathcal{D}, \det(I_q - M\Delta) = 0 \Big\}\Big]^{-1}$$

where $\bar{\sigma}(\Delta)$ is the largest singular value of $\Delta$. The following two facts are easily verified from the definition. If $\mathcal{D} = \mathbb{C}^{\ell \times q}$, then $\mu_{\mathcal{D}}(M) = \bar{\sigma}(M)$. If $q = \ell$, $\mathcal{D} = \mathbb{C} I_q$ then $\mu_{\mathcal{D}}(M)$ coincides with the spectral radius $\rho(M)$.

The following definition extends Doyle's notion allowing for more general perturbation sets (including sets of real perturbations only) and arbitrary perturbation norms. Throughout the paper we set $\inf \emptyset = \infty$, $\infty^{-1} = 0$, and $0^{-1} = \infty$.

**Definition 1.** Suppose that $M \in \mathbb{C}^{q \times l}$, $\emptyset \ne \mathcal{D} \subset \mathbb{C}^{l \times q}$ and $\operatorname{span} \mathcal{D}$ is provided with a norm $\|\cdot\|_{\mathcal{D}}$. Then

$$\mu_{\mathcal{D}}(M) = 1/\inf \Big\{ \|\Delta\|_{\mathcal{D}}; \Delta \in \mathcal{D},\ \det(I_q - M\Delta) = 0 \Big\} \tag{9}$$

is called the *$\mu$-value* of $M$ with respect to $\mathcal{D}$ and $\|\cdot\|_{\mathcal{D}}$.

If $M$ is nonnegative, it is of interest to compare $\mu_{\mathcal{D}}(M)$ with the $\mu$-values $\mu_{\mathcal{D}_{\mathbb{R}}}(M)$, $\mu_{\mathcal{D}_+}(M)$ obtained by replacing the perturbation set $\mathcal{D}$ by the sets of *real* or *nonnegative* matrices in $\mathcal{D}$, respectively

$$\mathcal{D}_{\mathbb{R}} = \mathcal{D} \cap \mathbb{R}^{l \times q}, \quad \mathcal{D}_+ = \mathcal{D} \cap \mathbb{R}_+^{l \times q} \tag{10}$$

Throughout the paper $\operatorname{span} \mathcal{D}_{\mathbb{R}}$ and $\operatorname{span} \mathcal{D}_+$ will always be endowed with the norm induced from $\mathcal{D}$. Clearly,

$$\mu_{\mathcal{D}}(M) \ge \mu_{\mathcal{D}_{\mathbb{R}}}(M) \ge \mu_{\mathcal{D}_+}(M)$$

The following lemma gives a technical criterion under which equalities hold.

**Lemma 4.** *Suppose that $M \in \mathbb{R}_+^{q \times l}$, $\mathcal{D} \subset \mathbb{C}^{l \times q}$ and $\|\cdot\|_{\mathcal{D}}$ is a norm on $\operatorname{span} \mathcal{D}$ such that for all $\Delta \in \mathcal{D}$ and $y \in \mathbb{C}^q$ there exists $\tilde{\Delta} \in \mathbb{C}^{l \times q}$ satisfying*

$$\tilde{\Delta} y = \Delta y, \quad |\tilde{\Delta}| \in \mathcal{D}, \quad \| |\tilde{\Delta}| \|_{\mathcal{D}} \le \|\Delta\|_{\mathcal{D}} \tag{11}$$

*If $\mathcal{D}_+$ is a cone, then*

$$\mu_{\mathcal{D}}(M) = \mu_{\mathcal{D}_{\mathbb{R}}}(M) = \mu_{\mathcal{D}_+}(M)$$

*Proof.* Since $\mu_{\mathcal{D}}(M) \ge \mu_{\mathcal{D}_{\mathbb{R}}}(M) \ge \mu_{\mathcal{D}_+}(M)$, it suffices to show that for any $\Delta \in \mathcal{D}$ such that $\det(I_q - M\Delta) = 0$ there exists $\Delta_+ \in \mathcal{D}_+$ satisfying $\det(I_q - M\Delta_+) = 0$

and $||\Delta_+||_{\mathcal{D}} \leq ||\Delta||_{\mathcal{D}}$. Now let $(I_q - M\Delta)y = 0$ where $y \in \mathbb{C}^q$, $y \neq 0$. Choose $\tilde{\Delta}$ as in (11). Then $M\tilde{\Delta}y = y$ and, by (4),

$$\tilde{\delta} := \rho(M|\tilde{\Delta}|) \geq \rho(M\tilde{\Delta}) \geq 1$$

Hence $\Delta_+ = \tilde{\delta}^{-1}|\tilde{\Delta}| \in \mathcal{D}_+$ satisfies $||\Delta_+||_{\mathcal{D}} \leq ||\Delta||_{\mathcal{D}}$ and $\det(I_q - M\Delta_+) = 0$ (by the Perron-Frobenius Theorem). ■

The following assumption specifies the type of norms of block-diagonal perturbations for which the results in this section can be established. The assumption is trivially satisfied in classical $\mu$-analysis.

**Assumption 1.** $\mathcal{D}$ is a block-diagonal perturbation class, i.e. there exist integers $l_i \geq 1$, $q_i \geq 1$ for $i \in \underline{N} := \{1, \dots, N\}$ and a subset $J \subset \underline{N}$ such that

$$\mathcal{D} = \left\{ \operatorname{diag}(\Delta_1, \dots, \Delta_N);\ \Delta_i \in \mathcal{D}_i,\ i \in \underline{N} \right\}$$

$$\mathcal{D}_i = \begin{cases} \mathbb{C}^{l_i \times q_i} & \text{if} \quad i \in J \\ \mathbb{C}I_{q_i} & \text{if} \quad i \in \underline{N} \setminus J \end{cases} \tag{12}$$

The vector spaces $\mathbb{C}^{l_i}$, $\mathbb{C}^{q_i}$ are provided with absolute norms and $\mathcal{D}_i$ with the associated operator norm $||\cdot||_{\mathcal{D}_i}$, for each $i \in \underline{N}$. $\mathcal{D}$ is endowed with the norm

$$||\Delta||_{\mathcal{D}} = \left\| \left( ||\Delta_i||_{\mathcal{D}_i} \right)_{i \in \underline{N}} \right\|_{\mathbb{R}^N} \tag{13}$$

where $||\cdot||_{\mathbb{R}^N}$ is a given absolute norm on $\mathbb{R}^N$.

$\mathcal{D}$ given by (12) is a linear subspace of $\mathbb{C}^{l \times q}$, where $l := \sum_{i=1}^N l_i$, $q := \sum_{i=1}^N q_i$. In the case of a single full block (i.e. $N = 1$, $J = \{1\}$ in (12)) the above assumption means that $\mathbb{C}^l, \mathbb{C}^q$ are provided with absolute norms and $\mathcal{D} = \mathbb{C}^{l \times q}$ is endowed with the associated operator norm (if the absolute value is used as the norm on $\mathbb{R}^N = \mathbb{R}^1$). Note that, in general, $||\cdot||_{\mathcal{D}}$ defined by (13) is not an operator norm on $\mathcal{D}$ if $||\cdot||_{\mathbb{R}^N} \neq ||\cdot||_\infty$.

**Proposition 1.** *Suppose that $M \in \mathbb{R}_+^{q \times l}$, $\mathcal{D} \subset \mathbb{C}^{l \times q}$ satisfies Assumption 1 and $\mathcal{D}_{\mathbb{R}}, \mathcal{D}_+$ are defined by (10). Then*

$$\mu_{\mathcal{D}}(M) = \mu_{\mathcal{D}_{\mathbb{R}}}(M) = \mu_{\mathcal{D}_+}(M)$$

*Proof.* It is clear that $\mathcal{D}_+$ is a cone. It suffices to prove that Assumption 1 implies the condition (11) of Lemma 4. Suppose $\Delta = \operatorname{diag}(\Delta_1, \dots, \Delta_N) \in \mathcal{D}$, $y = (y_i)_{i \in \underline{N}} \in \mathbb{C}^q$, $y \neq 0$, $u = (u_i)_{i \in \underline{N}} = \Delta y$. Then $u_i = \Delta_i y_i$, $i \in \underline{N}$, and we define

$$\tilde{\Delta} = \operatorname{diag}(\tilde{\Delta}_1, \dots, \tilde{\Delta}_N) \in \mathcal{D}, \quad \tilde{\Delta}_i = \begin{cases} \Delta_i & \text{if} \quad i \in \underline{N} \setminus J \\ 0 & \text{if} \quad i \in J,\ y_i = 0 \\ u_i y_i^D / ||y_i|| & \text{if} \quad i \in J,\ y_i \neq 0 \end{cases}$$

where $y_i^D \in (\mathbb{C}^{q_i})^*$ is a linear form on $\mathbb{C}^{q_i}$ such that $\|y_i^D\|_{(\mathbb{C}^{q_i})^*} = 1$ and $y_i^D y_i = \|y_i\|_{\mathbb{C}^{q_i}}$ (Hahn-Banach Theorem). Here $(\mathbb{C}^{q_i})^*$ denotes the dual space of $\mathbb{C}^{q_i}$ and $\|\cdot\|_{(\mathbb{C}^{q_i})^*}$ the dual norm of $\|\cdot\|_{\mathbb{C}^{q_i}}$. Then

$$\tilde{\Delta}_i y_i = u_i \quad \text{and} \quad \|\tilde{\Delta}_i\|_{\mathcal{D}_i} \leq \|\Delta_i\|_{\mathcal{D}_i}, \quad i \in \underline{N}$$

and so $\tilde{\Delta} y = \Delta y$. By definition $|\tilde{\Delta}| \in \mathcal{D}$. Moreover, by the fact that all $\Delta_i$ are either in $\mathbb{C}I_{q_i}$ or of rank one and the norm $\|\cdot\|_{\mathbb{R}^N}$ is absolute (and hence monotonic, see (Horn and Johnson, 1985)) it follows from Lemma 3(ii) that

$$\begin{aligned} \|\,|\tilde{\Delta}|\,\|_{\mathcal{D}} &= \left\| \left( \|\,|\tilde{\Delta}_i|\,\|_{\mathcal{D}_i} \right)_{i \in \underline{N}} \right\|_{\mathbb{R}^N} \\ &= \left\| \left( \|\tilde{\Delta}_i\|_{\mathcal{D}_i} \right)_{i \in \underline{N}} \right\|_{\mathbb{R}^N} \leq \left\| \left( \|\Delta_i\|_{\mathcal{D}_i} \right)_{i \in \underline{N}} \right\|_{\mathbb{R}^N} = \|\Delta\|_{\mathcal{D}} \end{aligned}$$

Thus (11) is satisfied and the proposition is proved. ■

We will now apply the above results to analyse stability radii of positive systems with respect to arbitrary real, complex or nonnegative parameter disturbances. For this purpose, we first adapt the definition of stability radius to the more general perturbation class considered here (Hinrichsen and Pritchard, 1986; 1990). The definition is given for arbitrary complex triplets $(A, D, E)$.

Suppose that $A \in \mathbb{C}^{n \times n}$ is a given Hurwitz stable matrix, i.e. $\gamma(A) < 0$. We view $A$ as the nominal system matrix and consider arbitrary affine parameter perturbations of the type

$$A \rightsquigarrow A(\Delta) = A + D\Delta E, \qquad \Delta \in \mathcal{D} \tag{14}$$

Here $D \in \mathbb{C}^{n \times l}$ and $E \in \mathbb{C}^{q \times n}$ are given matrices and $\mathcal{D} \subset \mathbb{C}^{l \times q}$ is an arbitrary given subset of disturbance matrices. The *structure matrices* $D$, $E$ and the *disturbance class* $\mathcal{D}$ together determine the structure of the perturbations $D\Delta E$ of $A$.

**Definition 2.** Given a subset $\mathcal{D} \subset \mathbb{C}^{l \times q}$ and norm $\|\cdot\|_{\mathcal{D}}$ on $\operatorname{span}\mathcal{D} \subset \mathbb{C}^{l \times q}$, the *complex stability radius* of a Hurwitz stable matrix $A \in \mathbb{C}^{n \times n}$ with respect to perturbations of the form (14) is defined by

$$r_{\mathcal{D}} = r_{\mathcal{D}}(A; D, E) = \inf \left\{ \|\Delta\|_{\mathcal{D}};\ \Delta \in \mathcal{D},\ \gamma(A + D\Delta E) \geq 0 \right\} \tag{15}$$

If the disturbance matrices $\Delta$ in (15) are restricted to the sets $\mathcal{D}_{\mathbb{R}}$ or $\mathcal{D}_+$, then we obtain the *real stability radius* $r_{\mathcal{D}_{\mathbb{R}}}$ and the *nonnegative stability radius* $r_{\mathcal{D}_+}$, respectively.

In the following examples we illustrate the above definition by discussing stability radii for various perturbation classes. We begin with the case of single block perturbations $(N = 1)$.

**Example 1.** (Single block perturbations) Suppose that $\mathcal{D} = \mathbb{C}^{l \times q}$, i.e. we consider perturbations of the form

$$A \rightsquigarrow A(\Delta) = A + D\Delta E, \qquad \Delta \in \mathbb{C}^{l \times q} \tag{16}$$

Let $\|\cdot\|_{\mathcal{D}}$ be a given norm on $\mathbb{C}^{l\times q}$. The corresponding complex stability radius is denoted by

$$r_{\mathbb{C}}(A;D,E) := r_{\mathcal{D}}(A;D,E) = \inf\left\{\|\Delta\|_{\mathcal{D}};\ \Delta\in\mathbb{C}^{l\times q},\ \gamma(A+D\Delta E)\geq 0\right\} \quad (17)$$

If $(A,\ D,\ E)$ are real matrices, it is natural to consider only real disturbances, i.e. $\Delta\in\mathbb{R}^{l\times q}=\mathcal{D}_{\mathbb{R}}$. The associated *real stability* is denoted by

$$r_{\mathbb{R}}(A;D,E) := r_{\mathcal{D}_{\mathbb{R}}}(A;D,E) = \inf\left\{\|\Delta\|_{\mathcal{D}};\ \Delta\in\mathbb{R}^{l\times q},\ \gamma(A+D\Delta E)\geq 0\right\} \quad (18)$$

In a similar manner, the nonnegative stability radius $r_{\mathbb{R}_+}(A;(D_i,E_i)_{i\in\underline{N}}) := r_{\mathcal{D}_+}(A;D,E)$ is defined. In general, the complex, real and nonnegative stability radii of $(A,D,E)$ are different (Hinrichsen and Pritchard, 1990).

In the unstructured case $(D=E=I_n)$ the real and complex stability radii $r_{\mathbb{K}}(A)=r_{\mathbb{K}}(A;I_n,I_n)$, $\mathbb{K}=\mathbb{R},\ \mathbb{C}$ represent the distances of $A\in\mathbb{K}^{n\times n}$ from the set of non Hurwitz stable matrices in $\mathbb{K}^{n\times n}$, with respect to the given norm $\|\cdot\|_{\mathcal{D}}$ (Hinrichsen and Pritchard, 1990). ♦

**Remark 1.** At first sight it seems unnatural to consider complex perturbations of a positive system. However, in the case of single block perturbations (16), it was shown in (Hinrichsen and Pritchard, 1992) that the complex stability radius (in contrast to the real one) remains invariant when more general (e.g. nonlinear, time-varying and/or dynamic) perturbations are considered. In fact, it was proved that the complex stability radius $r_{\mathbb{C}}(A;D,E)$ is equal to the stability radius with respect to *real dynamic* perturbations (ibid., Theorem 4.4). Therefore the complex stability radius is of interest when nonlinear, time-varying or dynamic perturbations of the nominal system $\Sigma$ have to be dealt with.

The following two examples describe two special classes of block-diagonal disturbances corresponding to the two extreme cases $J=\underline{N}$ and $J=\emptyset$ in Assumption 1. The first one corresponds to $J=\underline{N}$ and plays an important role in $\mu$-analysis.

**Example 2.** (Multi-perturbations) Suppose that the system matrix $A$ is perturbed as follows:

$$A \rightsquigarrow A(\Delta) = A + \sum_{i=1}^{N} D_i\Delta_i E_i, \quad \Delta_i\in\mathbb{C}^{l_i\times q_i} \quad (19)$$

where $D_i\in\mathbb{C}^{n\times l_i}$, $E_i\in\mathbb{C}^{q_i\times n}$, $i\in\underline{N}$ are given matrices defining the scaling and structure of the parameter uncertainty. Setting

$$\mathcal{D} = \left\{\mathrm{diag}(\Delta_1,\ldots,\Delta_N);\ \Delta_i\in\mathbb{C}^{l_i\times q_i},\ i\in\underline{N}\right\} \quad (20)$$

$l=\sum_{i=1}^N l_i$, $q=\sum_{i=1}^N q_i$ and defining the structure matrices

$$D = [D_1\ldots D_N], \quad E = \begin{bmatrix} E_1 \\ \vdots \\ E_N \end{bmatrix} \quad (21)$$

we see that the multi-perturbations (19) can be written in the form (14). The corresponding complex stability radius will be denoted by

$$\begin{aligned} r_{\mathbb{C}}(A;(D_i,E_i)_{i\in\underline{N}}) &:= r_{\mathcal{D}}(A;D,E) \\ &= \inf\left\{ \|\Delta\|_{\mathcal{D}};\ \Delta\in\mathcal{D},\ \gamma\Big(A+\sum_{i=1}^{N} D_i\Delta_i E_i\Big)\geq 0\right\} \end{aligned}$$

In a similar manner, we can define the real stability radius $r_{\mathbb{R}}(A;(D_i,E_i)_{i\in\underline{N}}) := r_{\mathcal{D}_{\mathbb{R}}}(A;D,E)$ and the nonnegative stability radius $r_{\mathbb{R}_+}(A;(D_i,E_i)_{i\in\underline{N}}) := r_{\mathcal{D}_+}(A;D,E)$ if $A, D_i, E_i$ are real (resp. nonnegative). ♦

The next example corresponds to the special case $J=\emptyset$ in Assumption 1. Arbitrary affine parameter uncertainties can be represented in this way.

**Example 3.** Consider perturbations of the form:

$$A \rightsquigarrow A(\Delta) = A + \sum_{i=1}^{N}\delta_i A_i, \quad \delta_i\in\mathbb{C},\ i\in\underline{N} \tag{22}$$

where $A_i\in\mathbb{C}^{n\times n}$, $i\in\underline{N}$ are given matrices and $\delta_i\in\mathbb{C}$ are unknown scalar parameters. Let $A_i = D_iE_i$ be any factorization of $A_i$ into $D_i\in\mathbb{C}^{n\times q_i}$, $E_i\in\mathbb{C}^{q_i\times n}$, $i\in\underline{N}$ (e.g. $D_i=I_n$, $E_i=A_i$). Define $D$, $E$ as in (21) and set

$$\mathcal{D} = \left\{ \operatorname{diag}(\delta_1 I_{q_1},\ldots,\delta_N I_{q_N});\ \delta_i\in\mathbb{C},\ i\in\underline{N}\right\} \tag{23}$$

Then clearly the affine perturbations (22) can equivalently be described in the form (14). If we provide $\mathcal{D}$ with the operator norm induced by an arbitrary absolute norm on $\mathbb{C}^q$ then, for each $\Delta\in\mathcal{D}$,

$$\|\Delta\|_{\mathcal{D}} = \big\|\operatorname{diag}(\delta_1 I_{q_1},\ldots,\delta_N I_{q_N})\big\|_{\mathcal{D}} = \max_{i\in\underline{N}}|\delta_i| \tag{24}$$

(see e.g. (Horn and Johnson, 1985)) and Definition 2 yields the following complex stability radius:

$$\begin{aligned} r_{\mathbb{C}}\big(A;(A_i)_{i\in\underline{N}}\big) &:= r_{\mathcal{D}}(A;D,E) \\ &= \inf\left\{ \max_{i\in\underline{N}}|\delta_i|;\ \delta_i\in\mathbb{C},\gamma\Big(A+\sum_{i=1}^{N}\delta_iA_i\Big)\geq 0\right\} \end{aligned} \tag{25}$$

For real (resp. nonnegative) triplets $(A,\ D,\ E)$ the stability radii $r_{\mathbb{R}}(A;(A_i)_{i\in\underline{N}})$ and $r_{\mathbb{R}_+}(A;(A_i)_{i\in\underline{N}})$ are defined in a similar way, by taking $\delta_i\in\mathbb{R}$ and $\delta_i\in\mathbb{R}_+$, respectively, in the above definition.

The following proposition establishes a relationship between $\mu$-values and stability radii.

**Proposition 2.** *Suppose that* $(A, D, E) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times l} \times \mathbb{C}^{q\times n}$, $G(s) = E(sI - A)^{-1}D$ *is the associated transfer matrix,* $A$ *is Hurwitz stable,* $\mathcal{D} \subset \mathbb{C}^{l\times q}$ *is a cone and* $\operatorname{span}\mathcal{D}$ *is provided with a norm* $\|\cdot\|_{\mathcal{D}}$. *Then*

$$r_{\mathcal{D}}(A; D, E) = \left[\sup_{\omega\in\mathbb{R}} \mu_{\mathcal{D}}(G(i\omega))\right]^{-1} \tag{26}$$

*Proof.* Suppose that $\Delta \in \mathcal{D}$ is destabilizing, i.e. $\gamma(A + D\Delta E) \geq 0$. By continuity of $\gamma(\cdot)$ there exists $\alpha \in (0, 1]$ such that $\gamma(A + \alpha D\Delta E) = 0$. Let $\omega \in \mathbb{R}$ and $x \in \mathbb{C}^n$, $x \neq 0$, such that $(A + \alpha D\Delta E)x = i\omega x$. Then $x = (i\omega I - A)^{-1}D\alpha\Delta Ex$ and multiplying this equation by $E$ from the left we obtain $y = G(i\omega)\alpha\Delta y$ for $y = Ex \neq 0$, hence $\det(I_q - G(i\omega)\alpha\Delta) = 0$. Therefore $\|\Delta\| \geq \alpha\|\Delta\| \geq [\mu_{\mathcal{D}}(G(i\omega))]^{-1}$ (since $\alpha\Delta \in \mathcal{D}$) and it follows that

$$r_{\mathcal{D}}(A; D, E) \geq \left[\sup_{\omega\in\mathbb{R}} \mu_{\mathcal{D}}(G(i\omega))\right]^{-1}$$

Conversely, suppose that $\Delta \in \mathcal{D}$ satisfies $\det(I_q - G(i\omega)\Delta) = 0$ for some $\omega \in \mathbb{R}$ and let $y \in \mathbb{C}^q$, $y \neq 0$ be such that $G(i\omega)\Delta y = y$. Then by setting $x = (i\omega I - A)^{-1}D\Delta y$ we get $Ex = y$ and hence $(A + D\Delta E)x = i\omega x$. Thus $\gamma(A + D\Delta E) \geq 0$ and $\Delta$ is destabilizing. It follows that, for every $\omega \in \mathbb{R}$, $\mu_{\mathcal{D}}(G(i\omega))^{-1} \geq r_{\mathcal{D}}(A; D, E)$ and this concludes the proof. ∎

We now turn to positive systems.

**Proposition 3.** *Suppose that* $A \in \mathbb{R}^{n\times n}$ *is a Hurwitz stable Metzler matrix,* $(D, E) \in \mathbb{R}_+^{n\times l} \times \mathbb{R}_+^{q\times n}$ *are given nonnegative structure matrices and the perturbation class* $\mathcal{D} \subset \mathbb{C}^{l\times q}$ *satisfies Assumption 1. Then*

$$r_{\mathcal{D}}(A; D, E) = r_{\mathcal{D}_{\mathbb{R}}}(A; D, E) = r_{\mathcal{D}_+}(A; D, E)$$

*Proof.* The case $r_{\mathcal{D}}(A; D, E) = \infty$ is trivial. Suppose $r_{\mathcal{D}}(A; D, E) < \infty$. Clearly, it suffices to show that $r_{\mathcal{D}_+}(A; D, E) \leq r_{\mathcal{D}}(A; D, E)$. As shown in the proof of Proposition 1, Assumption 1 implies that the condition of Lemma 4 is satisfied. Let $\Delta \in \mathcal{D}$ be a minimal norm destabilizing disturbance so that $\gamma(A + D\Delta E) \geq 0$ and $\|\Delta\|_{\mathcal{D}} = r_{\mathcal{D}}(A; D, E)$. Let $(A + D\Delta E)x = \alpha x$, $\operatorname{Re}\alpha \geq 0$ for some $x \in \mathbb{C}^n$, $x \neq 0$, $\alpha \in \mathbb{C}$. Then there exists $\tilde{\Delta} \in \mathcal{D}$ such that

$$\tilde{\Delta}Ex = \Delta Ex, \quad |\tilde{\Delta}| \in \mathcal{D}, \quad \|\,|\tilde{\Delta}|\,\|_{\mathcal{D}} \leq \|\Delta\|_{\mathcal{D}}$$

Hence $(A + D\tilde{\Delta}E)x = \alpha x$. Since $|D\tilde{\Delta}E| \leq D|\tilde{\Delta}|E$, it follows from Lemma 2 that

$$\gamma(A + D|\tilde{\Delta}|E) \geq \gamma(A + D\tilde{\Delta}E) \geq \operatorname{Re}\alpha \geq 0$$

Thus $|\tilde{\Delta}| \in \mathcal{D}_+$ is destabilizing for $(A, D, E)$ and

$$r_{\mathcal{D}_+}(A; D, E) \leq \|\,|\tilde{\Delta}|\,\|_{\mathcal{D}} \leq \|\Delta\|_{\mathcal{D}} = r_{\mathcal{D}}(A; D, E)$$

∎

The following example illustrates that the nonnegativity of the structure matrices $D$, $E$ is essential for the validity of Proposition 3.

**Example 4.** Consider perturbations of the form (22) where $N = 1$

$$A = \begin{bmatrix} -1 & \alpha \\ \alpha & -1 \end{bmatrix}, \quad 0 \le \alpha < 1, \quad A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Note that in this case $A_1$ cannot be factorized into nonnegative matrices $D$, $E$. The perturbed matrix is

$$A(\delta) = \begin{bmatrix} -1 & \alpha - \delta \\ \alpha + \delta & -1 \end{bmatrix}$$

Since $\sigma(A(\delta)) = \{-1 \pm \sqrt{\alpha^2 - \delta^2}\}$, it follows that there does not exist a real destabilizing perturbation $\delta$, i.e. $r_{\mathbb{R}}(A; A_1) = \infty$. On the other hand, it is easily seen that $\delta = \sqrt{1 - \alpha^2}\iota$ is a complex destabilizing perturbation of the smallest absolute value. Thus $r_{\mathbb{C}}(A; A_1) = \sqrt{1 - \alpha^2}$. ♦

## 4. Stability Radii and Transfer Matrices

In this section, we derive a computable formula for the stability radii of positive systems $\Sigma$ under arbitrary affine perturbations. Suppose that $A$ is a Hurwitz stable Metzler matrix, $(D, E) \in \mathbb{R}_+^{n \times l} \times \mathbb{R}_+^{q \times n}$ and

$$G(s) = E(sI - A)^{-1}D \tag{27}$$

is the associated transfer matrix. If $\mathbb{R}^l$, $\mathbb{R}^q$ are provided with absolute norms, $\|\cdot\|$ is the corresponding operator norm and $\gamma(A) < t_1 < t_2$, then (Son and Hinrichsen, 1996)

$$G(t_1) \ge G(t_2) \ge 0, \quad \|G(t_1)\| \ge \|G(t_2)\| \ge 0 \tag{28}$$

**Lemma 5.** *A disturbance matrix* $\Delta \in \mathbb{R}_+^{l \times q}$ *is destabilizing, i.e.* $\gamma(A + D\Delta E) \ge 0$, *if and only if there exist a nonzero vector* $y \in \mathbb{R}_+^q$ *and* $t \ge 0$ *such that*

$$y = G(t)\Delta y \tag{29}$$

*If* $\Delta \in \mathbb{R}_+^{l \times q}$ *is a minimum-norm destabilizing perturbation, then* (29) *holds with* $t = 0$.

*Proof.* If $\Delta \in \mathbb{R}_+^{l \times q}$ is destabilizing, then $t := \gamma(A + D\Delta E) \ge 0$. If $\Delta$ is minimum norm, then additionally $t = 0$. By Lemma 1(i) $t$ is an eigenvalue of $A + D\Delta E$ and there exists an eigenvector $x \in \mathbb{R}_+^n$, $x \ne 0$ such that $(A + D\Delta E)x = tx$. Hence $x = (tI - A)^{-1}D\Delta Ex$ and multiplying this equation by $E$ from the left we obtain (29) for $y = Ex \ge 0$, $y \ne 0$. Conversely, if (29) holds for some $t \ge 0$ and $y \in \mathbb{R}_+^q$, $y \ne 0$ then by setting $x = (tI - A)^{-1}D\Delta y$ we get $Ex = y$ and hence $(A + D\Delta E)x = tx$. Thus $\gamma(A + D\Delta E) \ge t \ge 0$ and $\Delta$ is destabilizing. ■

In the case of a *single block perturbation structure* ($\mathcal{D} = \mathbb{C}^{l\times q}$), the following theorem yields a simple formula for the three stability radii of a positive system (we use the notation of Example 1).

**Theorem 1.** *Suppose that $A \in \mathbb{R}^{n\times n}$ is a Hurwitz-stable Metzler matrix, $(D, E) \in \mathbb{R}_+^{n\times l} \times \mathbb{R}_+^{q\times n}$ are given nonnegative structure matrices and $\mathbb{C}^l$, $\mathbb{C}^q$ are provided with absolute norms. Then, with respect to the induced operator norms $\|\cdot\|$ on $\mathbb{K}^{l\times q}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$,*

$$r_{\mathbb{C}}(A; D, E) = r_{\mathbb{R}}(A; D, E) = r_{\mathbb{R}_+}(A; D, E) = \left\|G(0)\right\|^{-1} \tag{30}$$

*where $G(s) = E(sI - A)^{-1}D$, and $\|G(0)\|$ is the operator norm of $G(0) : \mathbb{C}^l \to \mathbb{C}^q$.*

*Proof.* The first two equalities in (30) follow from Proposition 3. If $\Delta \in \mathbb{R}_+^{l\times q}$ is destabilizing, then by Lemma 5 and (28) we get for some $t \geq 0$

$$\left\|G(0)\right\| \left\|\Delta\right\| \geq \left\|G(t)\right\| \left\|\Delta\right\| \geq \left\|G(t)\Delta\right\| \geq 1$$

(the norms being understood as operator norms). Hence $r_{\mathbb{R}_+}(A; D, E) \geq \|G(0)\|^{-1}$. On the other hand,

$$r_{\mathbb{C}}(A; D, E) = \left[\max_{\omega\in\mathbb{R}} \left\|G(i\omega)\right\|\right]^{-1} \tag{31}$$

(Hinrichsen and Pritchard, 1990) and this implies $r_{\mathbb{C}}(A; D, E) \leq \|G(0)\|^{-1}$. ■

Under the conditions of the theorem, the function $s \mapsto \|G(s)\|$ attains its maximum value on the imaginary axis at $s = 0$ and $G(0) \geq 0$, see (30) and (31).

We now return to block-diagonal perturbations. We only consider the unmixed cases where the perturbations are either of the form (22) ($J = \emptyset$ in Assumption 1) or of the form (19) ($J = \underline{N}$ in Assumption 1).

First, consider perturbations of the form (22) and recall that in the case of perturbations (22) the structure matrices $D_i, E_i$ are obtained by choosing arbitrary factorizations $A_i = D_iE_i$ of the matrices $A_i$, $i \in \underline{N}$. If $D, E$ are defined by (21), the associated transfer matrix $G(s) = E(sI - A)^{-1}D$ is of the form

$$G(s) = \begin{bmatrix} G_{11}(s) & \dots & G_{1N}(s) \\ \vdots & & \vdots \\ G_{N1}(s) & \dots & G_{NN}(s) \end{bmatrix}, \quad G_{ij}(s) = E_i(sI - A)^{-1}D_j, \quad i, j \in \underline{N} \tag{32}$$

Making use of Proposition 3 one can prove the following general formula for the stability radius of positive linear systems under arbitrary affine perturbations of the form (22). Here $r_{\mathbb{C}}(A; (A_i)_{i\in\underline{N}})$, $r_{\mathbb{R}}(A; (A_i)_{i\in\underline{N}})$ and $r_{\mathbb{R}_+}(A; (A_i)_{i\in\underline{N}})$ are defined as in Example 3.

**Theorem 2.** *Suppose that $A \in \mathbb{R}^{n\times n}$ is a Hurwitz-stable Metzler matrix and $A$ is subjected to perturbations of the form*

$$A \rightsquigarrow A(\Delta) = A + \sum_{i=1}^{N} \delta_i A_i, \quad \delta_i \in \mathbb{C}, \ i \in \underline{N}$$

*If*

$$A_i = D_i E_i, \quad D_i \in \mathbb{R}_+^{n\times q_i}, \quad E_i \in \mathbb{R}_+^{q_i \times n}$$

*are given factorizations of* $A_i$, $i \in \underline{N}$ *and* $G(s)$ *is defined by* (32), *then*

$$r_{\mathbb{C}}\big(A; (A_i)_{i\in\underline{N}}\big) = r_{\mathbb{R}}\big(A; (A_i)_{i\in\underline{N}}\big) = r_{\mathbb{R}_+}\big(A; (A_i)_{i\in\underline{N}}\big) = \big[\gamma(G(0))\big]^{-1} \tag{33}$$

*Proof.* Define $\mathcal{D}$ by (23), $D, E$ by (21) and endow $\mathcal{D}$ with the norm (24). Then $\mathcal{D}$ satisfies Assumption 1, see Example 3, and so the first two equalities in (33) follow from Proposition 3. By the Perron-Frobenius Theorem $\gamma = \gamma(G(0))$ is an eigenvalue of $G(0)$ and there exists an eigenvector $u \in \mathbb{R}_+^l = \mathbb{R}_+^q$ of $G(0) \geq 0$ such that $G(0)u = \gamma u$. When choosing $\Delta = \gamma^{-1} I_q \in \mathcal{D}$, it follows that $G(0)\Delta u = u$. By Lemma 5 $\Delta \in \mathcal{D}_+$ is a destabilizing disturbance matrix of norm $\|\Delta\|_{\mathcal{D}} = \gamma^{-1}$. This shows $r_{\mathbb{R}_+} := r_{\mathcal{D}_+} \leq [\gamma(G(0))]^{-1}$. Conversely, suppose that $\Delta = \mathrm{diag}(\delta_1 I_{q_1}, \ldots, \delta_N I_{q_N}) \in \mathcal{D}_+$ is a minimum-norm destabilizing disturbance. Then, by Lemma 5, there exists a nonzero vector $y = (y_i)_{i\in\underline{N}} \in \mathbb{R}_+^q$ such that $y = G(0)\Delta y$. Since

$$\|\Delta\|_{\mathcal{D}} = \gamma(\Delta) = \max\{\delta_1, \ldots, \delta_N\}$$

it follows that $\|\Delta\|\, y \geq (\delta_i y_i)_{i\in\underline{N}} = \Delta y$ and therefore $G(0)\|\Delta\| y \geq y$. Applying Lemma 1(ii) we conclude that $\|\Delta\| \geq [\gamma(G(0))]^{-1}$. This completes the proof. ■

Note that the stability radii in (33) only depend on $A_i$ and not on the individual factors. However, the transfer matrix $G(s)$ depends on the specific factorizations chosen.

We now turn to multiperturbations (19). For the stability radius with respect to this perturbation class, a computable formula is not yet known. In order to derive estimates, we make use of the following balancing result due to Stoer and Witzgall (1962).

**Lemma 6.** *Suppose that* $M \in \mathbb{R}_+^{N\times N}$ *is positive and* $\|\cdot\|$ *is the operator norm induced by any p-norm on* $\mathbb{R}^N$, $1 \leq p \leq \infty$. *Then*

$$\min_{\alpha>0} \big\| \mathrm{diag}(\alpha_i) M \,\mathrm{diag}(\alpha_i^{-1}) \big\| = \rho(M) \tag{34}$$

*where the minimum is taken over all scaling vectors* $\alpha = (\alpha_1, \ldots, \alpha_N) > 0$.

With the help of this lemma, the following estimates can be derived.

**Proposition 4.** *Suppose that* $A \in \mathbb{R}^{n\times n}$ *is a Hurwitz-stable Metzler matrix and* $A$ *is subjected to perturbations of the form*

$$A \rightsquigarrow A(\Delta) = A + \sum_{i=1}^{N} D_i \Delta_i E_i, \quad \Delta_i \in \mathbb{C}^{l_i \times q_i}, \quad i \in \underline{N}$$

*where* $D_i \in \mathbb{R}_+^{n\times l_i}$, $E_i \in \mathbb{R}_+^{q_i \times n}$, $i \in \underline{N}$, *are given. Suppose that* $\mathcal{D}$ *is defined by* (20) *and is provided with the norm*

$$\|\Delta\|_{\mathcal{D}} = \max_{i\in\underline{N}} \|\Delta_i\|, \quad \Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_N) \in \mathcal{D} \tag{35}$$

*where* $\|\Delta_i\|$ *is the operator norm of* $\Delta_i$ *induced by given absolute norms on* $\mathbb{C}^{l_i}, \mathbb{C}^{q_i}$. *Then*

$$r_{\mathbb{C}}(A;(D_i,E_i)_{i\in\underline{N}}) = r_{\mathbb{R}}(A;(D_i,E_i)_{i\in\underline{N}}) = r_{\mathbb{R}_+}(A;(D_i,E_i)_{i\in\underline{N}}) \tag{36}$$

*and*

$$r_{\mathbb{R}}(A;(D_i,E_i)_{i\in\underline{N}}) \geq \left[\inf_{\alpha>0} \|G^{\alpha}(0)\|\right]^{-1} \geq [\rho(H)]^{-1} \tag{37}$$

*where* $G^{\alpha}(0) = (\alpha_i G_{ij}(0)\alpha_j^{-1})_{i,j\in\underline{N}}$ *and* $H = (\|G_{ij}(0)\|)_{i,j\in\underline{N}}$.

The proof is similar to the discrete-time case (Hinrichsen and Son, 1996) and is omitted. In the single perturbation case ($N = 1$) the right-hand side of (37) is equal to $\|G(0)\|^{-1}$. Hence, in this case, the second lower bound is tight by Theorem 1. However, there are counterexamples which demonstrate that already for $N = 2$ the second inequality in (37) is, in general, *not* an equality. Whether or not the first inequality is in fact an equality is an open question. We conclude the paper with an example illustrating Theorem 2.

**Example 5.** Consider the perturbed matrix

$$A(\Delta) = \begin{bmatrix} -1 & \delta_1 \\ 1+\delta_2 & -3+\delta_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & \delta_1 \\ \delta_2 & \delta_3 \end{bmatrix}, \quad \delta_i \in \mathbb{C}, \quad i = 1,2,3 \tag{38}$$

where $\delta_1, \delta_2, \delta_3$ denote unknown parameters. The unperturbed matrix $A = A(0)$ is a Hurwitz-stable Metzler matrix. The uncertainty cannot be described by a single block perturbation structure, i.e. there do not exist structure matrices $D \in \mathbb{C}^{n\times l}$, $E \in \mathbb{C}^{q\times n}$ such that the set of perturbed matrices (38) coincides with $\{A + D\tilde{\Delta}E; \tilde{\Delta} \in \mathbb{C}^{l\times q}\}$. However, since the perturbation is affine, it can be represented in the form (22). In fact, if

$$D_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad D_2 = D_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E_1 = E_3 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

then

$$A(\Delta) = A + \sum_1^3 \delta_i D_i E_i$$

In order to determine the stability radius of $A$ with respect to this perturbation structure, we first have to choose a perturbation norm. If the size of a perturbation $\Delta = \mathrm{diag}(\delta_1, \delta_2, \delta_3)$ is measured by the operator norm $\|\Delta\| = \max\{|\delta_1|, |\delta_2|, |\delta_3|\}$, we can apply Theorem 2. The transfer matrix (32) is given by

$$G(s) = \begin{bmatrix} E_1(sI-A)^{-1}D_1 & E_1(sI-A)^{-1}D_2 & E_1(sI-A)^{-1}D_3 \\ E_2(sI-A)^{-1}D_1 & E_2(sI-A)^{-1}D_2 & E_2(sI-A)^{-1}D_3 \\ E_3(sI-A)^{-1}D_1 & E_3(sI-A)^{-1}D_2 & E_3(sI-A)^{-1}D_3 \end{bmatrix}$$

Any easy calculation yields

$$(-A)^{-1}=\begin{bmatrix}1 & 0\\ 1/3 & 1/3\end{bmatrix},\quad G(0)=\begin{bmatrix}1/3 & 1/3 & 1/3\\ 1 & 0 & 0\\ 1/3 & 1/3 & 1/3\end{bmatrix},\quad \sigma(G(0))=\{0,-1/3,1\}$$

so that $\gamma(G(0))=1$. It follows from Theorem 2 that $r_{\mathbb{K}}=1$. In fact, the perturbation $\Delta=\mathrm{diag}(\delta_1,\delta_2,\delta_3)=\mathrm{diag}(1,1,1)$ is destabilizing and of norm 1, and it is easily verified that no smaller perturbation of the given structure can destabilize the matrix $A$.

♦

**Remark 2.** Suppose that $(A,D,E)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times l}\times\mathbb{R}^{q\times n}$ is a given non-positive stable system such that the associated transfer matrix

$$G(s)=E(sI-A)^{-1}D$$

admits a positive realization

$$(\tilde{A},\tilde{D},\tilde{E})\in\mathbb{R}^{\tilde{n}\times\tilde{n}}\times\mathbb{R}_+^{\tilde{n}\times l}\times\mathbb{R}_+^{q\times\tilde{n}},\quad \sigma(\tilde{A})\subset\mathbb{C}_-$$

$\tilde{A}$ being a Metzler matrix. Then $G(s)=\tilde{E}(sI-\tilde{A})^{-1}\tilde{D}$ and $\Delta\in\mathcal{D}$ destabilizes $(A,D,E)$ if and only if it destabilizes $(\tilde{A},\tilde{D},\tilde{E})$. Thus, for any class $\mathcal{D}$ of block-diagonal perturbations we have by Proposition 3

$$r_{\mathcal{D}}(A,D,E)=r_{\mathcal{D}_{\mathbb{R}}}(A,D,E)=r_{\mathcal{D}_+}(A,D,E)$$

If $\mathcal{D}=\mathbb{C}^{l\times q}$, then by Theorem 1

$$r_{\mathbb{C}}(A,D,E)=r_{\mathbb{R}}(A,D,E)=r_{\mathbb{R}_+}(A,D,E)=\left\|G(0)\right\|^{-1}$$

In order to apply the above formulae to the data $(A,D,E)$ it is not necessary to *construct* a positive realization of $G(s)$. It is sufficient to know that it does exist. In the single-input single-output case ($l=q=1$), a complete characterization of positively realizable rational transfer functions $g(s)\in\mathbb{R}(s)$ can be found in (Anderson *et al.*, 1996).

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