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ADMISSIBLE DISTURBANCE SETS FOR DISCRETE PERTURBED SYSTEMS

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We consider a discrete disturbed system given by the difference bilinear equation

$$x_{i+1}^w = Ax_i^w + De_i + \sum_{j=1}^{q} f_i^j B_j x_i^w, \quad i \geq 0,$$

where $w = ((e_i)_{i\geq 0},\ (f_i)_{i\geq 0})$ are disturbances which excite the system in a linear and a bilinear form. We assume that the system is augmented with the output function $y_i^w = Cx_i^w$, $i \geq 0$. Let ε be a tolerance index on the output. The disturbance w is said to be ε-admissible if $\|y_i^w - y_i\| \leq \varepsilon$, $\forall\ i \geq 0$, where $(y_i)_{i\geq 0}$ is the output signal associated with the case of an uninfected system. The set of all ε-admissible disturbances is the admissible set $\mathcal{W}(\varepsilon)$. The characterization of $\mathcal{W}(\varepsilon)$ is investigated and numerical simulations are given.

Keywords: discrete systems, bilinear disturbance, admissibility index

1. Introduction

The principle of action and counter-action which exists between a system and its environment necessarily involves, in the mathematical modelling, the presence of certain parameters. In this case, the study of the system output can be conducted by means of various approaches in systems theory. This is the case of identifiability (Suzuki and Murayama, 1980; Kitamura and Nakagiri, 1977), detectability (Afifi and El Jai, 1994), filtering theory (Balakrishnan, 1976; Bensoussan and Viot, 1975; Curtain and Pritchard, 1978; Wonham, 1968), sentinels theory (Lions, 1988; Lions, 1990), or H^∞-control theory (Curtain and Zwart, 1995; Francis, 1987).

In this paper we explore a technique which allows us to determine among a class of disturbances those which are said to be ε-admissible. The considered disturbed system

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will be called the infected system. Consider the discrete infected system governed by the difference equation

$$\begin{cases} x_{i+1}^w = Ax_i^w + De_i + \sum_{j=1}^{q} f_i^j B_j x_i^w, \quad \forall i \geq 0, \\ x_0^w = x_0 \in \mathbb{R}^n, \end{cases} \tag{1}$$

and an associated output function

$$y_i^w = Cx_i^w, \quad \forall i \geq 0, \tag{2}$$

where A, B_j, D and C are respectively $(n{\times}n)$, $(n{\times}n)$, $(n{\times}p)$ and $(m{\times}n)$ matrices, $x_i^w \in \mathbb{R}^n$ is the state variable, $(e_i)_{i\geq 0} \in U(\mathbb{N}, \mathbb{R}^p) = \{(e_i)_{i\geq 0} : e_i = (e_i^j)_{1\leq j\leq p} \in \mathbb{R}^p\}$ and $(f_i)_{i\geq 0} \in V(\mathbb{N}, \mathbb{R}^q) = \{(f_i)_{i\geq 0} : f_i = (f_i^j)_{1\leq j\leq q} \in \mathbb{R}^q\}$ denote disturbances which excite the system respectively in a linear and a bilinear form. In the case where the system is autonomous (uninfected), this reduces to

$$\begin{cases} x_{i+1} = Ax_i, \quad i \geq 0, \\ x_0 \in \mathbb{R}^n, \end{cases} \tag{3}$$

and the associated output function is

$$y_i = Cx_i, \quad i \geq 0. \tag{4}$$

We will say that a disturbance $w = ((e_i)_{i\geq 0},\ (f_i)_{i\geq 0})$ is admissible if for every integer i, the output variable y_i^w remains close to the output of the uninfected system. Given a positive parameter ε, a disturbance $w = ((e_i)_{i\geq 0}, (f_i)_{i\geq 0})$ is said to be ε-admissible if $\|y_i^w - y_i\| \leq \varepsilon,\ \forall\ i \geq 0$. The parameter $\varepsilon \geq 0$ is the admissibility index. We assume that the disturbance w is only persistent on a given time interval $\{0, 1, \ldots, I\}$. The final time I is called the age of the disturbance w. This is motivated by some practical applications. Consequently, in this work we suppose that the disturbances w which infect the system are given by

$$(e_i)_{i\geq 0} \in U^I = \big\{(e_i)_{i\geq 0} \in U(\mathbb{N}, \mathbb{R}^p) : e_i = 0,\ \forall\ i > I\big\},$$

$$(f_i)_{i\geq 0} \in V^I = \big\{(f_i)_{i\geq 0} \in V(\mathbb{N}, \mathbb{R}^q) : f_i = 0,\ \forall\ i > I\big\}.$$

The principal objective is then to characterize the set $\mathcal{W}(\varepsilon)$ of all disturbances $w \in U^I \times V^I$ which are ε-admissible. $\mathcal{W}(\varepsilon)$ is called the ε-admissible set.

Remark 1. In order to motivate the formulation examined in this paper, consider the temperature distribution in an industrial oven (see Fig. 1), whose simplified mathematical model is

$$\frac{\partial T}{\partial t}(x,t) = \alpha \frac{\partial^2 T}{\partial x^2}(x,t) + \beta T(x,t) + \gamma u(t) T(x,t), \quad \forall\ t \geq 0, \tag{5}$$

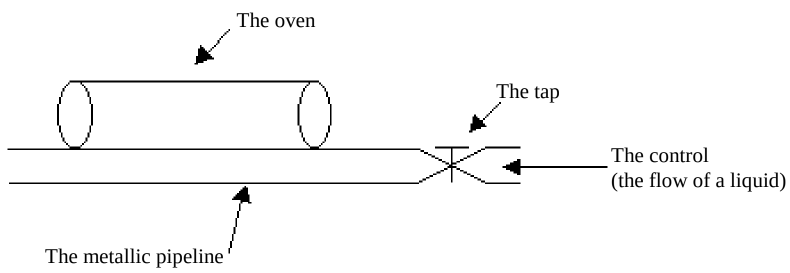

Fig. 1. Scheme of an industrial oven.

where $T(\cdot, t)$ is the temperature profile at time t. We suppose that the system is controlled via the flow of a liquid $u(\cdot)$ in an adequate metallic pipeline. The objective is to attain

$$y_u(t) = y_d(t), \quad \forall\, t \geq t_f,$$

where $y_u(\cdot)$ is the output function corresponding to the control $u(\cdot)$, and $y_d(\cdot)$ is the desired output.

The associated initial condition is supposed to be homogeneous

$$T(x, 0) = T_0(x), \quad \forall x \in [0, 1],$$

and so is the boundary condition,

$$T(0, t) = T(1, t) = 0, \quad \forall t \geq 0.$$

If we suppose that the system (5) is thermically isolated, we should stop supervising the system as soon as we achieve our objective, i.e. at time t_f. Then, starting from instant t_f, the evolution equation becomes

$$\frac{\partial T}{\partial t}(x, t) = \alpha \frac{\partial^2 T}{\partial x^2}(x, t) + \beta T(x, t), \quad \forall\, t > t_f. \tag{6}$$

In practice, however, we should not ignore the fact that there are some disturbances '$e(t)$' which affect the system and which stem essentially from

- the amount of heat preserved by the pipeline metal during the limited time interval $[t_f, t_1]$, and
- a delay 'h' existing between the stop control '$u(t)$' and its effects on the system.

Hence the evolution equation of the system can be written as follows:

$$\frac{\partial T}{\partial t}(x, t) = \alpha \frac{\partial^2 T}{\partial x^2}(x, t) + \beta T(x, t) + \gamma e(t) T(x, t), \quad \forall\, t > t_f, \tag{7}$$

where

$$e(t) = 0 \;\text{ for }\; t \geq \max(t_1, t_f + h).$$

State-space description. Equation (7) can be written down as

$$\frac{\partial T}{\partial t}(x,t) = AT(x,t) + \gamma e(t)T(x,t), \quad \forall\, t > t_f, \tag{8}$$

where A is the operator $\alpha(\partial^2/\partial x^2) + \beta$ whose domain $\mathcal{D}(A)$ and spectrum $\sigma(A)$ are respectively given by

$$\mathcal{D}(A) = \big\{ f \in L^2(0,1) \ : \ f^{''} \in L^2(0,1) \ \text{ and } \ f(0) = f(1) = 0 \big\},$$

$$\sigma(A) = \big\{ \lambda_n = \beta - \alpha n^2 \pi \ : \ n \in \mathbb{N}^* \big\},$$

and the associated eigenfunctions are

$$\varphi_n(x) = \sqrt{2}\sin(n\pi x), \quad n = 1, 2, \dots\ .$$

Then we have

$$T(x,t) = \sum_{i=1}^{\infty} a_i(t)\varphi_i(x). \tag{9}$$

Substituting (9) in (8), we obtain

$$\sum_{n=1}^{\infty} \dot{a}_n(t)\varphi_n(x) = \sum_{n=1}^{\infty} \lambda_n a_n(t)\varphi_n(x) + \sum_{n=1}^{\infty} \gamma e(t) a_n(t)\varphi_n(x),$$

which implies

$$\dot{a}_m(t) = \lambda_m a_m(t) + \gamma e(t) a_m(t); \quad m = 1, 2, \dots\ .$$

If we introduce the notation $\mathcal{A} = \operatorname{diag}(\lambda_1, \lambda_2, \dots)$, $a(t) = (a_1(t), a_2(t), \dots)^T$ and $\mathcal{B} = \gamma Id$, then eqn. (8) can be written as follows:

$$\begin{cases} \dot{a}(t) = \mathcal{A}a(t) + e(t)\mathcal{B}a(t), \ \ \forall t > t_f, \\ a(0) = a_0, \end{cases} \tag{10}$$

where $a_0 = (a_1(0), a_2(0), \dots)^T$ and $a_i(0) = \langle T(x,0), \varphi_i(x)\rangle_{L^2(0,1)}$.

Spatial approximation. Projecting the system (10) onto a finite dimensional subspace, we obtain

$$\begin{cases} \dot{a}^N(t) = \mathcal{A}_N a^N(t) + e(t)\mathcal{B}_N a^N(t), \quad \forall t > t_f, \\ a^N(0) = \big(a_1(0), a_2(0), \dots, a_N(0)\big)^T, \end{cases} \tag{11}$$

with $\mathcal{A}_N = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$, $\mathcal{B}_N = \gamma I_N$, where I_N is the $N \times N$ identity matrix.

Time sampling. In order to make the problem tractable by a computer, we partition the time interval as follows:

$$[t_f, \infty[= \bigcup_{i=0}^{\infty} [t_i, t_{i+1}],$$

where

$$\begin{cases} t_0 = t_f \\ t_{i+1} = t_i + \Delta, \quad \forall\, i \geq 0 \end{cases}$$

with Δ being sufficiently small. If we use the approximation

$$\dot{a}^N(t_i) \approx \frac{a^N(t_{i+1}) - a^N(t_i)}{\Delta},$$

we will have

$$\begin{cases} a_{i+1}^N = (\Delta \mathcal{A}_N + I_N) a_i^N + e_i \mathcal{B}_N a_i^N, \quad \forall i \geq 0, \\ a_0^N \quad \in \mathbb{R}^N, \end{cases} \tag{12}$$

where $a_i^N = a(t_i)$ and $e_i = e(t_i)$. Since $e(t) = 0$, $\forall\, t > \max(t_1, t_f + h)$, we get that $(e_i)_{i\geq 0}$ is zero starting from a certain integer I (I is called the age of disturbance). Consequently, the system (12) can be considered as a motivation of the systems studied in this paper.

2. Problem Statement

In order to formulate appropriately the problem, we consider the following approach. For every disturbance $w = ((e_i)_{i\geq 0},\ (f_i)_{i\geq 0}) \in U^I \times V^I$, the signal $s = (s_i)_{i\geq 0}$ denotes

$$s_i = \begin{pmatrix} s_i^1 \\ \vdots \\ s_i^p \\ s_i^{p+1} \\ \vdots \\ s_i^{p+q} \end{pmatrix} = \begin{pmatrix} e_i^1 \\ \vdots \\ e_i^p \\ f_i^1 \\ \vdots \\ f_i^q \end{pmatrix} \in \mathbb{R}^{p+q}$$

with $s_i^r = e_i^r,\ \forall\, r \in \{1, \dots, p\}$ and $s_i^{p+r} = f_i^r,\ \forall\, r \in \{1, \dots, q\}$. Then the system (1) can be rewritten in the form

$$\begin{cases} x_{i+1}^w = A x_i^w + [D \mid 0_{n,q}]\, s_i + \displaystyle\sum_{j=1}^{p+q} s_i^j D_j x_i^w, \quad i \geq 0, \\ x_0^w = x_0 \in \mathbb{R}^n, \end{cases} \tag{13}$$

where $0_{n,q}$ is the $(n \times q)$ zero matrix and

$$D_j = \begin{cases} 0_{n\times n} & \text{if } \ j \in \{1, \dots, p\}, \\ B_{j-p} & \text{if } \ j \in \{p+1, \dots, p+q\}. \end{cases}$$

Since the disturbance $w = ((e_i)_{i\geq 0}, (f_i)_{i\geq 0})$ is identified with $s = (s_i)_{i\geq 0}$, it is convenient to rewrite the system (13) as follows:

$$\begin{cases} x^s_{i+1} = Ax^s_i + \bar{D}s_i + \sum_{j=1}^{p+q} s^j_i D_j x^s_i, \quad i \geq 0, \\ x^s_0 = x_0 \in \mathbb{R}^n, \end{cases} \tag{14}$$

where $\bar{D}$ is the $n\times(p+q)$ matrix given by $\bar{D} = [D \mid 0_{n,q}]$. Denote by $(\phi_1, \phi_2, \dots, \phi_n)$ the canonical basis of $\mathbb{R}^n$ and let $(\psi_1, \psi_2, \dots, \psi_{p+q})$ stand for the canonical basis of $\mathbb{R}^{p+q}$. Hence for every $s_i \in \mathbb{R}^{p+q}$, we have

$$\begin{aligned} \bar{D}s_i \ &= \ \sum_{k=1}^{n} \left\langle \bar{D}s_i, \phi_k \right\rangle \phi_k = \sum_{k=1}^{n} \left\langle s_i, \bar{D}^T\phi_k \right\rangle \phi_k = \sum_{k=1}^{n} \left\langle \sum_{j=1}^{p+q} s^j_i \psi_j, \bar{D}^T\phi_k \right\rangle \phi_k \\ &= \sum_{j=1}^{p+q} s^j_i \sum_{k=1}^{n} \left\langle \psi_j, \bar{D}^T\phi_k \right\rangle \phi_k = \sum_{j=1}^{p+q} s^j_i V_j, \end{aligned} \tag{15}$$

where V_j is the vector of $\mathbb{R}^n$ given by

$$V_j = \sum_{k=1}^{n} \left\langle \psi_j, \bar{D}^T\phi_k \right\rangle \phi_k, \quad \forall \ j \in \{1, \dots, p+q\}. \tag{16}$$

Using state-space techniques, we will show that the system (14) is equivalent to a simpler one which contains a bilinear disturbance. For that purpose, let $s \in \mathcal{S}(\mathbb{N}, \mathbb{R}^{p+q}) = \{(s_i)_{i\geq 0} : s_i = (s^j_i)_{1\leq j\leq p+q} \in \mathbb{R}^{p+q}\}$ be a disturbance and consider the augmented state defined in $\mathbb{R}^{n+1}$ by

$$z^s_i = \begin{pmatrix} x^s_i \\ 1 \end{pmatrix} \quad \text{and} \quad z_i = \begin{pmatrix} x_i \\ 1 \end{pmatrix}, \quad \forall \ i \geq 0.$$

Using the $(n+1)\times(n+1)$ matrix $\mathcal{A} = \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and

$$\begin{aligned} \mathcal{K}_j : \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \ &\longrightarrow \ \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}, \\ \begin{pmatrix} x \\ r \end{pmatrix} \qquad &\mapsto \ \begin{pmatrix} D_j x + rV_j \\ 0 \end{pmatrix}, \end{aligned}$$

we show easily that $(z^s_i)_{i\geq 0}$ and $(z_i)_{i\geq 0}$ are the solutions of the following difference equations:

$$\begin{cases} z^s_{i+1} = \mathcal{A}z^s_i + \sum_{j=1}^{p+q} s^j_i \mathcal{K}_j z^s_i, \quad \forall \ i \geq 0, \\ z^s_0 = \begin{pmatrix} x_0 \\ 1 \end{pmatrix}, \end{cases} \tag{17}$$

and

$$\begin{cases} z_{i+1} = \mathcal{A} z_i, \quad \forall\ i \geq 0, \\ z_0 = \begin{pmatrix} x_0 \\ 1 \end{pmatrix}, \end{cases} \tag{18}$$

respectively. Consequently, the output functions (2) and (4) become

$$y_i^s = \widetilde{C} z_i^s \quad \text{and} \quad y_i = \widetilde{C} z_i,$$

respectively, where $\widetilde{C} = [C \mid 0_{m,1}]$ and $0_{m,1}$ is the $(m \times 1)$ zero matrix. This new formulation makes the solution of the initial problem possible. We are to determine the set $\mathcal{D}(\varepsilon)$ of all the disturbances $s = (s_i)_{i \geq 0}$ such that

$$s = (s_i)_{i \geq 0} \in \mathcal{S}^I = \left\{ (s_i)_{i \geq 0} \in \mathcal{S}(\mathbb{N}, \mathbb{R}^{p+q}) : s_i = 0,\ \forall\, i > I \right\},$$

and which satisfy $\|y_i^s - y_i\| \leq \varepsilon,\ \ \forall\, i \geq 0$. The set of ε-admissible disturbances $\mathcal{W}(\varepsilon)$ can be identified with the set $\mathcal{D}(\varepsilon)$ via the canonical isomorphism

$$\begin{array}{rcl} \Gamma : \mathcal{D}(\varepsilon) & \longrightarrow & \mathcal{W}(\varepsilon) \\ s_i = \begin{pmatrix} s_i^1 \\ \vdots \\ s_i^{p+q} \end{pmatrix}_{i \geq 0} & \mapsto & w = \left(\begin{pmatrix} s_i^1 \\ \vdots \\ s_i^p \end{pmatrix}, \begin{pmatrix} s_i^{p+1} \\ \vdots \\ s_i^{p+q} \end{pmatrix} \right)_{i \geq 0} \end{array}$$

and in the next sections the study will be focused on the set $\mathcal{D}(\varepsilon)$.

3. Admissible Set $\boldsymbol{\mathcal{D}}(\boldsymbol{\varepsilon})$

3.1. Preliminary Results

We start this section with some technical results which will be used in the sequel. Let $s = (s_i)_{i \geq 0}$. Then we can easily show that the solution to (17), corresponding to s, satisfies

$$z_i^s = \mathcal{H}_{i-1}^s \mathcal{H}_{i-2}^s \dots \mathcal{H}_0^s z_0, \quad i \geq 1, \tag{19}$$

where $z_0 = z_0^s = \left(\begin{smallmatrix} x_0 \\ 1 \end{smallmatrix}\right)$ and $\mathcal{H}_i^s$ is the $(n+1) \times (n+1)$ matrix defined by

$$\mathcal{H}_k^s = \mathcal{A} + \sum_{j=1}^{p+q} s_k^j \mathcal{K}_j, \quad \forall\ k \geq 0. \tag{20}$$

Then

$$\begin{aligned} \mathcal{D}(\varepsilon) &= \left\{ s = (s_k)_{k \geq 0} \in \mathcal{S}^I \ : \ \|y_i^s - y_i\| \leq \varepsilon, \quad \forall\, i \geq 1 \right\} \\ &= \left\{ s = (s_k)_{k \geq 0} \in \mathcal{S}^I \ : \ \|\widetilde{C} \mathcal{H}_{i-1}^s \dots \mathcal{H}_0^s z_0 - \widetilde{C} \mathcal{A}^i z_0\| \leq \varepsilon, \quad \forall\, i \geq 1 \right\} \\ &= \mathcal{D}_1(\varepsilon) \cap \mathcal{D}_2(\varepsilon), \end{aligned}$$

where

$$\mathcal{D}_1(\varepsilon) = \big\{s = (s_k)_{k\geq 0} \in \mathcal{S}^I \ : \ \|\widetilde{C}\mathcal{H}^s_{i-1}\cdots\mathcal{H}^s_0 z_0 - \widetilde{C}\mathcal{A}^i z_0\| \leq \varepsilon, \ \ \forall \ i \in \{1,\dots,I\}\big\},$$

$$\mathcal{D}_2(\varepsilon) = \big\{s = (s_k)_{k\geq 0} \in \mathcal{S}^I \ : \ \|\widetilde{C}\mathcal{H}^s_{i-1}\cdots\mathcal{H}^s_0 z_0 - \widetilde{C}\mathcal{A}^i z_0\| \leq \varepsilon, \ \ \forall \, i \geq I+1\big\}.$$

Furthermore, identifying $\mathcal{S}^I$ with $\mathbb{R}^{(p+q)(I+1)}$, we obtain

$$\mathcal{D}_1(\varepsilon) = \big\{s^I \in \mathbb{R}^{(p+q)(I+1)} \ : \ \|\widetilde{C}\mathcal{H}^s_{i-1}\cdots\mathcal{H}^s_0 z_0 - \widetilde{C}\mathcal{A}^i z_0\| \leq \varepsilon, \ \ \forall \, i \in \{1,\dots,I\}\big\},$$

$$\mathcal{D}_2(\varepsilon) = \big\{s^I \in \mathbb{R}^{(p+q)(I+1)} \ : \ \|\widetilde{C}\mathcal{H}^s_{i-1}\cdots\mathcal{H}^s_0 z_0 - \widetilde{C}\mathcal{A}^i z_0\| \leq \varepsilon, \quad \forall \ i \geq I+1\big\}.$$

where s^I is given by $s^I = \begin{pmatrix} s_0 \\ \vdots \\ s_I \end{pmatrix}$. Let us define the functional $\mathcal{L}$ by

$$\begin{array}{rcl} \mathcal{L} : \mathbb{R}^{(p+q)(I+1)} & \longrightarrow & \mathbb{R}^{n+1} \\ s^I = \begin{pmatrix} s_0 \\ \vdots \\ s_I \end{pmatrix} & \mapsto & \Big(\ \mathcal{H}^s_I \mathcal{H}^s_{I-1} \dots \mathcal{H}^s_0 z_0 - \mathcal{A}^{I+1} z_0 \ \Big) \end{array}$$

and the functional $(\mathcal{L}_i)_{i\in\{1,\dots,I\}}$ by

$$\begin{array}{rcl} \mathcal{L}_i : \mathbb{R}^{(p+q)(I+1)} & \longrightarrow & \mathbb{R}^{n+1} \\ s^I = \begin{pmatrix} s_0 \\ \vdots \\ s_I \end{pmatrix} & \mapsto & \Big(\ \mathcal{H}^s_{i-1} \mathcal{H}^s_{i-2} \dots \mathcal{H}^s_0 z_0 - \mathcal{A}^{i} z_0 \ \Big). \end{array}$$

Hence

$$\mathcal{D}_1(\varepsilon) = \big\{s^I \in \mathbb{R}^{(p+q)(I+1)} \ : \ \|\widetilde{C}\mathcal{L}_i(s^I)\| \leq \varepsilon, \quad \forall \ i \in \{1,\dots,I\}\big\}$$

and, since $s_i = 0$ for every $i \geq I+1$, we have $\mathcal{H}^s_k = \mathcal{A}, \quad \forall k \geq I+1$. Finally,

$$\begin{aligned} \mathcal{D}_2(\varepsilon) &= \big\{s^I \in \mathbb{R}^{(p+q)(I+1)} \ : \ \|\widetilde{C}\mathcal{A}^{i-I-1}\mathcal{H}^s_I \dots \mathcal{H}^s_0 z_0 - \widetilde{C}\mathcal{A}^i z_0\| \leq \varepsilon, \ \ \forall \, i \geq I+1\big\} \\ &= \big\{s^I \in \mathbb{R}^{(p+q)(I+1)} \ : \ \|\widetilde{C}\mathcal{A}^{i-I-1}(\mathcal{H}^s_I \dots \mathcal{H}^s_0 z_0 - \mathcal{A}^{I+1} z_0)\| \leq \varepsilon, \ \forall \, i \geq I+1\big\} \\ &= \big\{s^I \in \mathbb{R}^{(p+q)(I+1)} \ : \ \|\widetilde{C}\mathcal{A}^k \mathcal{L}(s^I)\| \leq \varepsilon, \quad \forall \ k \geq 0\big\}. \end{aligned}$$

Remark 2. Before trying to characterize the set $\mathcal{D}(\varepsilon)$, it is natural to justify that this set is not reduced to zero. ($w = 0$ corresponds to the case where the system is not disturbed.) In the following proposition, we show that, under the hypothesis of the Lyapunov stability of A, both $\mathcal{D}(\varepsilon)$'s contain a ball centred at $w = 0$.

Proposition 1. *We have the following results:*

(i) $\mathcal{D}(\varepsilon)$ *is a closed set.*

(ii) If A *is Lyapunov stable (i.e., the eigenvalues* λ_i *of* A *satisfy the condition* $|\lambda_i| \leq 1$ *for all* i *and* $|\lambda_i| = 1$ *implies that* λ_i *is simple), then* $0 \in \operatorname{int} \mathcal{D}(\varepsilon)$.

Proof.

(i) Since

$$\mathcal{D}_1(\varepsilon) = \bigcap_{i\in\{1,\dots,I\}} \mathcal{L}_i^{-1}\left(\{x \in \mathbb{R}^{n+1} \ : \ \|\widetilde{C}x\| \le \varepsilon\}\right)$$

and $(\mathcal{L}_i)_{i\in\{1,\dots,I\}}$ are continuous, the set $\mathcal{D}_1(\varepsilon)$ is closed because $\{x \in \mathbb{R}^{n+1} : \|\widetilde{C}x\| \le \varepsilon\}$ is closed.

On the other hand, $\mathcal{D}_2(\varepsilon) = \mathcal{L}^{-1}(\mathcal{U})$, where $\mathcal{U}$ is the closed set given by $\mathcal{U} = \{x \in \mathbb{R}^{n+1} \ : \ \|\widetilde{C}\mathcal{A}^k x\| \le \varepsilon, \ \forall k \ge 0\}$. The continuity of $\mathcal{L}$ implies that $\mathcal{D}_2(\varepsilon)$ is closed. Thus we conclude that so is $\mathcal{D}(\varepsilon) = \mathcal{D}_1(\varepsilon) \cap \mathcal{D}_2(\varepsilon)$.

(ii) It is clear that $0 = \mathcal{L}_i(0) \in \operatorname{int}(\{x \in \mathbb{R}^{n+1} \ : \ \|x\| \le \varepsilon\})$. Moreover, $(\mathcal{L}_i)_{i\in\{1,\dots,I\}}$ are continuous functions. Consequently, for every integer $i \in \{1,\dots,I\}$, there exists an open set $\mathcal{O}_i$ such that $0 \in \mathcal{O}_i$ and $\mathcal{L}_i(\mathcal{O}_i) \subset \{x \in \mathbb{R}^{n+1} \ : \ \|x\| \le \varepsilon\}$. This implies that for every integer $i \in \{1,\dots,I\}$, we have $\mathcal{O}_i \subset \mathcal{L}_i^{-1}(\{x \in \mathbb{R}^{n+1} : \|x\| \le \varepsilon\})$. Consequently, $0 \in \cap_{i\in\{1,\dots,I\}} \mathcal{O}_i \subset \cap_{i\in\{1,\dots,I\}} \mathcal{L}_i^{-1}(\{x \in \mathbb{R}^{n+1} \|\widetilde{C}x\| \le \varepsilon\} = \mathcal{D}_1(\varepsilon)$, so we deduce that $0 \in \operatorname{int}\mathcal{D}_1(\varepsilon)$.

On the other hand, the Lyapunov stability of A implies the existence of $\alpha > 0$ such that

$$\|CA^i x\| \le \alpha\|x\|, \quad \forall\, x \in \mathbb{R}^n \quad \text{and} \quad \forall\, i \in \mathbb{N}.$$

Hence there exists a constant ρ which satisfies

$$\|\widetilde{C}\mathcal{A}^i X\| \le \rho\|X\|, \quad \forall\, X \in \mathbb{R}^{n+1} \quad \text{and} \quad \forall\, i \in \mathbb{N}.$$

This implies

$$\|\widetilde{C}\mathcal{A}^i \mathcal{L}(s^I)\| \le \rho\|\mathcal{L}(s^I)\|, \quad \forall\, i \ge 0, \quad \forall\, s^I \in \mathbb{R}^{(p+q)(I+1)},$$

and then

$$\left\{s^I \in \mathbb{R}^{(p+q)(I+1)} \ : \ \|\mathcal{L}(s^I)\| \le \varepsilon/\rho\right\} \subset \mathcal{D}_2(\varepsilon).$$

Finally, from the continuity of $\mathcal{L}$ we deduce that $0 \in \operatorname{int}\mathcal{D}_2(\varepsilon)$ and, consequently, $0 \in \operatorname{int}\mathcal{D}_1(\varepsilon) \cap \operatorname{int}\mathcal{D}_2(\varepsilon) \Longrightarrow 0 \in \operatorname{int}\mathcal{D}(\varepsilon)$. ■

Remark 3. It is obvious that the set $\mathcal{D}_1(\varepsilon)$ can be completely obtained by solving a finite number of functional inequalities. However, the set $\mathcal{D}_2(\varepsilon)$ is defined by an infinite number of inequalities, and so it can be hardly obtained. As in (Rachik *et al.*, 1997, page 174, Th.3.3; Zabczyk, 1995, page 1012, Th.41), the fundamental hypothesis on which we are going to base our argument is the asymptotic stability of A. In fact, although A is disturbed by the term $\sum_{j=1}^{q} f_i^j B_j$, we demonstrate that the hypothesis of the asymptotic stability of A remains a very effective means of establishing that $\mathcal{D}_2(\varepsilon)$ is finitely accessible.

3.2. Characterization

In order to improve the structure of $\mathcal{D}_2(\varepsilon)$, we introduce the following sets:

$$\mathcal{U}(\varepsilon) = \big\{x \in \mathbb{R}^{n+1} \ : \ \|\widetilde{C}\mathcal{A}^i x\| \le \varepsilon \ \ \forall\, i \ge 0\big\},$$

$$\mathcal{U}^k(\varepsilon) = \big\{x \in \mathbb{R}^{n+1} \ : \ \|\widetilde{C}\mathcal{A}^i x\| \le \varepsilon \ \ \forall\, i \in \{0,1,\dots,k\}\big\}, \ \ k \ge 0$$

and

$$\mathcal{D}_2^k(\varepsilon) = \big\{(s^I) \in \mathbb{R}^{(p+q)(I+1)} \ : \ \|\widetilde{C}\mathcal{A}^i \mathcal{L}(s^I)\| \le \varepsilon, \ \ \forall \ \ i \in \{0,1,\dots,k\}\big\}, \ \ k \ge 0.$$

Definition 1. The set $\mathcal{D}_2(\varepsilon)$ is said to be finitely accessible if there exists an integer k such that $\mathcal{D}_2(\varepsilon) = \mathcal{D}_2^k(\varepsilon)$. We denote by $k^\star$ the smallest integer which satisfies $\mathcal{D}_2(\varepsilon) = \mathcal{D}_2^{k^\star}(\varepsilon)$. Similarly, the set $\mathcal{U}(\varepsilon)$ is said to be finitely accessible if there exists $k \in \mathbb{N}$ such that $\mathcal{U}(\varepsilon) = \mathcal{U}^k(\varepsilon)$.

Remark 4.
(a) For integers i and j such that $i \ge j$, we have

$$\mathcal{U}(\varepsilon) \subset \mathcal{U}^i(\varepsilon) \subset \mathcal{U}^j(\varepsilon) \quad \text{and} \quad \mathcal{D}_2(\varepsilon) \subset \mathcal{D}_2^i(\varepsilon) \subset \mathcal{D}_2^j(\varepsilon).$$

(b) It is easy to show that

$$\mathcal{D}_2^i(\varepsilon) = \mathcal{L}^{-1}(\mathcal{U}^i(\varepsilon)) \quad \text{and} \quad \mathcal{D}_2(\varepsilon) = \mathcal{L}^{-1}\big(\mathcal{U}(\varepsilon)\big).$$

Then we have

$$\mathcal{U}(\varepsilon) \ \text{ is finitely accessible } \ \Longrightarrow \mathcal{D}_2(\varepsilon) \ \text{ is finitely accessible.}$$

Proposition 2. *The set $\mathcal{U}(\varepsilon)$ is finitely accessible if and only if $\mathcal{U}^{i+1}(\varepsilon) = \mathcal{U}^i(\varepsilon)$ for some $i \in \mathbb{N}$.*

Proof. If $\mathcal{U}(\varepsilon)$ is finitely accessible, the equality holds for all $i \ge k^\star$ ($k^\star$ is the smallest integer such that $\mathcal{U}(\varepsilon) = \mathcal{U}^k(\varepsilon)$).

Conversely, from $\mathcal{U}^{i+1}(\varepsilon) = \mathcal{U}^i(\varepsilon)$ we deduce that $\mathcal{A}(\mathcal{U}^i(\varepsilon)) \subset \mathcal{U}^i(\varepsilon)$, which implies the inclusion $\mathcal{A}^k(\mathcal{U}^i(\varepsilon)) \subset \mathcal{U}^i(\varepsilon)$ for every integer k. Consequently, $\mathcal{U}^i(\varepsilon) \subset \mathcal{U}(\varepsilon)$, and then from Remark 4(a) we see that $\mathcal{U}^i(\varepsilon) = \mathcal{U}(\varepsilon)$. ■

As a natural consequence of the previous proposition, in Subsection 4.1 we give an algorithm which allows us to determine the smallest integer $k^\star$ such that $\mathcal{U}(\varepsilon) = \mathcal{U}^{k^\star}(\varepsilon)$ and then, by Remark 4(b), $\mathcal{D}_2(\varepsilon) = \mathcal{D}_2^{k^\star}(\varepsilon)$.

3.3. Finite Accessibility of $\mathcal{D}(\varepsilon)$

The aim of this section is to build up sufficient conditions which make $\mathcal{D}_2(\varepsilon)$ accessible.

Theorem 1. *Assume that the following assumptions hold:*
(i) A *is asymptotically stable* ($|\lambda| < 1$ *for every* λ *in the spectrum of* A).
(ii) The system defined by the pair (C, A) *is observable (in other words,* $[C^T|A^TC^T|\dots|(A^T)^{n-1}C^T]$ *has rank* n*, see (Zabczyk, 1995, p. 25)).*

Then the set $\mathcal{D}_2(\varepsilon)$ *is finitely accessible.*

Proof. Consider the set

$$\mathcal{N}^i(\varepsilon) = \big\{x \in \mathbb{R}^n \ : \ \|CA^k x\| \le \varepsilon, \ \ \forall\, k \in \{0, 1 \dots, i\}\big\}.$$

It is clear that $\mathcal{U}^i(\varepsilon) = \mathcal{N}^i(\varepsilon) \times \mathbb{R}$. On the other hand, for every $x \in \mathcal{N}^{n-1}(\varepsilon)$, we have

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x \in \overbrace{B_m(0,\varepsilon) \times \cdots \times B_m(0,\varepsilon)}^{n\text{-times}},$$

where

$$B_m(0,\varepsilon) = \big\{x \in \mathbb{R}^m \ : \ \|x\| \le \varepsilon\big\}.$$

Then

$$(\Lambda^T\Lambda)\big[\mathcal{N}^{n-1}(\varepsilon)\big] \subset \Lambda^T\big(\overbrace{B_m(0,\varepsilon) \times \cdots \times B_m(0,\varepsilon)}^{n\text{-times}}\big), \tag{21}$$

where Λ is the $(n \times nm)$ matrix given by

$$\Lambda = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Since (A, C) is observable, there exists $\alpha > 0$ such that

$$\big\langle \Lambda^T\Lambda x, x\big\rangle \ge \alpha\|x\|^2, \quad \forall x \in \mathbb{R}^n.$$

Hence from (21) we deduce that for every $x \in \mathcal{N}^{n-1}(\varepsilon)$ there exists $z \in \overbrace{B_m(0,\varepsilon) \times \cdots \times B_m(0,\varepsilon)}^{n\text{-times}}$ such that $\Lambda^T\Lambda x = \Lambda^T z$, and then $\langle \Lambda^T\Lambda x, x\rangle = \langle \Lambda^T z, x\rangle$. Thus

$$\alpha\|x\|^2 \le \|\Lambda^T\|\ \|x\|\ \|z\|, \quad \forall x \in \mathcal{N}^{n-1}(\varepsilon).$$

Since $\overbrace{B_m(0,\varepsilon)\times\cdots\times B_m(0,\varepsilon)}^{n\text{-times}}$ is a bounded set, there exists a constant $r>0$ such that

$$\mathcal{N}^{n-1}(\varepsilon)\subset B_n(0,r)=\{x\in\mathbb{R}^n\ :\ \|x\|\le r\}.$$

Hence

$$\mathcal{N}^{i}(\varepsilon)\subset B_n(0,r),\quad \forall\, i\ge n-1.$$

Using the asymptotic stability of A, we see that there exists $k_0\ge n-1$ such that $\|CA^{k_0+1}\|\le\varepsilon/r$. Thus

$$CA^{k_0+1}\big(B_n(0,r)\big)\subset B_m(0,\varepsilon),$$

which implies

$$\mathcal{N}^{k_0}(\varepsilon)=\mathcal{N}^{k_0+1}(\varepsilon).$$

Hence $\mathcal{U}^{k_0}(\varepsilon)=\mathcal{U}^{k_0+1}(\varepsilon)$, so by Proposition (2), we conclude that $U(\varepsilon)$ is finitely accessible, and consequently $\mathcal{D}_2(\varepsilon)$ is finitely accessible. ■

4. Algorithmic Approach and Examples

From the previous results we can deduce an algorithm for determination of the ε-admissible set.

4.1. Algorithm

The main problem is to achieve $\mathcal{U}^{i+1}(\varepsilon)=\mathcal{U}^{i}(\varepsilon)$, and this will naturally produce the value of $k^\star$. For that purpose, we consider the following approach:

Let $\mathbb{R}^m$ be normed by

$$\|x\|=\max_{1\le i\le m}|x_i|,\quad x=(x_1,\dots,x_m)\in\mathbb{R}^m.$$

For every integer p, the set $\mathcal{U}^p(\varepsilon)$ can be written down as follows:

$$\mathcal{U}^p(\varepsilon)=\big\{x\in\mathbb{R}^{n+1}\ :\ h_j(\widetilde{C}\mathcal{A}^i x)\le 0 \text{ for } j=1,2,\dots,2m \text{ and } i=0,1,\dots,p\big\},$$

where $h_j:\mathbb{R}^m\longrightarrow\mathbb{R}$ are such that for every $x=(x_1,\dots,x_m)\in\mathbb{R}^m$ and for every $t\in\{1,2,\dots,m\}$ we have

$$\begin{cases} h_{2t-1}(x)=x_t-\varepsilon,\\ \quad h_{2t}(x)=-x_t-\varepsilon.\end{cases}$$

Since

$$\mathcal{U}^{p+1}(\varepsilon)=\mathcal{U}^{p}(\varepsilon)\Longleftrightarrow\mathcal{U}^{p}(\varepsilon)\subset\mathcal{U}^{p+1}(\varepsilon),$$

it follows that

$$\mathcal{U}^{p+1}(\varepsilon) = \mathcal{U}^{p}(\varepsilon) \Longleftrightarrow \forall x \in \mathcal{U}^{p}(\varepsilon), \quad \forall j \in \{1, 2, \ldots, 2m\} \;\; h_j(\widetilde{C}\mathcal{A}^{p+1}x) \le 0$$

or, equivalently,

$$\sup_{x \in \mathcal{U}^{p}(\varepsilon)} h_j(\widetilde{C}\mathcal{A}^{p+1}x) \le 0 \;\text{ for }\; j \in \{1, 2, \ldots, 2m\}.$$

Thus the algorithm can be implemented as follows:

Step 1 : Let $p := 0$;

Step 2 : Repeat

- For $i := 1, 2, \ldots, 2m$ do

  Solve the constrained optimization problem

  $$\begin{cases} \text{maximize } \; J_i(x) = h_i(\widetilde{C}\mathcal{A}^{p+1}x) \text{ subject to} \\ h_j(\widetilde{C}\mathcal{A}^{\ell}x) \le 0; \quad j = 1, 2, \ldots, 2m, \quad \ell = 0, 1, \ldots, p; \end{cases}$$

  Let $J_i^\star$ be the maximum value of J_i;

- Let $p := p + 1$;

Until $(J_1^\star \le 0$ and $J_2^\star \le 0$ and $\cdots$ and $J_{2m}^\star \le 0)$;

Step 3 : Let $k^\star := p - 1$;

Remark 5. The success of the proposed algorithm depends on the existence of effective algorithms for solving rather large mathematical programming problems which arise in Step 2. This presents some difficulty because global optima are needed. Even when the functions $h_i, i = 1, \ldots, 2m$ are convex, the difficulty remains because the programming problems require the maximun of a convex function subject to convex constraints. When the set $\{z \in \mathbb{R}^m \;:\; h_j(z) \le 0;\; j = 0, 1, \ldots, 2m\}$ is a polyhedron, the difficulty disappears as the programming problems are linear and efficient computer codes for obtaining global maxima abound.

4.2. Examples

In Examples 1–5 we assume that the disturbances are given by $(e_0, f_0) \in \mathbb{R}^2$ $(I = 0)$. Using the algorithm developed in Section 4, we give in Table 1 the values of $k^\star$ corresponding to various choices of the matrices defining the system.

Comment 1. We notice that $k^\star$ is finite for Example 5, although the matrix A has one eigenvalue equal to 1. This illustrates the fact that the condition of the asymptotic stability of A in Theorem 1 is sufficient but not necessary.

Table 1. Data for Examples 1–5.

	A	D	B	C	ε	$k^\star$
Example 1	$\begin{pmatrix} 0.9 & 0 \\ 0.6 & 0.3 \end{pmatrix}$	$\begin{pmatrix} 0.7 \\ 0.5 \end{pmatrix}$	$\begin{pmatrix} 1.5 & -2 \\ -1 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \end{pmatrix}$	0.5	2
Example 2	$\begin{pmatrix} 0.6 & 0.5 \\ 0 & 0.5 \end{pmatrix}$	$\begin{pmatrix} 0.7 \\ 0.5 \end{pmatrix}$	$\begin{pmatrix} 0.5 & 0 \\ 1 & 0.5 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$	0.3	2
Example 3	$\begin{pmatrix} 0.5 & -1 \\ 0 & 0.3 \end{pmatrix}$	$\begin{pmatrix} -0.8 \\ 0.5 \end{pmatrix}$	$\begin{pmatrix} 1.5 & -2 \\ 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \end{pmatrix}$	0.1	1
Example 4	$\begin{pmatrix} 0.6 & 0.7 \\ 0 & 0.5 \end{pmatrix}$	$\begin{pmatrix} 0.7 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0.6 & 0 \\ 1 & 0.7 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	0.2	2
Example 5	$\begin{pmatrix} 1 & 0 \\ 0 & 0.8 \end{pmatrix}$	$\begin{pmatrix} 2 \\ -3 \end{pmatrix}$	$\begin{pmatrix} 0.5 & 0 \\ 1 & 0.5 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$	0.5	2

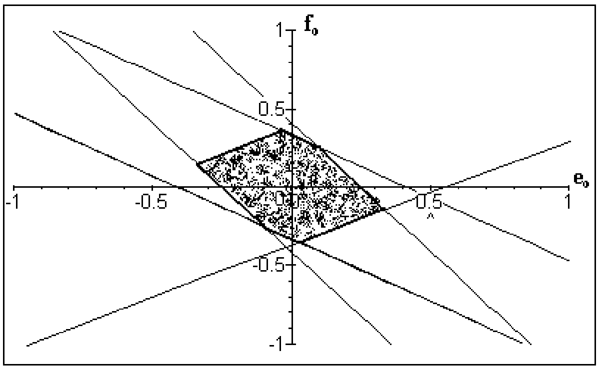

Fig. 2. The dotted region is the set $\mathcal{W}(\varepsilon)$ corresponding to Example 1.

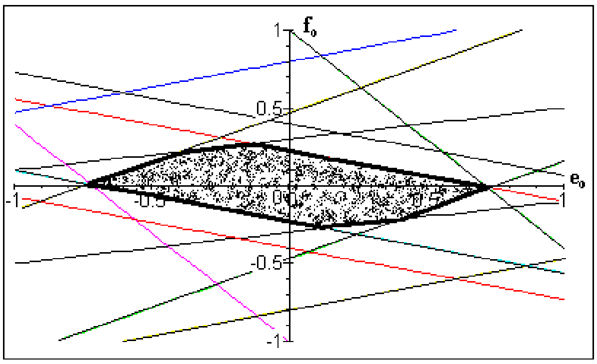

Fig. 3. The dotted region is the set $\mathcal{W}(\varepsilon)$ corresponding to Example 2.

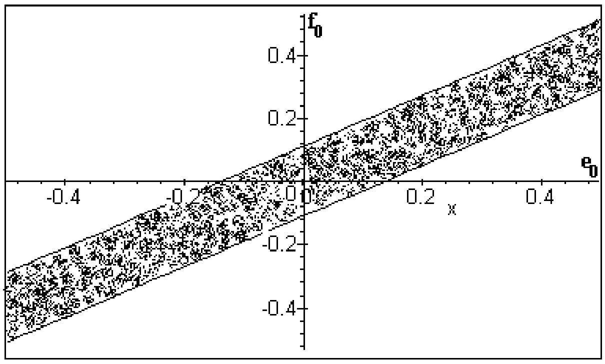

Fig. 4. The dotted region is the set $\mathcal{W}(\varepsilon)$ corresponding to Example 3.

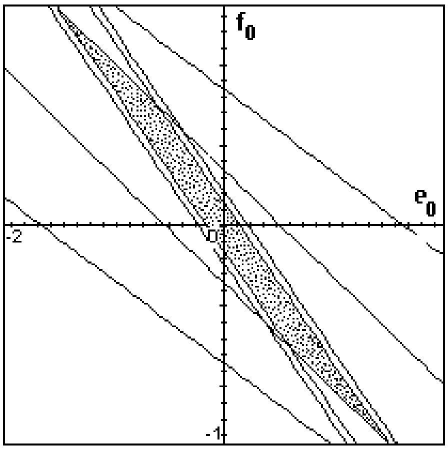

Fig. 5. The dotted region is the set $\mathcal{W}(\varepsilon)$ corresponding to Example 4.

In Examples 6–10 we assume that the system is disturbed only by a bilinear term $w = (f_0, f_1)$ with age $I = 1$. Table 2 gives the calculated values of $k^\star$.

Comment 2. The fact of writing $k^\star = \infty$ in Example 8 does not mean that we have proved that the algorithm is not convergent, but, by $k^\star = \infty$, we explain the fact that at the time of the analysis of this example on a computer, we obtained higher values of $k^\star$ without achieving the condition $(J_1^\star \leq 0 \text{ and } J_2^\star \leq 0 \text{ and } \dots \text{ and } J_{2m}^\star \leq 0)$

Comment 3. In Example 11 we showed by hand calculation that $k^\star = \infty$. In fact, the perturbed system is

$$\begin{cases} x_{i+1}^w = a x_i^w + e_i + f_i x_i^w, \ \ \forall i \geq 0, \\ \quad x_0 \in \mathbb{R}. \end{cases}$$

Table 2. Data for Examples 6–11.

	A	B	C	ε	$k^\star$
Example 6	$\begin{pmatrix} 0.7 & 0 \\ -1 & 0.3 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \end{pmatrix}$	0.4	1
Example 7	$\begin{pmatrix} 0.7 & 0 \\ -1 & 0.4 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$	0.8	2
Example 8	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 1 \\ 0 & 0 & 0.6 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$	1	∞
Example 9	$\begin{pmatrix} 0.7 & 0 \\ -1 & 0.4 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \end{pmatrix}$	0.4	1
Example 10	$\begin{pmatrix} 0.7 & 0 \\ -1 & 0.4 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$	0.3	2
Example 11	$a > 1$	1	1	ε	∞

The associated output function is

$$y_i^w = x_i^w, \quad \forall i \geq 0.$$

We suppose that a, e_i and $f_i \in \mathbb{R}$ satisfy $a > 1$ and $e_i = f_i = 0$, $\forall\ i \geq 1$. Here $y_i = x_i$ is the output signal corresponding to the case of the uninfected system $x_{i+1} = ax_i$. Then

$$\mathcal{W}(\varepsilon) = \left\{ \begin{pmatrix} e_0 \\ f_0 \end{pmatrix} \in \mathbb{R}^2 \ : \ |y_i^w - y_i| \leq \varepsilon, \quad \forall\ i \geq 1 \right\} = \mathcal{D}(\varepsilon) \cap \bar{\mathcal{D}}(\varepsilon),$$

where

$$\mathcal{D}(\varepsilon) = \left\{ \begin{pmatrix} e_0 \\ f_0 \end{pmatrix} \in \mathbb{R}^2 \ : \ |f_0 x_0 + e_0| \leq \varepsilon, \quad \forall\ i \geq 1 \right\},$$

$$\bar{\mathcal{D}}(\varepsilon) = \left\{ \begin{pmatrix} e_0 \\ f_0 \end{pmatrix} \in \mathbb{R}^2 \ : \ |f_0 x_0 + e_0| \leq \varepsilon, \quad \forall\ i \geq 1 \right\},$$

if we set

$$\mathcal{D}_k(\varepsilon) = \left\{ \begin{pmatrix} e_0 \\ f_0 \end{pmatrix} \in \mathbb{R}^2 \ : \ a^i |f_0 x_0 + e_0| \leq \varepsilon, \quad \forall\ i \in \{1, 2, \ldots, k\} \right\}, \quad k \geq 1.$$

It is easy to show that

$$\mathcal{D}_{k+1}(\varepsilon) \subset \mathcal{D}_k(\varepsilon), \quad \forall\ k \geq 1,$$

$$\begin{aligned} \mathcal{W}(\varepsilon) \text{ is finitely accessible} &\Longrightarrow \bar{\mathcal{D}}(\varepsilon) \text{ is finitely accessible} \\ &\Longrightarrow \exists\ k^* \ :\ \bar{\mathcal{D}}(\varepsilon) = \mathcal{D}_{k^*}(\varepsilon) \\ &\Longrightarrow \mathcal{D}_k(\varepsilon) = \mathcal{D}_{k^*}(\varepsilon)\ \ \forall\ \ k \geq k^*. \end{aligned}$$

Hence the fact that $a > 1$ implies

$$\mathcal{D}_k(\varepsilon) = \left\{ \begin{pmatrix} e_0 \\ f_0 \end{pmatrix} \in \mathbb{R}^2 \ :\ a^k |f_0 x_0 + e_0| \leq \varepsilon \right\}.$$

Then, if we suppose that $\mathcal{W}(\varepsilon)$ is finitely accessible, there exists $k^\star \in \mathbb{N}$ such that

$$\mathcal{D}_k(\varepsilon) = \mathcal{D}_{k^\star}(\varepsilon), \quad \forall\ k \geq k^\star \Longrightarrow \mathcal{D}_{k^\star+1}(\varepsilon) = \mathcal{D}_{k^\star}(\varepsilon),$$

which constitutes a contradiction because

$$w = \begin{pmatrix} e_0 = \varepsilon(a+1)/2a^{k^*+1} \\ \\ f_0 = 0 \end{pmatrix} \in \mathcal{D}_{k^\star}(\varepsilon) \quad \text{but} \quad w \notin \mathcal{D}_{k^\star+1}.$$

In fact,

$$a^{k^\star} |f_0 x_0 + e_0| = a^{k^\star} \varepsilon \frac{a+1}{2a^{k^\star}+1} = \varepsilon \frac{a+1}{2a} \leq \varepsilon \ \text{ because } \ a+1 \leq 2a,$$

$$a^{k^\star+1} |f_0 x_0 + e_0| = a^{k^\star+1} \varepsilon \frac{a+1}{2a^{k^\star+1}} = \varepsilon \frac{a+1}{2} > \varepsilon \ \text{ because } a > 1.$$

Therefore we conclude that it is impossible to find $k^\star \in \mathbb{N}$ such that

$$\mathcal{D}_{k^\star}(\varepsilon) = \mathcal{D}_{k^\star+1}(\varepsilon).$$

Consequently, $\mathcal{W}(\varepsilon)$ is not finitely accessible, i.e. $k^* = \infty$.

5. Conclusion

In this paper we have considered a discrete finite-dimensional system infected by some disturbances which are only persistent on a given time interval $\{0, 1, \ldots, I\}$. By fixing a degree of tolerance ε, we have given a characterization of all perturbations whose effect is ε-acceptable. The results obtained in this paper can be considered as a generalization of those established in the linear case (see Rachik *et al.*, 2000). As a logical continuation of this work, the following problems are under investigation:

Problem 1. How to solve the problem for perturbations which are not limited in time? For example, consider pertubations $((e_i)_{i\geq 0}, (f_i)_{i\geq 0})$ such that

$$\lim_{i\to\infty} e_i = 0, \quad \lim_{i\to\infty} f_i = 0,$$

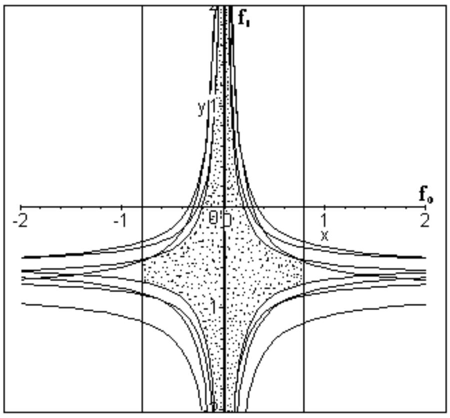

Fig. 6. The dotted region is the set $\mathcal{W}(\varepsilon)$ corresponding to Example 7.

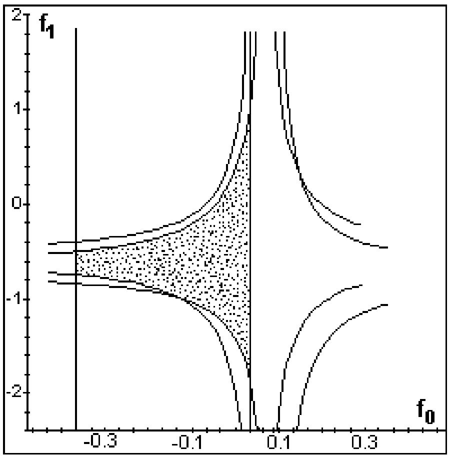

Fig. 7. The dotted region is the set $\mathcal{W}(\varepsilon)$ corresponding to Example 9.

$$\sum_{i=0}^{\infty} e_i^2 < \infty, \quad \sum_{i=0}^{\infty} f_i^2 < \infty,$$

$$\lim_{i\to\infty} i^k e_i = 0, \quad \lim_{i\to\infty} i^h f_i = 0,$$

where k and h are in $\mathbb{R}_*^+$.

Problem 2. Given a disturbance $\mathcal{E} = ((e_i)_{i\geq 0}, (f_i)_{i\geq 0})$, determine a control which eliminates or reduces the effect of the disturbance $\mathcal{E}$, with a minimal energy in an optimal time.

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