*Int. J. Appl. Math. Comput. Sci., 2001, Vol.11, No.1, 189–204*

# QUASISTATIC PROBLEMS IN CONTACT MECHANICS

Meir SHILLOR*

We describe some of our recent results concerning the modeling and analysis of quasistatic contact problems between a deformable body and a foundation. We concentrate mainly on frictional contact, and in some of the problems thermal effects and the wear of the contacting surfaces are also taken into account. We describe the physical processes involved, the mathematical models, their variational formulation and then present statements of our results. We conclude with a description of some unresolved problems.

**Keywords:** quasistatic contact, unilateral contact, friction, wear, thermoriscoelastic, normal compliance

## 1. Introduction

Contact phenomena are abundant in everyday life and play a very important role in engineering structures and systems. They include *friction*, *wear*, *adhesion* and *lubrication*, among other things; are inherently complex and time dependent; take place on the outer surfaces of parts and components, and involve thermal, physical and chemical processes. The need for a comprehensive well posed mathematical theory, based on fundamental physical principles, that can predict reliably the evolution of the contact process in different situations and under various conditions, was recognized long ago, but the tools needed for realizing this goal have been developed only in recent years.

Such a mathematical theory is emerging currently. It deals with the rigorous modeling of these processes using new and advanced concepts and results from such topics as variational inequalities, convex and non-convex analysis, and set-valued operators. It is concerned with the construction of models based on first principles, as much as possible, as well as advanced mathematical concepts, well-posedness of the models, their analysis, and the numerical analysis of their approximations. A considerable progress has been achieved recently in the modeling, variational analysis and numerical approximations of contact problems involving viscoelastic and viscoplastic materials. Some of it is described in the sequel.

A variety of unilateral or bilateral contact conditions have been used. Unilateral contact has been modeled by the Signorini, normal compliance or damped response conditions. Friction has been described by versions of the Coulomb law. Very recently,

* Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309, USA, e-mail: `shillor@oakland.edu`, web: `www.oakland.edu/~shillor`

the coefficient of friction has been assumed to depend on the slip velocity, the temperature and even allowed to have a jump from a static to a dynamic value at the onset of slip motion. The wear has been modeled by the Archard law, in which the surface wear rate is proportional to the normal stress and to the slip speed. New models of surface adhesion were proposed, which use the notion of an additional dependent variable, the bonding function.

There exists a considerable body of engineering literature on contact with or without friction. On the other hand, mathematical publications are quite recent. Static problems were considered in many publications. However, contact and friction are evolutionary processes, that is, they are time dependent.

In the last three years a number of publications have appeared dealing with various models for the different aspects of quasistatic frictional contact. These are processes where the applied forces vary slowly, and therefore, the system response is relatively slow, so that the inertial terms in the equations of motion can be neglected. This means that the propagation of waves is neglected, i.e., the system is observed on a time scale much longer than that necessary for any waves to travel through it and to decay.

Only recently the various aspects of the dynamics of frictional contact have been addressed in comprehensive mathematical models. We survey some models and recent mathematical results dealing with these problems. We concentrate on quasistatic problems which can be formulated as elliptic variational inequalities. We also describe a number of important open problems.

The mathematical, mechanical and numerical state of the art for contact problems can be found in the proceedings (Raous *et al.*, 1995), in the special issue (Shillor, 1998), in the survey (Fernández-García *et al.*, 2000a) and in the short survey (Fernández-García *et al.*, 2000b) in this issue, among other places.

This paper is based on our plenary talk at the MTNS 2000 meeting, and our purpose is to present the physical and engineering settings for the mathematical studies of contact processes, and to indicate their current trends. This survey is not exhaustive or comprehensive, since the discipline of mathematical contact mechanics is already too large for a summary in a short paper. However, this discipline does not yet have a coherent and mature structure, and as we indicate in the last section, there is much to be done.

## 2. Motivation and Physics

An estimated 0.5% of the US GNP is lost (see, e.g., (Rabinowicz, 1995)) because of insufficient control of contact processes in machines, cars and mechanical equipment. This loss is mainly due to frictional wear, frictional heat generation which causes the softening of the contacting surfaces, frictional heat losses and the possible damage of the contacting surfaces.

Therefore, the accurate prediction of the evolution of frictional contact processes and their control is of major economic importance. However, the contact between deformable bodies is very complicated and it is not yet well understood. This is mainly due to the fact that it is very difficult to measure directly the evolution of surface quantities during contact processes.

Any coherent theory of contact has to deal with surfaces that may be:

(i) clean, or contaminated with oil or water or gas;

(ii) contain a layer of adsorbed gas, or an oxide layer;

(iii) are very smooth, smooth, rough, or very rough.

Moreover, the layers of material adjacent to the contacting surface have a major influence on the process, and themselves often have a different structure than the parent material (see a very detailed exposition in (Rabinowicz, 1995)).

During the contact of parts and components, a number of processes take place on the contacting surfaces. Some of these are related to the interior of the contacting bodies, such as elastic or plastic deformations and the strains and stresses that accompany them. Other processes take place only on the contacting surfaces, such as deformation (elastic or plastic) of asperities, squeezing of oil or other fluid layers, breaking of tips of the asperities, motion of debris between the surfaces, welding and formation of junctions, creeping, etc.

It is clear that the processes are diverse and to describe them we need a considerable insight, and quite sophisticated mathematics. To gain some insight, we next consider the usual high school explanation of friction. We consider the setting depicted in Fig. 1, where a rigid body is placed on a rigid foundation. To describe the contact of a body with a foundation, we need to consider the *normal* approach and the *tangential* process.

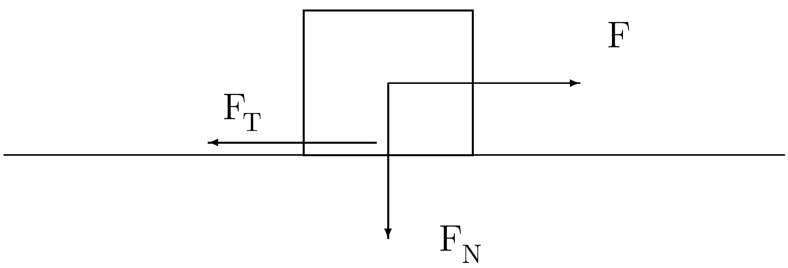


Fig. 1. Contact forces.

We consider a body acted upon by a tangential force $F$ and a normal force $F_N$, which includes gravity. As a result, a frictional resistance force is set, denoted by $F_T$. As $F$ is increased, the frictional resistance force increases, until it reaches a critical value $F_T^*$ which is the maximal resistance the surfaces can produce, beyond which the body starts a slip motion.

It is customary to define the *friction coefficient* $\mu$ as the ratio

$$\mu = \frac{|F_T^*|}{|F_N|}.$$

However, there is no fundamental reason for $\mu$ to be a property of the system. And, indeed, there are cases where it does not make sense. Surprisingly, in many applications it does behave as a system parameter, and can be assigned a numerical value. Since it is not a fundamental quantity, its value must be found experimentally for each system.

We expect $\mu$ to depend on the bulk material properties of the contacting bodies and on the surface properties. It is customary to define two friction coefficients, the static coefficient $\mu_s$ and the dynamic coefficient $\mu_d$, where

$$\mu_s = \frac{|F_T^*|}{|F_N|}.$$

The body remains motionless as long as $|F| \le \mu_s|F_N|$, which constitutes the basis for its experimental determination, and $\mu_d$ is obtained from sliding experiments.

It is often found that $\mu_d$ is substantially lower than $\mu_s$. In part, the difference is caused by the chemical bonding that takes place while the surfaces are in relative rest, in the so-called *stick* state. This is often called adhesion in the engineering literature, although we reserve the term *adhesion* for the process where a separate agent, such as glue, is spread on the surfaces. The chemical bonding takes place in the junctions, as depicted in Fig. 2.

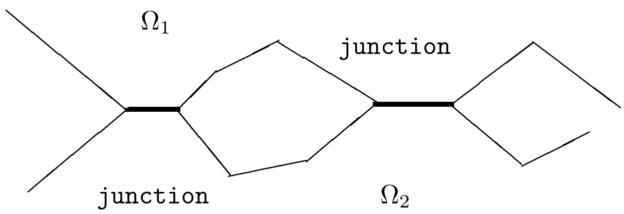


Fig. 2. Contacting surfaces; junctions.

It turns out, as is to be expected, that $\mu$ depends on the lubrication of the surfaces, on their roughness, on the slip velocity, and on the temperature. The last one is very complicated to measure, since frictional sliding causes considerable heat generation which, in turn, causes uneven temperature rises. Moreover, it is observed that as the melting temperature of one of the surfaces is approached, the friction coefficient raises rapidly, and then decreases sharply, once melting commences (see, e.g., (Rabinowicz, 1995) for full details).

Friction is accompanied by: heat generation, wear, damage, and adhesion.

The dependence of friction on temperature may be considerable. While braking a fast moving car, over 100 HP of heat may be generated at the brakes. This causes the raise of the temperature at the contact area, and may cause local melting of the surface material, which may lead, in turn, to the loss of braking power. Experimentally, the coefficient of friction does depend on the temperature.

*Wear* is a major issue in technological applications. The contacting surfaces undergo structural changes which may be detrimental to the systems performance. Even if the changes are small, they may influence the efficiency and durability of the system.

*Damage* of the surfaces is a severe case of wear, as well as structural damage in the form of cracks on the surface. This may lead to a catastrophic failure of the system.

*Adhesion* may take place between parts of the contacting surfaces. It may be intentional, when surfaces are bonded with glue, or unintentional, as a seizure between very clean surfaces.

Since frictional contact is so important in industry, there is a need to model and predict it accurately. However, the main industrial need is to effectively control the process of frictional contact. In applications there are four main areas of friction control:

1. Low friction—lubrication. Needed in machinery and moving equipment to reduce wear, tear and loss of energy.
2. High friction. Needed in braking systems of cars, trains and moving equipment.
3. Friction within specified bounds. Needed in braking systems to avoid jerks and sudden accelerations/decelerations.
4. Slip dependence. Needed to avoid slip/stick transitions and the associated unpleasant noise—squeaks and squeals.

It is clear from the abundance and diversity of the types of behavior of systems with friction, that it is unlikely that one single model will be capable of describing, in a predictive way, all these phenomena. One may be tempted to put them all together in one extremely complicated and abstract model, but the usefulness of such a model is in doubt. Therefore, we present below a number of models for different aspects of frictional contact. The unifying element is that we can formulate them as variational inequalities, and apply the methods of this theory to their investigation.

This motivates the different models we present below.

## 3. Mathematical Models

We now describe mathematical models for frictional contact between a deformable body and a rigid foundation. The physical setting is depicted in Fig. 3. A deformable body occupies an open, bounded and connected set $\Omega \subset \mathbb{R}^d$, $d = 1, 2$ or 3. The boundary $\Gamma = \partial\Omega$ is assumed to be Lipschitz continuous and is composed of three mutually disjoint, relatively open sets $\Gamma_D$, $\Gamma_N$ and $\Gamma_C$. The set $\Gamma_C$ represents the potential contact surface, and we assume that $\operatorname{meas}(\Gamma_D) > 0$, which is essential

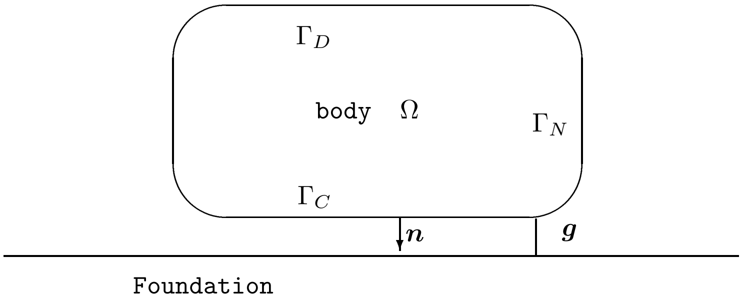


Fig. 3. The setting: $\Gamma_C$ denotes the contact surface, $\boldsymbol{g}$ is the gap.

in quasistatic problems. As the boundary is Lipschitz continuous, the unit outward normal vector $\boldsymbol{n}$ exists a.e. on $\Gamma$.

We are interested in the evolution of the body's mechanical state over the time interval $[0, T]$ $(T > 0)$. The body is clamped on $\Gamma_D$ and so the displacement field vanishes there. A surface traction of density $\mathbf{f}_S$ acts on $\Gamma_N$, and a volume force of density $\mathbf{f}_B$ acts in $\Omega$. Each force may vary in time.

The mechanical state of the system is described by the displacement vector

$$\boldsymbol{u} = \boldsymbol{u}(x,t) = \big(u_1(x,t), u_2(x,t), u_3(x,t)\big),$$

the velocity vector $\mathbf{v} = \mathbf{v}(x,t) = \boldsymbol{u}' = (v_1, v_2, v_3)$, the strain tensor $\epsilon = \epsilon(\boldsymbol{u})$, with components $\epsilon_{ij} = \partial u_i/\partial x_j$, and the stress tensor $\sigma = (\sigma_{ij})$. Here and subsequently, $i, j, k, l = 1, \ldots, d$, and the summation convention is employed.

The material properties of the body enter via the constitutive law which relates the stress tensor to the strain tensor. The constitutive relation of a viscoelastic material is

$$\sigma = A\big(\varepsilon(\boldsymbol{u})\big) + B\big(\varepsilon(\mathbf{v})\big),$$

where $A$ represents the elastic part and $B$ the viscosity part of the stress. Here and subsequently, a prime indicates a time derivative. In linearized viscoelasticity

$$\sigma_{ij} = a_{ijkl}\frac{\partial u_k}{\partial x_l} + b_{ijkl}\frac{\partial^2 u_k}{\partial t \partial x_l},$$

where the $a$'s and $b$'s are the coefficients which may depend on $x$ but are independent of the stresses or strains.

The equations of motion of the material are

$$\rho \boldsymbol{u}'' - \mathrm{Div}\,(\sigma) = \mathbf{f}_B,$$

where $\rho$ stands for the material density and $\mathrm{Div}\,(\sigma) = \partial\sigma_{ij}/\partial x_j$. The boundary conditions are

$$\boldsymbol{u} = 0 \ \text{ on } \ \Gamma_D, \quad \sigma\,\boldsymbol{n} = \mathbf{f}_S \ \text{ on } \ \Gamma_N.$$

The initial conditions take the form

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \mathbf{v} = \mathbf{v}_0, \ \text{ in } \ \Omega.$$

The quasistatic approximation consists in neglecting the inertial terms in the equations of motion, and it means that the system is observed over a long-time scale. It is justified when the external forces vary slowly with time. However, a rigorous mathematical justification is not available yet. Then the equations of motion are

$$-\mathrm{Div}\,(\sigma) = \mathbf{f}_B.$$

We use the same boundary conditions, but now we cannot impose the initial conditions. Indeed, the problem is transformed from being hyperbolic to being elliptic. In particular, it does not allow for elastic waves propagation in the system.

Frictional contact, when slip takes place, is accompanied by frictional heat generation. To take thermal effects into account, we need to study thermoviscoelastic contact. To this end, we introduce the temperature $\theta$.

The constitutive law is now

$$\sigma = A\big(\epsilon(\boldsymbol{u})\big) + B\big(\epsilon(\mathbf{v})\big) + D(\theta),$$

where $D$ is the tensor of thermal expansion. In linearized thermoviscoelasticity we have

$$\sigma_{ij} = a_{ijkl}\frac{\partial u_k}{\partial x_l} + b_{ijkl}\frac{\partial^2 u_k}{\partial t \partial x_l} - d_{ij}\theta.$$

The equations of motion need to be supplemented with the energy equation. Thus,

$$\rho\boldsymbol{u}'' - \mathrm{Div}\,(\sigma) = \mathbf{f}_B,$$

$$\rho c\theta' - \mathrm{Div}\,(k\nabla\theta) = Q + Q^*\big(\varepsilon(\mathbf{v})\big),$$

where $Q$ is a given volume heat source and $Q^*$ is the term representing the heat generated by the work of internal forces. Also, initial and boundary conditions for the temperature are needed.

To describe the contact conditions we need additional notation. The setting is depicted in Fig. 4. We use the notation $\boldsymbol{n} = (n_1, n_2, n_3)$ for the outer normal on $\Gamma_C$. The normal component of the displacements is $u_n = \boldsymbol{u}\,\boldsymbol{n}$, and the tangential component is $\boldsymbol{u}_T = \boldsymbol{u} - (\boldsymbol{u}\,\boldsymbol{n})\boldsymbol{n}$, the normal stress (pressure) is $\sigma_n = \sigma_{ij}n_i n_j$, and the tangential stress vector is $\sigma_{Ti} = \sigma_{ij}n_j - (\sigma_{ij}n_i n_j)n_i$.

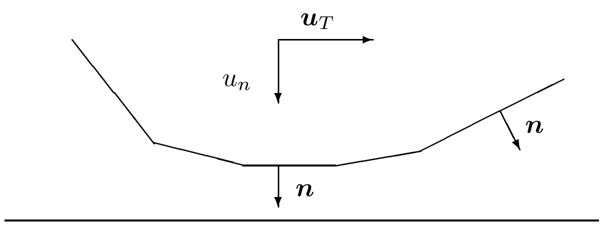


Fig. 4. Normal and tangential components.

We begin with the description of the normal approach. Recently, four different conditions have been used. The first one is the *bilateral contact*, when there is no separation during the process. Then

$$u_n = 0, \quad (g = 0). \tag{1}$$

Next, in most papers on contact problems, especially in engineering literature, the *Signorini (non-penetration) condition* is used. It assumes that the foundation is perfectly rigid. We have

$$u_n \le g, \quad \sigma_n \le 0, \quad \sigma_n(g - u_n) = 0. \tag{2}$$

The *normal compliance condition* has been used in a number of mathematical publications. It allows for the interpenetration of surface asperities, but penalizes it. Its form is

$$\sigma_n = -p(u_n - g), \tag{3}$$

and it has frequently been used in the form $p(r) = \kappa(r)_+^m$, where $0 < \kappa$ is an experimental constant and $m$ the exponent. The function $p$ has to vanish for negative values of the argument, since then there is no contact, and the contact pressure vanishes.

A new condition, the so-called *damped response*, has been employed recently in the form

$$\sigma_n = -\phi(v_n) + \phi_0. \tag{4}$$

Here, the surface resistance is proportional to the normal velocity $v_n$, and the functions $\phi$ and $\phi_0$ are assumed to be known. This may describe the sinking of an object in the sand.

**Remark 1.** It is customary to consider the normal compliance as an approximation of the Signorini condition, and indeed, as $\kappa \to \infty$, the first converges to the second (at least formally). But, especially when dealing with dynamic situations, the concept of a rigid foundation is only a high idealization. It seems that the normal compliance is the more physically correct condition, and Signorini's is the approximation. Moreover, Signorini's condition is not very useful in view of the unsurmountable mathematical difficulties it introduces.

We now turn to describe the tangential process. Here we have three conditions. When the surfaces are fully lubricated, it is customary to use

$$\sigma_T = 0. \tag{5}$$

However, this is only an approximation, valid when the relative slip is small. Otherwise, a shear stress is set in the lubricating layer which contributes to the tangential stress.

The usual description of friction is by using the Coulomb law of dry friction

$$|\sigma_T| \le \mu|\sigma_n|, \tag{6}$$

and, if $\mathbf{v}_T \ne 0$, then

$$|\sigma_T| = \mu|\sigma_n| \quad \text{and} \quad \frac{\sigma_T}{|\sigma_T|} = -\frac{\mathbf{v}_T}{|\mathbf{v}_T|}. \tag{7}$$

A serious mathematical difficulty arises with the term $|\sigma_n|$ which is either a distribution, defined via an extension of the Green formula, or does not make much sense as a trace on the boundary of a function in $L^2(\Omega)$.

When the normal compliance is used in the normal direction, the friction condition is modified as

$$|\sigma_T| \le \mu p(u_n - g), \tag{8}$$

and, if $\mathbf{v}_T \ne 0$, then

$$|\sigma_T| = \mu p(u_n - g) \quad \text{and} \quad \frac{\sigma_T}{|\sigma_T|} = -\frac{\mathbf{v}_T}{|\mathbf{v}_T|}. \tag{9}$$

Here, $p$ is the contact pressure, cf. (3).

Recently, the *wear* of contacting surfaces has been taken into account, in view of its industrial and engineering importance. It is usually represented as an additional displacement $w$ of the contacting surfaces, and $u_n$ is replaced with $u_n - w$. Thus, $w$ measures the volume, per unit area, of the material removed by the wear process. The rate of wear of the surfaces is assumed to be proportional to the contact pressure and to the relative slip speed, which is the so-called *Archard law*:

$$\frac{\partial w}{\partial t} = k_w|\sigma_n||\mathbf{v}_T - v^*|. \tag{10}$$

Here $k_w$ is the wear rate coefficient, $v^*$ denotes the velocity of the foundation, and then $|\mathbf{v}_T - v^*|$ is the relative slip speed.

We have all the ingredients to discuss the mathematical models for the various processes of interest.

## 4. Mathematical Formulations and Results

First mathematical results on quasistatic frictional contact were obtained by L.-E. Andersson (Andersson, 1991; Andersson and Klarbring, 1997) and Klarbring, Mikelic and Shillor (Klarbring *et al.*, 1991), using normal compliance, as well as by Cocu (1984) using regularization.

In (Shillor and Sofonea, 2001) the following bilateral viscoelastic frictional problem with regularized contact stress was studied. Find a displacement field $u$ : $\Omega \times [0,T] \to \mathbb{R}^3$ and a stress field $\sigma\colon\ \Omega\times[0,T]\to\mathcal{S}_3$ such that

$$\sigma = A(\epsilon(\boldsymbol{u})) + B(\epsilon(\mathbf{v})),$$

$$-\mathrm{Div}\,(\sigma) = \mathbf{f}_B,$$

$$\boldsymbol{u} = 0, \ \text{ on } \Gamma_D, \quad \sigma\,\boldsymbol{n} = \mathbf{f}_S, \ \text{ on } \Gamma_N,$$

$$u_n = 0, \ \text{ on } \Gamma_C,$$

$$|\sigma_T| \le \mu|\mathcal{R}\sigma_n|, \ \text{ on } \Gamma_C,$$

$$\text{if } \mathbf{v}_T \ne 0, \ \text{ then } |\sigma_T| = \mu|\mathcal{R}\sigma_n| \ \text{ and } \ \frac{\sigma_T}{|\sigma_T|} = -\frac{\mathbf{v}_T}{|\mathbf{v}_T|},$$

$$u(0) = u_0, \quad w(0) = 0.$$

Here, $\mathcal{R}$ is a boundary regularization operator.

Generally, such problems do not have solutions smooth enough to be classical, that is, having all the necessary continuous derivatives. Thus, the derivatives are understood in the *sense of distributions*, and *weak solutions* are sought.

To present the weak or variational formulation, we define the *friction functional*

$$j(\sigma,\mathbf{w}) = \int_{\Gamma_C} \mu|\mathcal{R}\sigma_n||\mathbf{w}_T|\,\mathrm{d}S.$$

The following is the weak formulation. We denote by $V$ the subspace of $(H^1(\Omega))^3$

$$V = \Big\{\mathbf{w}\in\big(H^1(\Omega)\big)^3\colon\ \mathbf{w}=0 \ \text{ on } \Gamma_D\Big\}.$$

**Problem 1.** Find a displacement field $\boldsymbol{u}\colon\ [0,T]\to V$, $\mathbf{v}=\boldsymbol{u}'$ and a stress field $\sigma\colon\ [0,T]\to L^2(0,T;L^2(\Omega))$, such that $\boldsymbol{u}(0)=\boldsymbol{u}_0$, and for all $t\in[0,T]$,

$$\sigma(t) = A\big(\epsilon(\boldsymbol{u}(t))\big) + B\big(\epsilon(\mathbf{v}(t))\big),$$

$$\big\langle\sigma(t),\epsilon(\mathbf{w})-\epsilon(\mathbf{v}(t))\big\rangle + j\big(\sigma(t),\mathbf{w}\big) - j\big(\sigma(t),\mathbf{v}(t)\big) \ge \big\langle F(t),\mathbf{w}-\mathbf{v}(t)\big\rangle_V, \quad \mathbf{w}\in V.$$

**Theorem 1.** (Shillor and Sofonea, 2001) *There exists* $\mu_0>0$*, which depends only on* $\Omega$*,* $\Gamma$ *and* $B$*, such that if* $|\mu|_{L^\infty(\Gamma_C)}\le\mu_0$*, then there exists a unique solution* $\{\boldsymbol{u},\sigma\}$ *to Problem 1. Moreover, the solution satisfies*

$$\boldsymbol{u}\in C^1(0,T;V), \quad \sigma\in C\big(0,T;L^2(\Omega)\big).$$

The problem with sliding contact and wear was considered there too, and a similar existence and uniqueness theorem was established.

A related problem was investigated in (Rochdi *et al.*, 1998b), where the contact was described by normal compliance, the foundation was assumed to move at speed $v^*$, and the wear due to friction has been included. The problem is:

Find a displacement field $\boldsymbol{u}: \ \Omega \times [0,T] \to \mathbb{R}^3$, a stress field $\sigma: \ \Omega \times [0,T] \to \mathcal{S}_3$ and the wear function $w: \ [0,T] \to \mathbb{R}$ such that

$$\begin{aligned}
\sigma &= A\big(\epsilon(\boldsymbol{u})\big) + B\big(\epsilon(\mathbf{v})\big),\\
-\mathrm{Div}(\sigma) &= \mathbf{f}_B,\\
\boldsymbol{u} &= 0, \quad \text{on} \quad \Gamma_D, \quad \sigma\,\boldsymbol{n} = \mathbf{f}_S, \quad \text{on} \quad \Gamma_N,\\
-\sigma_n &= p_n(u_n - w - g),\\
|\sigma_T| &\le \mu p_T(u_n - w - g),\\
\text{if} \ \ \mathbf{v}_T \neq 0, \ \ \text{then } |\sigma_T| &= \mu p_T(u_n - w - g) \quad \text{and} \quad \frac{\sigma_T}{|\sigma_T|} = -\frac{\mathbf{v}_T}{|\mathbf{v}_T|},\\
w' &= k_w v^* p_n(u_n - w - g),\\
\boldsymbol{u}(0) &= \boldsymbol{u}_0, \quad w(0) = 0.
\end{aligned}$$

To obtain the weak formulation, we introduce the friction functional

$$h(\boldsymbol{u}, \mathbf{z}, w) = \int_{\Gamma_C} \big(p_n(u_n - w - g) z_n + p_T(u_n - w - g)|\mathbf{z}_T|\big)\,\mathrm{d}S.$$

**Problem 2.** Find a displacement field $\boldsymbol{u}: \ [0,T] \to V$, $\mathbf{v} = \boldsymbol{u}'$, a stress field $\sigma: \ [0,T] \to L^2(0,T; L^2(\Omega))$ and $w: \ [0,T] \to L^2(\Gamma_C)$, such that

$$\sigma(t) = A\big(\epsilon(\boldsymbol{u}(t))\big) + B\big(\epsilon(\mathbf{v}(t))\big), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad w' = -k_w v^* \sigma_n, \quad w(0) = 0,$$

$$\big\langle \sigma(t), \epsilon(\mathbf{z}) - \epsilon\big(\mathbf{v}(t)\big)\big\rangle + h\big(\boldsymbol{u}, \mathbf{z}, w\big) - h\big(\boldsymbol{u}, \mathbf{v}(t), w\big) \ge \big\langle F(t), \mathbf{z} - \mathbf{v}(t)\big\rangle_V, \quad \mathbf{z} \in V.$$

**Theorem 2.** (Rochdi *et al.*, 1998b) *There exists a unique solution* $\{\boldsymbol{u}, \sigma, w\}$ *to Problem 2. Moreover, the solution satisfies*

$$\boldsymbol{u} \in C^1(0,T;V), \quad \sigma \in C\big(0,T; L^2(\Omega)\big), \quad w \in C^1\big(0,T; L^2(\Gamma_C)\big).$$

The proof is based on results from the theory of elliptic variational inequalities and fixed point arguments. We note that there is no restriction on the size of $\mu$ and the solution is unique.

The dual formulation of the quasistatic viscoelastic bilateral contact problem with Tresca's friction law was investigated in (Awbi *et al.*, 2001). In Tresca's law the *friction bound* $\mu|\sigma_n|$ is replaced with $G$ which is assumed to be known.

A problem of bilateral contact for a viscoelastic material with slip-dependent friction coefficient can be found in (Amassad *et al.*, 1999b; 2001b). In the context of quasistatic problems this is the first result in which $\mu$ is not a constant. The friction coefficient $\mu$ was assumed to depend either on the *current slip rate*

$$\mu = \mu\big(|\mathbf{v}_T - v^*|\big),$$

or on the *total slip history*

$$\mu = \mu\left(\int_0^t |\mathbf{v}_T(s) - v^*(s)|\,\mathrm{d}s\right).$$

**Theorem 3.** *(Amassad* et al.*, 1999b) There exists $\mu_0 > 0$ which depends only on the problem data, such that if $\mu(\cdot) \le \mu_0$, then there exists a unique solution $\{u, \sigma\}$ to the problem. Moreover, the solution satisfies*

$$\boldsymbol{u} \in C^1(0, T; V), \quad \sigma \in C\big(0, T; L^2(\Omega)\big).$$

A quasistatic contact problem with directional friction and damped response was investigated in (Rochdi *et al.*, 1998a). Variational analysis of quasistatic *viscoplastic* contact problems with friction can be found in (Sofonea and Shillor, 2001), where a general existence and uniqueness theorem and a number of examples with different friction conditions, as well as the addition of the damage of the material, are presented. A contact problem for the *Bingham fluid* with friction was considered in (Awbi *et al.*, 1999). In (Amassad *et al.*, 1999a), we investigated a quasistatic contact problem for an *elastic perfectly plastic* body with Tresca's friction. A nonlinear evolution inclusion in *perfect plasticity* with friction can be found in (Amassad *et al.*, 2001a). A number of quasistatic problems were investigated recently in the Ph.D. thesis of Chau (2000). A review of related quasistatic problems can be found in (Garcia *et al.*, 2000) and in the review article by (Shillor *et al.*, 2001).

We turn to a quasistatic frictional bilateral contact problem in *thermoviscoelasticity*, investigated in (Rochdi and Shillor, 2001).

$$\begin{aligned}
\sigma_{ij} &= a_{ijkl}u_{k,l} + b_{ijkl}v_{k,l} - c_{ij}\theta \;\text{ in }\; \Omega \times (0,T),\\
\sigma_{ij,j} + \mathbf{f}_B &= 0 \;\text{ in }\; \Omega \times (0,T),\\
\theta' - (k_{ij}\theta_{,j})_{,i} &= -c_{ij}u'_{i,j} + q \;\text{ in }\; \Omega \times (0,T),\\
\boldsymbol{u} &= 0 \;\text{ on }\; \Gamma_D \times (0,T),\\
\sigma\boldsymbol{n} &= \mathbf{f}_S \;\text{ on }\; \Gamma_N \times (0,T),\\
\theta &= \theta_b \;\text{ on }\; (\Gamma_D \cup \Gamma_N) \times (0,T),\\
u_n &= 0 \;\text{ on }\; \Gamma_C \times (0,T),
\end{aligned}$$

$$|\sigma_T| \le \mu|\mathcal{R}\sigma_n|\big(1-\delta|\mathcal{R}\sigma_n|\big)_+ \ \text{ on } \ \Gamma_C\times(0,T),$$

$$\text{if } \ \mathbf{v}_T \ne v^* \Rightarrow |\sigma_T| = \mu|\mathcal{R}\sigma_n|\,(1-\delta|\mathcal{R}\sigma_n|)_+ \ \text{ and } \ \frac{\sigma_T}{|\sigma_T|} = -\frac{\mathbf{v}_T - v^*}{|\mathbf{v}_T - v^*|},$$

$$k_{ij}\theta_{,i}n_j = \mu|\mathcal{R}\sigma_n|\,(1-\delta|\mathcal{R}\sigma_n|)_+\, s_c(\cdot\,,|\mathbf{v}_T - v^*|) - k_e(\theta-\theta_R) \ \text{ on } \ \Gamma_C\times(0,T),$$

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0, \quad \theta(\cdot,0) = \theta_0 \ \text{ in } \ \Omega.$$

Here we used the linearized thermoviscoelastic constitutive law.

The problem can be set in an abstract way (see (Rochdi and Shillor, 2001) for details) as follows:

**Problem 3.** Find $\{\boldsymbol{u},\sigma,\theta\}$, such that

$$\theta' + K_1\theta + K_2\theta + C_1\mathbf{v} + S(\sigma,\mathbf{v}) = Q \quad \text{in} \quad \mathcal{V}',$$

$$B\mathbf{v}' + A\boldsymbol{u} + C_2\theta + \partial_2 j(\sigma,\mathbf{v}) \ni F \quad \text{in} \quad \mathcal{E}'.$$

Here $F\in\mathcal{E}'$ and $Q\in\mathcal{V}'$ are given by

$$\langle F,w\rangle = \int_0^T\int_\Omega \mathbf{f}_B\ \mathbf{w}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega c_{ij}\Theta w_{i,j}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Gamma_N}\mathbf{f}_S\ w\,\mathrm{d}S\,\mathrm{d}t$$

and

$$\langle Q,\eta\rangle = \int_0^T\langle q,\eta\rangle\,\mathrm{d}t - \int_0^T\int_\Omega \Theta'\eta\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\Gamma_C} k_e(\Theta-\theta_R)\eta\,\mathrm{d}S\,\mathrm{d}t + \int_0^T\int_\Omega k_{ij}\Theta_{,i}\eta_{,j}\,\mathrm{d}x\,\mathrm{d}t,$$

respectively, and $\partial_2 j(\sigma,\mathbf{z})$ denotes the partial subdifferential of

$$j(\sigma,\mathbf{z}) = \int_0^T\int_{\Gamma_C}\mu|\mathcal{R}\sigma_n|\,(1-\delta|\mathcal{R}\sigma_n|)_+\,|\mathbf{z}_T - v^*|\,\mathrm{d}S\,\mathrm{d}t$$

with respect to $\mathbf{z}$. We have established the following result:

**Theorem 4.** (Rochdi and Shillor, 2001) *Problem 3 has a unique solution when $|\mu|_{L^\infty(\Gamma_C)}$ is sufficiently small.*

General results on contact problems can be found in the monograph by Raous *et al.* (1995) and in the special issue of the journal of Mathematical and Computer Modelling, (Shillor, 1998), as well as in the references below.

## 5. Conclusions

Considerable progress has been made in the modelling, variational analysis and numerical analysis of quasistatic contact of viscoelastic bodies. Our understanding of the behavior of the models for these processes has deepened. However, many open problems remain. There is an urgent need for regularity theory for contact problems. There is intrinsic mathematical interest in the optimal regularity of the solutions. From the numerical analysis point of view, smoother solutions allow for better error estimates. Moreover, we would like to remove the regularization operator $\mathcal{R}$.

Our analytical tools, at this stage, are insufficient to provide any detailed information on the structure of the solutions. Indeed, the most important applied aspects of the problems are the structures of the contact zones and the distribution of the contact stress in them. New mathematical tools need to be developed for this task. Since theoretical investigations are beyond our current capabilities, we must resort to numerical approximations and simulations of the models. Recent progress, indicated in the literature below, in the numerical analysis and the establishing of rigorous error estimates allows for confidence in the results of the computer simulations.

In the following list we mention briefly major directions in which mathematical contact mechanics will move in the near future, now that the existence theory is achieving a mature status:

- *Optimal control of frictional problems.* Indeed, in applications, the control of the frictional stresses, heat generation and wear are the main issues. Related is the optimal shape design of frictional settings. The goal is to design the parts and components in an optimal way with respect to wear, durability, etc.
- *Numerical analysis and error estimates.* These put the numerical simulations on solid footing. Moreover, efficient and convergent numerical methods for contact problems are needed.
- It is of interest to investigate *settings with large friction coefficient*. Mathematically these are likely to turn out to be very hard problems. There is the possibility of the nonexistence or nonuniqueness of solutions.
- A related issue is the *need to incorporate the various types of behavior of* $\mu$; its dependence on temperature, slip speed, roughness of the surface, and its evolutions as a result of the wear of the surface.
- There is a *need to investigate thermal effects in more detail* since it is well-known that these problems lead to thermal instabilities.
- *Dynamic noise generation, the squeaks and squeals of brakes and other friction devices* need a deep mathematical investigation. It entails the study of wave propagation in the contacting bodies and surfaces, which result from the slip/stick on the surfaces.
- *Detailed investigations of more restricted problems* are likely to give a better insight into the more general problems. Indeed, considerable progress has been made in the study of one-dimensional contact problems.

- As our recent research experience shows, *including additional phenomena into contact models* leads to new and interesting types of variational inequalities, creating thus impetus to develop and expand our current mathematical tools.

We conclude that considerable progress has been achieved, but also that much remains to be done to construct a comprehensive mathematical theory of quasistatic frictional contact problems.

## Acknowledgment

The author would like to thank the organizers of the MTNS 2000 Conference for their hospitality.

## References

Amassad A., Shillor M. and Sofonea M. (1999a): *A quasistatic contact problem for an elastic perfectly plastic body with Tresca's friction.* — Nonlin. Anal., Vol.35, pp.95–109.

Amassad A., Shillor M. and Sofonea M. (1999b): *A quasistatic contact problem with slip-dependent coefficient of friction.* — MMAS, Vol.22, pp.267–284.

Amassad A., Shillor M. and Sofonea M. (2001a): *A nonlinear evolution inclusion in perfect plasticity with friction.* — (preprint).

Amassad A., Rochdi M. and Shillor M. (2001b): *A dynamic thermoviscoelastic contact with slip and temperature dependent friction.* — (in preparation).

Andersson L.-E. (1991): *A quasistatic frictional problem with normal compliance.* — Nonlin. Anal., Vol.16, No.4, pp.347–370.

Andersson L.-E. and Klarbring A. (1997): *On a class of limit states of frictional joints: formulation and existence theorem.* — Quart. Appl. Math., Vol.LV, No.1, pp.9–87.

Andrews K.T., Kuttler K.L. and Shillor M. (1997): *On the dynamic behaviour of a thermo-viscoelastic body in frictional contact with a rigid obstacle.* — Europ. J. Appl. Math., Vol.8, pp.417–436.

Andrews K.T., Klarbring A., Shillor M. and Wright S. (1997): *A dynamic thermoviscoelastic contact problem with friction and wear.* — Int. J. Eng. Sci., Vol.35, No.14, pp.1291–1309.

Awbi B., M. Shillor and M. Sofonea (1999): *A contact problem for Bingham fluid with friction.* — Applic. Analysis, Vol.72, Vol.3–4, pp.469–484.

Awbi B., Shillor M. and Sofonea M. (2001): *Dual formulation of a quasistatic viscoelastic contact problem with Tresca's friction law.* — Applic. Anal. (to apper).

Chau O. (2000): *Analyse Variationnelle et Numérique de Quelques Problèmes aux Limites en Mécanique du Contact.* — Ph.D. Thesis, University of Perpignan.

Cocu M. (1984): *Existence of solutions of Signorini problems with friction.* — Int. J. Eng. Sci., Vol.22, No.7, pp.567–581.

Cocu M., Pratt E. and Raous M. (1996): *Formulation and approximation of quasistatic frictional contact.* — Int. J. Eng. Sci., Vol.34, No.7, pp.783–798.

Fernández-García J., Han W., Shillor M. and Sofonea M. (2000): *Recent advances on variational and numerical analysis in contact mechanics.* — Proc. *MTNS 2000*, Perpignan, France, June 19–23, published on CD-ROM.

Han W., Shillor M. and Sofonea M., *Variational and numerical analysis of a quasistatic viscoelastic problem with normal compliance, friction and damage.* — (to appear).

Jarusek J. and Eck C. (1996): *Dynamic contact problems with friction in linear viscoelasticity.* — C.R. Acad. Sci., Paris, Vol.322, pp.497–502.

Kikuchi N. and Oden J.T. (1988): *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods.* — Philadelphia: SIAM.

Klarbring A., Mikelic A. and Shillor M. (1988): *Frictional contact problems with normal compliance.* — Int. J. Eng. Sci., Vol.26, No.8, pp.811–832.

Klarbring A., Mikelic A. and Shillor M. (1991): *A global existence result for the quasistatic frictional contact problem with normal compliance*, In: Unilateral Problems in Structural Analysis, Vol.4 (G. Del Piero and F. Maceri, Eds.). — Boston: Birkhäuser, pp.85–111.

Kuttler K.L. and Shillor M. (1999): *Set-valued pseudomonotone maps and degenerate evolution inclusions.* — Comm. Contemp. Math., Vol.1, No.1, pp.87–123.

Martins J.A.C. and Oden J.T. (1987): *Existence and uniqueness results for dynamic contact problems with nonlinear normal and friction interface laws.* — Nonlin. Anal., Vol.11, No.3, pp.407–428.

Raous M., Jean M. and Moreau J.J. (Eds.) (1995): *Contact Mechanics.* — New York: Plenum Press.

Rabinowicz E. (1995): *Friction and Wear of Materials, 2nd Ed.* — New York: Wiley.

Rochdi M. and Shillor M. (2001): *Existence and uniqueness for a quasistatic frictional bilateral contact problem in thermoviscoelasticity.* — Q. Appl. Math. (to appear).

Rochdi M., Shillor M. and Sofonea M., (1998a): *A quasistatic contact problem with directional friction and damped response.* — Applic. Anal., Vol.68, No.3–4, pp.409–422.

Rochdi M., Shillor M. and Sofonea M. (1998b): *Quasistatic viscoelastic contact with normal compliance and friction.* — J. Elasticity, Vol.51, pp.105–126.

Shillor M. (Ed.) (1998): *Recent Advances in Contact Mechanics.* — Special issue of Math. Comput. Modelling, Vol.28, No.4–8.

Shillor M. and Sofonea M. (2001): *A quasistatic viscoelastic contact problem with friction.* — Inl. J. Eng. Sci. (to appear).

Sofonea M. and Shillor M. (2001): *Variational analysis of quasistatic viscoplastic contact problems with friction.* — Comm. Appl. Analysis (to appear).

Shillor M., Sofonea M. and Telega J.J. (2001): *Review of recent results on quasistatic contact* (in preparation).