Int. J. Appl. Math. Comput. Sci., 2002, Vol.12, No.2, 181–195

CONTROLLABILITY, OBSERVABILITY AND OPTIMAL CONTROL OF CONTINUOUS-TIME 2-D SYSTEMS

GERHARD JANK*

* Lehrstuhl II für Mathematik, RWTH Aachen, D-52056 Aachen, Germany
e-mail: jank@math2.rwth-aachen.de

We consider linear 2-D systems of Fornasini-Marchesini type in the continuous-time case with non-constant coefficients. Using an explicit representation of the solutions by utilizing the Riemann-kernel of the equation under consideration, we obtain controllability and observability criteria in the case of the inhomogeneous equation, where control is obtained by choosing the inhomogeneity appropriately, but also for the homogeneous equation, where control is obtained by steering with Goursat data. The optimal control problem with a quadratic cost functional is also solved.

Keywords: 2-D continuous-time systems, controllability, observability, optimal control, quadratic cost

1. Introduction

We study controllability and observability properties of the linear system described by the following hyperbolic system:

$$\frac{\partial^2}{\partial s \partial t} x = A_0(s,t)x + A_1(s,t)\frac{\partial}{\partial s}x + A_2(s,t)\frac{\partial}{\partial t}x + B(s,t)u, \tag{1}$$

where $x(s,t) \in \mathbb{R}^n$, $u(s,t) \in \mathbb{R}^m$ together with $A_0(s,t), A_1(s,t), A_2(s,t) \in \mathbb{R}^{n\times n}$, $B(s,t) \in \mathbb{R}^{n\times m}$, A_0 and B being assumed to be piecewise continuous on some interval $\mathcal{I} := [S_0, S_1] \times [T_0, T_1]$, while A_1, A_2 are assumed to be continuously differentiable on the same interval. Hereby we call a function in $\mathcal{I}$ piecewise continuous if there exists a rectangular subdivision $R_1, \dots, R_N$ of $\mathcal{I}$ such that the restriction of the function to the open rectangles $\overset{\circ}{R}_i$, $i = 1, \dots, N$ has a continuous extension to their closure. This linear 2-D system is considered together with the initial-boundary or Goursat conditions

$$\begin{aligned} x(s,t_0) &= x_1(s) \in \mathbb{R}^n, \quad (s,t_0) \in \mathcal{I}, \\ x(s_0,t) &= x_2(t) \in \mathbb{R}^n, \quad (s_0,t) \in \mathcal{I}, \\ x_1(s_0) &= x_2(t_0), \end{aligned} \tag{2}$$

where x_1, x_2 are piecewise continuously differentiable.

For applications in image processing, see, e.g., (Jain and Jain, 1977; 1978). For this type of control problems there has also been developed a Pontryagin maximum principle (see, e.g., (Wolfersdorf, 1978) and the references therein) for applications in optimization of quasistationary chemical reactors.

The main aim of this paper is to obtain conditions on the parameters of the system (1), (2) for unconstrained controllability. Therefore, analogously to (Sontag, 1998, p. 83), we start with the following definition:

Definition 1. The system (1), (2), is said to be *(completely) controllable* in a given interval $\mathcal{J} = [s_0, s_1] \times [t_0, t_1]$ if for any initial-boundary condition (2) and any $x \in \mathbb{R}^n$ there exists a piecewise continuous function u in $\mathcal{J}$, $u(s,t) \in \mathbb{R}^m$, such that $x(s_1, t_1) = x$.

There are several papers concerning the controllability of systems of type (1), (2). Here we only make a reference to (Bergmann *et al.*, 1989; Pulvirenti and Santagati, 1975; Kaczorek, 1995; 1996). In (Pulvirenti and Santagati, 1975) the scalar case is treated, in (Bergmann *et al.*, 1989) and (Kaczorek, 1995) the case of constant coefficients is studied. In (Kaczorek, 1996) the general case with non-constant coefficients is treated and a necessary and sufficient controllability criterion is obtained by demanding the so-called Gramian matrix to be positive definite (see Theorem 4). In the present paper we formulate controllability conditions for the case of non-constant coefficients by making use of the solutions of the adjoint equation, which are similar to the one-dimensional situation. This then also gives rise to observability results. Besides, we also study control by Goursat data and solve an optimal control problem. The controllability problem for the system (1), (2) under certain restrictions on the steering function u is treated in (Gyurkovics and Jank, 2001).

In order to find conditions for controllability and observability as in the classical one-dimensional case, we recall a representation formula for solutions of (1), (2) obtained with the *matrix Riemann function* for (1). This formula yields an operator, mapping any admissible function u to a solution x of (1) and (2).

Although this representation formula is used, e.g., in (Bergmann *et al.*, 1989; Kaczorek, 1995; 1996), a proof is available only for the scalar case in (Pulvirenti and Santagati, 1975). In principle, one can deduce the desired representation also from (Vekua, 1967, p. 15). However, the presentation there is oriented towards the representation of solutions of elliptic differential equations with analytic coefficients using a complex transformation into a formal hyperbolic system.

Since we consider neither analytic coefficients nor elliptic equations, for the reader's convenience we shortly recall that representation theory, which is based on a method introduced by Riemann. Readers not interested in the construction of solutions to (1), (2) can directly start with Theorem 3.

The paper is organized as follows: After the introductory section, in Section 2 we briefly recall the representation theory for solutions of equation (1) using the matrix Riemann kernel function. Then in Section 3 we obtain controllability and observability results for systems with non-constant coefficients using solutions of the adjoint equation. In Sections 4 and 5 we briefly address the issue of control of the system by initial boundary values and optimal control, respectively.

In the next section we introduce the Riemann kernel function for equations of type (1).

2. Riemann Kernel and a Representation Formula

Before introducing the Riemann kernel function, we prove a lemma concerning the solvability of an integral equation of Volterra type. Let us first define the set of matrix-valued functions

$$\mathcal{S}_{n\times k}(\mathcal{J}) := \Big\{U : \mathcal{J} \to \mathbb{R}^{n\times k} \mid U(s,t),\ \frac{\partial}{\partial s}U(s,t),\ \frac{\partial}{\partial t}U_t(s,t),\ \frac{\partial^2}{\partial s\partial t}U(s,t) \text{ are piecewise continuous in } \mathcal{J}\subset\mathcal{I}\Big\}.$$

Lemma 1. *Let $A(s,t,\xi,\eta)\in\mathcal{S}_{n\times n}(\mathcal{I})$. Then the following integral equation of Volterra type:*

$$R_0(s,t,\sigma,\tau) - \int_\sigma^s\int_\tau^t R_0(\xi,\eta,\sigma,\tau)A(s,t,\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = I_n \quad (3)$$

has a unique continuous solution such that $\frac{\partial^2}{\partial s\partial t}R_0(s,t,\sigma,\tau)$ is piecewise continuous and $\frac{\partial}{\partial s}R_0$ and $\frac{\partial}{\partial t}R_0$ are continuous in $\mathcal{I}\times\mathcal{I}$.

Proof. The operator T defined by

$$(TF)(s,t,\sigma,\tau) := \int_\sigma^s\int_\tau^t F(\xi,\eta)A(s,t,\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

maps any matrix-valued function $F(\xi,\eta)\in\mathbb{R}^{n\times n}$ continuous in $\mathcal{I}$ to $\mathbb{R}^{n\times n}$-valued functions continuous in $\mathcal{I}\times\mathcal{I}$. There exists a constant $C>0$ such that

$$\begin{aligned}\|(TF)(s,t,\sigma,\tau)\| &\le \max_{(s,t,\xi,\eta)\in\mathcal{I}\times\mathcal{I}}\|A(s,t,\xi,\eta)\| \\ &\quad\times\max_{(\xi,\eta)\in\mathcal{I}}\|F(\xi,\eta)\|\,|s-\sigma|\,|t-\tau| \\ &\le C|s-\sigma|\,|t-\tau|\max_{(\xi,\eta)\in\mathcal{I}}\|F(\xi,\eta)\|, \end{aligned} \quad (4)$$

and for $k=2,3,\dots$ we have

$$\|(T^kF)(s,t,\sigma,\tau)\| \le \frac{C^k}{(k!)^2}|s-\sigma|^k|t-\tau|^k\max_{(\xi,\eta)\in\mathcal{I}}\|F(\xi,\eta)\|.$$

Equation (3) can now be written as

$$R_0 = I_n + TR_0. \quad (5)$$

Then it follows from the Picard iteration (Arnol'd, 1980, p. 212), setting $R_0^0=I_n,\ R_0^k=I_n+TR_0^{k-1},\quad k=1,2,\dots,$

$$R_0^l=\sum_{k=0}^l T^kI_n, \quad (6)$$

where $T^0=\mathrm{id}$, and for $k=1,2,\dots: T^k=TT^{k-1}$. Here id denotes the identity mapping the space of matrix-valued functions $F(\xi,\eta)\in\mathbb{R}^{n\times n}$ continuous in $\mathcal{I}$ to $\mathbb{R}^{n\times n}$-valued functions continuous in $\mathcal{I}\times\mathcal{I}$.

For $\varepsilon>0$ and m,l sufficiently large, $m>l$, together with (4), (6) we obtain the estimate

$$\|R_0^m-R_0^l\| = \Big\|\sum_{k=l+1}^m T^kI_n\Big\| \le \sum_{k=l+1}^m C^k\frac{|s-\sigma|^k|t-\tau|^k}{(k!)^2} < \varepsilon.$$

Hence we infer the uniform convergence of R_0^l towards R_0 and that R_0 is continuous on $\mathcal{I}\times\mathcal{I}$ and solves (3).

The uniqueness can also be concluded in the usual way, since any difference Δ of two solutions of equation (3) solves the homogenous equation $\Delta=T\Delta$, and hence also $\Delta = T^k\Delta, k = 2, 3, \ldots$. From this, together with (4), we see that $\Delta = 0$. Furthermore, from (5), (6) we obtain

$$\begin{aligned}\frac{\partial^2}{\partial s\partial t} & R_0^l(s,t,\sigma,\tau) \\ &= R_0^{l-1}(s,t,\sigma,\tau)A(s,t,s,t) \\ &\quad + \int_\tau^t R_0^{l-1}(s,\eta,\sigma,\tau)\frac{\partial}{\partial t}A(s,t,s,\eta)\,\mathrm{d}\eta \\ &\quad + \int_\sigma^s R_0^{l-1}(\xi,t,\sigma,\tau)\frac{\partial}{\partial s}A(s,t,\xi,t)\,\mathrm{d}\xi \\ &\quad + \int_\sigma^s\int_\tau^t R_0^{l-1}(\xi,\eta,\sigma,\tau)\frac{\partial^2}{\partial s\partial t}A(s,t,\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.\end{aligned}$$

Taking the limit as $l \to \infty$, we infer that $\frac{\partial^2}{\partial s\partial t}R_0$ is also piecewise continuous in $\mathcal{I}\times\mathcal{I}$. In a similar way, it can be seen that $\frac{\partial}{\partial s}R_0$ and $\frac{\partial}{\partial t}R_0$ are continuous. ■

Now we are ready to introduce the Riemann kernel function.

Theorem 1. (Riemann kernel function) *Let $A_0(s,t) \in \mathbb{R}^{n\times n}$ be piecewise continuous and $A_1(s,t)$, $A_2(s,t) \in \mathbb{R}^{n\times n}$ be continuously differentiable on $\mathcal{I}$. Then the following integral equation of Volterra type:*

$$\begin{aligned}R(s,t,\sigma,\tau) &+ \int_\sigma^s R(\xi,t,\sigma,\tau)A_2(\xi,t)\,\mathrm{d}\xi \\ &+ \int_\tau^t R(s,\eta,\sigma,\tau)A_1(s,\eta)\,\mathrm{d}\eta \\ &- \int_\sigma^s\int_\tau^t R(\xi,\eta,\sigma,\tau)A_0(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = I_n \quad (7)\end{aligned}$$

has a unique continuous solution $R(s,t,\sigma,\tau) \in \mathbb{R}^{n\times n}$ such that $\frac{\partial^2}{\partial s\partial t}R(s,t,\sigma,\tau)$ is piecewise continuous and $\frac{\partial}{\partial s}R(s,t,\sigma,\tau)$, as well as $\frac{\partial}{\partial t}R(s,t,\sigma,\tau)$, is continuous in $\mathcal{I}\times\mathcal{I}$.

This matrix-valued function $R(s,t,\sigma,\tau) \in \mathbb{R}^{n\times n}$ is called the *matrix Riemann function* or the *matrix Riemann kernel* of the equation

$$\frac{\partial^2}{\partial s\partial t}x - A_1\frac{\partial}{\partial s}x - A_2\frac{\partial}{\partial t}x - A_0x = 0. \quad (8)$$

Proof of Theorem 1. Iterating equation (7) in a similar way as we have done with (3) yields a sequence of matrix-valued functions $R^{(i)}(s,t,\sigma,\tau)$, but it seems to require an enormous effort to get appropriate estimates in order to obtain convergence. Therefore we follow the way proposed in (Vekua, 1967). We introduce the following integral equations:

$$\begin{aligned}R_1(s,t,\xi) &= A_2(\xi,t) - \int_\xi^s R_1(\xi_1,t,\xi)A_2(\xi_1,t)\,\mathrm{d}\xi_1, \\ R_2(s,t,\eta) &= A_1(s,\eta) - \int_\eta^t R_2(s,\eta_1,\eta)A_1(s,\eta_1)\,\mathrm{d}\eta_1.\end{aligned} \quad (9)$$

Defining the integral operators

$$(T_1R_1)(s,t,\xi) := -\int_\xi^s R_1(\xi_1,t,\xi)A_2(\xi_1,t)\,\mathrm{d}\xi_1$$

and

$$(T_2R_2)(s,t,\eta) := -\int_\eta^t R_2(s,\eta_1,\eta)A_1(s,\eta_1)\,\mathrm{d}\eta_1,$$

which map continuously differentiable functions into continuously differentiable functions, equations (9) can now formally be written as $R_1 = A_2 + T_1R_1$ and $R_2 = A_1 + T_2R_2$.

Iterating (9) in a similar way as we did before, i.e. calculating successively $R_1^0 = A_2$, $R_1^l = A_2 + TR_1^{l-1}$, $R_2^0 = A_1$, $R_2^l = A_1 + TR_2^{l-1}$, $l = 1, 2, \ldots$, we obtain

$$\begin{aligned}R_1(s,t,\xi) &= \sum_{j=0}^{\infty}(T_1^jA_2)(s,t,\xi), \\ R_2(s,t,\eta) &= \sum_{j=0}^{\infty}(T_2^jA_1)(s,t,\eta).\end{aligned} \quad (10)$$

The uniform convergence is obvious, since, as has been done before in the proof of Lemma 1, we have performed a simple Picard-iteration procedure, and hence R_1, R_2 are continuously differentiable functions and the unique solutions of (9). With these solutions we now determine a matrix-valued function $R_0(s,t,\sigma,\tau)$ such that

$$\begin{aligned}R(s,t,\sigma,\tau) &= R_0(s,t,\sigma,\tau) \\ &\quad - \int_\sigma^s R_0(\xi,t,\sigma,\tau)R_1(s,t,\xi)\,\mathrm{d}\xi \\ &\quad - \int_\tau^t R_0(s,\eta,\sigma,\tau)R_2(s,t,\eta)\,\mathrm{d}\eta. \quad (11)\end{aligned}$$

Inserting (11) into (7), while suppressing σ and τ, yields

$$\begin{aligned}I_n &= R_0(s,t) - \int_\sigma^s R_0(\xi,t)R_1(s,t,\xi)\,\mathrm{d}\xi \\ &\quad - \int_\tau^t R_0(s,\eta)R_2(s,t,\eta)\,\mathrm{d}\eta \\ &\quad + \int_\sigma^s \Big[R_0(\xi,t) - \int_\sigma^\xi R_0(\xi_1,t)R_1(\xi,t,\xi_1)\,\mathrm{d}\xi_1 \\ &\quad - \int_\tau^t R_0(\xi,\eta)R_2(\xi,t,\eta)\,\mathrm{d}\eta\Big]A_2(\xi,t)\,\mathrm{d}\xi \\ &\quad + \int_\tau^t \Big[R_0(s,\eta) - \int_\sigma^s R_0(\xi,\eta)R_1(s,\eta,\xi)\,\mathrm{d}\xi \\ &\quad - \int_\tau^\eta R_0(s,\eta_1)R_2(s,\eta,\eta_1)\,\mathrm{d}\eta_1\Big]A_1(s,\eta)\,\mathrm{d}\eta \\ &\quad - \int_\sigma^s\int_\tau^t \Big[R_0(\xi,\eta) - \int_\sigma^\xi R_0(\xi_1,\eta)R_1(\xi,\eta,\xi_1)\,\mathrm{d}\xi_1 \\ &\quad - \int_\tau^\eta R_0(\xi,\eta_1)R_2(\xi,\eta,\eta_1)\,\mathrm{d}\eta_1\Big]A_0(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.\end{aligned}$$

Together with (9) and (10), we then obtain, after a short calculation,

$$\begin{aligned}I_n &= R_0(s,t) - \int_\sigma^s\int_\tau^t R_0(\xi,\eta)R_2(\xi,t,\eta)A_2(\xi,t)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\quad - \int_\sigma^s\int_\tau^t R_0(\xi,\eta)R_1(s,\eta,\xi)A_1(s,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\quad - \int_\sigma^s\int_\tau^t \Big[R_0(\xi,\eta) - \int_\sigma^\xi R_0(\xi_1,\eta)R_1(\xi,\eta,\xi_1)\,\mathrm{d}\xi_1 \\ &\quad - \int_\tau^\eta R_0(\xi,\eta_1)R_2(\xi,\eta,\eta_1)\,\mathrm{d}\eta_1\Big]A_0(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.\end{aligned}$$

Interchanging the order of integration in the last two lines we finally obtain the following integral equation of Volterra type for R_0:

$$\begin{aligned}&R_0(s,t,\sigma,\tau) \\ &\quad - \int_\sigma^s\int_\tau^t R_0(\xi,\eta,\sigma,\tau)A(s,t,\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = I_n, \quad (12)\end{aligned}$$

where

$$\begin{aligned}&A(s,t,\xi,\eta) \\ &\quad := R_2(\xi,t,\eta)A_2(\xi,t) + R_1(s,\eta,\xi)A_1(s,\eta) \\ &\qquad + A_0(\xi,\eta) - \int_\xi^s R_1(\xi_1,\eta,\xi)A_0(\xi_1,\eta)\,\mathrm{d}\xi_1 \\ &\qquad - \int_\eta^t R_2(\xi,\eta_1,\eta)A_0(\xi,\eta_1)\,\mathrm{d}\eta_1\end{aligned}$$

and, together with $\frac{\partial^2}{\partial s\partial t}A = 0$, we have $A \in \mathcal{S}_{n\times n}(\mathcal{I})$. In order to establish Theorem 1, we need the existence and uniqueness of the solution of this latter integral equation. Setting A in (3) as defined in (12) and using Lemma 1, we infer that R as defined in (11) is a continuous solution of (7). Since R_0, R_1, and R_2 are unique, we also obtain the uniqueness of R. Moreover, since R_1 and R_2 are continuously differentiable, $\frac{\partial}{\partial s}R_0$ and $\frac{\partial}{\partial t}R_0$ are continuous and $\frac{\partial^2}{\partial s\partial t}R_0$ is piecewise continuous, we infer from (11) that also $\frac{\partial^2}{\partial\sigma\partial\tau}R$ is piecewise continuous. ■

The next step now is to prove some important properties of the matrix Riemann function.

Theorem 2. *The matrix Riemann function is a solution of the differential equation*

$$\begin{aligned}&\frac{\partial^2}{\partial s\partial t}R(s,t,\sigma,\tau) + \frac{\partial}{\partial s}\big(R(s,t,\sigma,\tau)A_1(s,t)\big) \\ &\qquad + \frac{\partial}{\partial t}\big(R(s,t,\sigma,\tau)A_2(s,t)\big) \\ &\qquad - R(s,t,\sigma,\tau)A_0(s,t) = 0, \quad (13)\end{aligned}$$

with $\det R(s,t,s,\tau) \neq 0$, $\det R(s,t,\sigma,t) \neq 0$ *and*

i) $\frac{\partial}{\partial s}R(s,t,\sigma,t) + R(s,t,\sigma,t)A_2(s,t) = 0$,

ii) $\frac{\partial}{\partial t}R_t(s,t,s,\tau) + R(s,t,s,\tau)A_1(s,t) = 0$,

iii) $\frac{\partial}{\partial \xi}R(s,t,\xi,t) - A_2(\xi,t)R(s,t,\xi,t) = 0$,

iv) $\frac{\partial}{\partial \eta}R(s,t,s,\eta) - A_1(s,\eta)R(s,t,s,\eta) = 0$,

v) $R(s,t,s,t) = I_n$.

Proof. That R is a solution of the *adjoint* differential equation (13) results from differentiating (7). Item (i) follows by setting $t = \tau$ in (7) and differentiating the result with respect to s. Analogously, we obtain (ii) and (v).

With the well-known Abel-Jacobi-Liouville formula (Gantmacher, 1986, p. 470), we obtain from, e.g., (ii) and (v)

$$\det R(s,t,s,\tau) = e^{\int_\tau^t \mathrm{trace}\,\left(A_1(s,\eta)\right)\,\mathrm{d}\eta},$$

which yields the desired property. Analogously, from (i) and (v) we obtain the second determinant. The remaining two items require some more effort. For any continuous matrix-valued function $X(s,t) \in \mathbb{R}^{n\times n}$, differentiable with respect to s for all $t \in \mathcal{I}$, together with (ii) we deduce that

$$\frac{\partial}{\partial t}\big(R(s,t,s,\tau)X(s,t)\big) - R(s,t,s,\tau)\Big(\frac{\partial}{\partial t}X(s,t) - A_1(s,t)X(s,t)\Big) = \Big(\frac{\partial}{\partial t}R(s,t,s,\tau) + R(s,t,s,\tau)A_1(s,t)\Big)X(s,t) = 0.$$

Replacing therein t by η and integrating the result with respect to η from τ to t together with (v) yields

$$R(s,t,s,\tau)X(s,t) - I_n X(s,\tau) = \int_\tau^t R(s,\eta,s,\tau)\Big[\frac{\partial}{\partial \eta}X(s,\eta) - A_1(s,\eta)X(s,\eta)\Big]\,\mathrm{d}\eta.$$

Setting now $X(s,\tau) := R(s,t,s,\tau)$ and applying again (v) yields

$$0 = \int_\tau^t R(s,\eta,s,\tau) \times \Big[\frac{\partial}{\partial \eta}R(s,t,s,\eta) - A_1(s,\eta)R(s,t,s,\eta)\Big]\mathrm{d}\eta.$$

Since $t,\tau \in \mathcal{I}$ are arbitrary, we infer that necessarily

$$R(s,\eta,s,\tau)\Big[\frac{\partial}{\partial \eta}R(s,t,s,\eta) - A_1(s,\eta)R(s,t,s,\eta)\Big] = 0$$

for all $\eta \in \mathcal{I}$. Since $R(s,t,s,\eta)$ is invertible, we obtain (iv). Analogously, we get (iii). ■

Having introduced the matrix Riemann function, we can now use it to obtain a general representation formula for all solutions to (1).

We start deriving an important identity.

Lemma 2. *Let $U(s,t) \in \mathcal{S}_{n\times k}(\mathcal{I})$ and let the matrices A_0, A_1, A_2 be defined as before. Then with*

$$F(U) := \frac{\partial^2}{\partial s \partial t}U - A_1\frac{\partial}{\partial s}U - A_2\frac{\partial}{\partial t}U - A_0 U$$

and R as the matrix Riemann function, we obtain the identity

$$\frac{\partial^2}{\partial s \partial t}(RU) - RF(U) = \frac{\partial}{\partial s}\Big[\Big(\frac{\partial}{\partial t}R + RA_1\Big)U\Big] + \frac{\partial}{\partial t}\Big[\Big(\frac{\partial}{\partial s}R + RA_2\Big)U\Big]. \tag{14}$$

Proof. It is easy to check that the left-hand side, together with (13) and the definition of $F(U)$, yields

$$\frac{\partial^2}{\partial s \partial t}(RU) - RF(U) = 2\frac{\partial^2}{\partial s \partial t}RU + \frac{\partial}{\partial s}R\frac{\partial}{\partial t}U + \frac{\partial}{\partial t}R\frac{\partial}{\partial s}U + RA_1\frac{\partial}{\partial s}U + RA_2\frac{\partial}{\partial t}U + \Big(\frac{\partial}{\partial t}(RA_2)\Big)U + \Big(\frac{\partial}{\partial s}(RA_1)\Big)U,$$

and this exactly equals the term on the right-hand side of (14). ■

The next identity yields an integrated version of (14).

Lemma 3. *Let $U(\sigma,\tau) \in \mathcal{S}_{n\times k}(\mathcal{J})$ and let $R(s,t,\sigma,\tau)$ be the matrix Riemann kernel of the differential equation (8). Then we obtain the identity*

$$\begin{aligned} U(s,t) &= R(s_0,t_0,s,t)U(s_0,t_0) \\ &+ \int_{t_0}^t R(s_0,\tau,s,t)\Big[\frac{\partial}{\partial \tau}U(s_0,\tau) - A_1(s_0,\tau)U(s_0,\tau)\Big]\mathrm{d}\tau \\ &+ \int_{s_0}^s R(\sigma,t_0,s,t)\Big[\frac{\partial}{\partial \sigma}U_\sigma(\sigma,t_0) - A_2(\sigma,t_0)U(\sigma,t_0)\Big]\mathrm{d}\sigma \\ &+ \int_{s_0}^s\int_{t_0}^t R(\sigma,\tau,s,t)F\big(U(\sigma,\tau)\big)\,\mathrm{d}\sigma\,\mathrm{d}\tau, \end{aligned} \tag{15}$$

where $(s_0,t_0) \in \mathcal{J}$.

Proof. Interchanging the first pair of variables (s,t) with the second pair (σ,τ) and integrating the identity (14) from s_0 to s with respect to σ and also from t_0 to t with respect to τ yields for the left-hand side of (14)

$$\begin{aligned} &\int_{s_0}^s\int_{t_0}^t \frac{\partial}{\partial \sigma \partial \tau}(RU)\,\mathrm{d}\sigma\,\mathrm{d}\tau - \int_{s_0}^s\int_{t_0}^t R(\sigma,\tau,s,t)F\big(U(\sigma,\tau)\big)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &= \int_{s_0}^s \frac{\partial}{\partial \sigma}\big(R(\sigma,t,s,t)U(\sigma,t)\big)\,\mathrm{d}\sigma - \int_{s_0}^s \frac{\partial}{\partial \sigma}\big(R(\sigma,t_0,s,t)U(\sigma,t_0)\big)\,\mathrm{d}\sigma \\ &\quad - \int_{s_0}^s\int_{t_0}^t R(\sigma,\tau,s,t)F\big(U(\sigma,\tau)\big)\,\mathrm{d}\sigma\,\mathrm{d}\tau \end{aligned}$$

$$= R(s,t,s,t)U(s,t) - R(s_0,t,s,t)U(s_0,t)$$
$$-R(s,t_0,s,t)U(s,t_0) + R(s_0,t_0,s,t)U(s_0,t_0)$$
$$-\int_{s_0}^{s}\int_{t_0}^{t} R(\sigma,\tau,s,t)F\big(U(\sigma,\tau)\big)\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

For the right-hand side, we obtain

$$\int_{t_0}^{t}\Big(\frac{\partial}{\partial t}R(s,\tau,s,t) + R(s,\tau,s,t)A_1(s,\tau)\Big)U(s,\tau)\,\mathrm{d}\tau$$
$$-\int_{t_0}^{t}\Big(\frac{\partial}{\partial t}R(s_0,\tau,s,t)+R(s_0,\tau,s,t)A_1(s_0,\tau)\Big)U(s_0,\tau)\mathrm{d}\tau$$
$$+\int_{s_0}^{s}\Big(\frac{\partial}{\partial s}R(\sigma,t,s,t)+R(\sigma,t,s,t)A_2(\sigma,t)\Big)U(\sigma,t)\,\mathrm{d}\sigma$$
$$-\int_{s_0}^{s}\Big(\frac{\partial}{\partial s}R(\sigma,t_0,s,t)+R(\sigma,t_0,s,t)A_2(\sigma,t_0)\Big)U(\sigma,t_0)\mathrm{d}\sigma.$$

Using now the properties of the Riemann kernel as stated in Theorem 2, we obtain

$$U(s,t) - R(s_0,t,s,t)U(0,t) - R(s,t_0,s,t)U(s,t_0)$$
$$+ R(s_0,t_0,s,t)U(s_0,t_0)$$
$$-\int_{s_0}^{s}\int_{t_0}^{t} R(\sigma,\tau,s,t)F\big(U(\sigma,\tau)\big)\,\mathrm{d}\sigma\,\mathrm{d}\tau$$
$$= -\int_{t_0}^{t}\frac{\partial}{\partial t}R(s_0,\tau,s,t)U(s_0,\tau)\,\mathrm{d}\tau$$
$$-\int_{t_0}^{t}R(s_0,\tau,s,t)A_1(s_0,\tau)U(s_0,\tau)\,\mathrm{d}\tau$$
$$-\int_{s_0}^{s}\frac{\partial}{\partial s}R(\sigma,t_0,s,t)U(\sigma,t_0)\,\mathrm{d}\sigma$$
$$-\int_{s_0}^{s}R(\sigma,t_0,s,t)A_2(\sigma,t_0)U(\sigma,t_0)\,\mathrm{d}\sigma$$
$$= -R(s_0,\tau,s,t)U(s_0,\tau)\big|_{\tau=t_0}^{t}$$
$$+\int_{t_0}^{t}R(s_0,\tau,s,t)\Big(\frac{\partial}{\partial\tau}U(s_0,\tau)-A_1(s_0,\tau)U(s_0,\tau)\Big)\mathrm{d}\tau$$
$$- R(\sigma,t_0,s,t)U(\sigma,t_0)\big|_{\sigma=s_0}^{s}$$
$$+\int_{s_0}^{s}R(\sigma,t_0,s,t)\Big(\frac{\partial}{\partial\sigma}U(\sigma,t_0)-A_2(\sigma,t_0)U(\sigma,t_0)\Big)\mathrm{d}\sigma.$$

This immediately yields the desired identity (15). ■

Remark 1. Notice that from Lemma 3 we can conclude that the matrix Riemann function with respect to the second pair of variables is a solution of the homogeneous equation (1) in $\mathcal{I}$, i.e.

$$\frac{\partial^2}{\partial\sigma\partial\tau}R(s,t,\sigma,\tau)$$
$$= A_0(\sigma,\tau)R(s,t,\sigma,\tau) + A_1(\sigma,\tau)\frac{\partial}{\partial\sigma}R(s,t,\sigma,\tau)$$
$$+ A_2(\sigma,\tau)\frac{\partial}{\partial\tau}R(s,t,\sigma,\tau).$$

Proof. Calculating $\frac{\partial^2}{\partial\sigma\partial\tau}R$, $\frac{\partial}{\partial\sigma}R$, $\frac{\partial}{\partial\tau}R$ from (7) and defining

$$\varphi(s,t,\sigma,\tau)$$
$$:= \frac{\partial^2}{\partial\sigma\partial\tau}R(s,t,\sigma,\tau) - A_1(\sigma,\tau)\frac{\partial}{\partial\sigma}R(s,t,\sigma,\tau)$$
$$- A_2(\sigma,\tau)\frac{\partial}{\partial\tau}R(s,t,\sigma,\tau) - A_0(\sigma,\tau)R(s,t,\sigma,\tau)$$

yields, together with Theorem 2, (iii), (iv) and (v),

$$\varphi(s,t,\sigma,\tau) = -\int_{\sigma}^{s}\varphi(\xi,t,\sigma,\tau)A_2(\xi,t)\,\mathrm{d}\xi$$
$$-\int_{\tau}^{t}\varphi(s,\eta,\sigma,\tau)A_1(s,\eta)\,\mathrm{d}\eta$$
$$+\int_{\sigma}^{s}\int_{\tau}^{t}\varphi(\xi,\eta,\sigma,\tau)A_0(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

With R_1, R_2 and (10), we perform again the transformation (11):

$$\varphi(s,t,\sigma,\tau) = \varphi_0(s,t,\sigma,\tau)$$
$$-\int_{\sigma}^{s}\varphi_0(\xi,t,\sigma,\tau)R_1(s,t,\xi)\,\mathrm{d}\xi$$
$$-\int_{\tau}^{t}\varphi_0(s,\eta,\sigma,\tau)R_2(s,t,\eta)\,\mathrm{d}\eta,$$

which yields the integral equation for φ_0:

$$\varphi_0(s,t,\sigma,\tau) = (T\varphi_0)(s,t,\sigma,\tau).$$

By iterating this equation we obtain $\varphi_0(s,t,\sigma,\tau) = (T^n\varphi_0)(s,t,\sigma,\tau)$ for all $n\in\mathbb{N}$. Using the estimate (4) for T, we see that $\varphi_0 = 0$ and hence $\varphi = 0$. ■

Now we can obtain a representation formula in much the same way as for the one-dimensional continuous-time case. This formula will then also enable us to derive similar controllability criteria.

Theorem 3. *(i) Let u be a piecewise continuous function, $u(s,t) \in \mathbb{R}^m$, and let $x \in \mathcal{S}_{n\times 1}(\mathcal{J})$ be a solution of (1) in $\mathcal{J}$. Then*

$$\begin{aligned} x(s,t) &= R(s_0,t_0,s,t)x_1(s_0) \\ &\quad + \int_{s_0}^{s} R(\sigma,t_0,s,t)\big(x_1'(\sigma) - A_2(\sigma,t_0)x_1(\sigma)\big)\,\mathrm{d}\sigma \\ &\quad + \int_{t_0}^{t} R(s_0,\tau,s,t)\big(x_2'(\tau) - A_1(s_0,\tau)x_2(\tau)\big)\,\mathrm{d}\tau \\ &\quad + \int_{s_0}^{s}\int_{t_0}^{t} R(\sigma,\tau,s,t)B(\sigma,\tau)u(\sigma,\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau, \end{aligned} \tag{16}$$

where $(s_0,t_0) \in \mathcal{J}$, $x_1(\sigma) := x(\sigma,t_0)$, $x_2(\tau) := x(s_0,\tau)$.

(ii) For any piecewise continuously differentiable functions x_1 (resp. x_2) in $[s_0,s_1]$ (resp. in $[t_0,t_1]$), with $x_1(s_0) = x_2(t_0)$, and a piecewise continuous function u, $u(s,t) \in \mathbb{R}^m$, in $\mathcal{J}$, x in (16) is a solution of the differential equation (1) (i.e. $x \in \mathcal{S}_{n\times 1}(\mathcal{J})$ and fulfils (1) a.e.), with the initial-boundary values (2).

Proof. If $x \in \mathcal{S}_{n\times 1}(\mathcal{J})$ is a solution of (1), then from the identity (15), setting $F(x) = Bu$, we infer the representation (16).

To show (ii), we first prove by direct computation that x as represented by (16) is in $\mathcal{S}_{n\times 1}$ and fulfils (1). There we have to use Remark 1 and properties (iii), (iv) and (v) of Theorem 2.

It remains to prove that $x(s,t)$ also has the desired boundary values. From (16) we obtain

$$\begin{aligned} x(s_0,t) &= R(s_0,t_0,s_0,t)x_1(s_0) \\ &\quad + \int_{t_0}^{t} R(s_0,\tau,s_0,t)\big[x_2'(\tau) - A_1(s_0,\tau)x_2(\tau)\big]\,\mathrm{d}\tau, \end{aligned}$$

which yields, after partial integration,

$$\begin{aligned} x(s_0,t) &= R(s_0,t_0,s_0,t)x_1(s_0) + R(s_0,\tau,s_0,t)x_2(\tau)\big|_{t_0}^{t} \\ &\quad - \int_{t_0}^{t}\Big[\frac{\partial}{\partial\tau}R(s_0,\tau,s_0,t) \\ &\quad + R(s_0,\tau,s_0,t)A_1(s_0,\tau)\Big]x_2(\tau)\,\mathrm{d}\tau. \end{aligned}$$

Using then (ii) and (v), from Theorem 2 we finally get $x(s_0,t) = x_2(t)$. Analogously, we obtain $x(s,t_0) = x_1(s)$. ■

Notice that (16) remains true if u is in the space of square integrable functions $\mathcal{L}_2^n(\mathcal{J})$ and x, x_1, x_2 are in some appropriate Sobolev space, since the representation operator is continuous. We used piecewise continuous functions having in mind only technical applications.

In some particular situations it is possible to apply a simple transformation of (1) in order to obtain a simpler form.

Remark 2. Let $A_1(s,t), A_2(s,t) \in \mathbb{R}^{n\times n}$ be piecewise continuously differentiable on $\mathcal{I}$ such that the integrability conditions

$$\begin{aligned} \frac{\partial A_1}{\partial s}(s,t) &= \frac{\partial A_2}{\partial t}(s,t), \\ A_1(s,t)A_2(\sigma,t) &= A_2(\sigma,t)A_1(s,t) \end{aligned} \tag{17}$$

hold for all $\sigma, s, t \in [S_0,S_1]$, $t \in [T_0,T_1]$.

If $V(s,t,s_0,t_0)$, $(s,t,s_0,t_0) \in \mathcal{I}$ is the solution of

$$\begin{aligned} \frac{\partial}{\partial s}V &= A_2(s,t)V, \\ \frac{\partial}{\partial t}V &= A_1(s,t)V \end{aligned} \tag{18}$$

with $V(s_0,t_0,s_0,t_0) = I_n$, then by the transformation

$$x = Vy \tag{19}$$

eqn. (1) with boundary values (2) is equivalent to

$$\begin{aligned} &\frac{\partial^2}{\partial s\partial t}y + V^{-1}(s,t)\Big(\frac{\partial}{\partial t}A_2(s,t) - A_2(s,t)A_1(s,t) \\ &\quad - A_0(s,t)\Big)V(s,t)y = V^{-1}(s,t)B(s,t)u(s,t) \end{aligned} \tag{20}$$

and

$$\begin{aligned} y(s,t_0) &= V^{-1}(s,t_0)x_1(s), \\ y(s_0,t) &= V^{-1}(s_0,t)x_2(t), \\ y(s_0,t_0) &= x_1(s_0) = x_2(t_0). \end{aligned} \tag{21}$$

Proof. First, notice that the required solution of (18) under the assumption of (17) can be written as

$$\begin{aligned} &V(s,t,s_0,t_0) \\ &\quad = \exp\Big(\int_{s_0}^{s}A_2(\sigma,t)\,\mathrm{d}\sigma\Big)\exp\Big(\int_{t_0}^{t}A_1(s_0,\tau)\,\mathrm{d}\tau\Big). \end{aligned} \tag{22}$$

Inserting (19) into (1) together with (17) and (18), we obtain

$$\begin{aligned} &V\frac{\partial^2}{\partial s\partial t}y + \Big(\frac{\partial}{\partial t}A_2 - A_2A_1 - A_0\Big)Vy \\ &\quad = V\frac{\partial^2}{\partial s\partial t}y + \Big(\frac{\partial}{\partial s}A_1 - A_1A_2 - A_0\Big)Vy = Bu, \end{aligned}$$

and hence (20). Clearly, from (19) we have (21). Since $V(s,t) \in \mathbb{R}^{n\times n}$ is regular in $\mathcal{I}$, we conclude the equivalence.

3. Controllability and Observability

The first controllability condition can now be obtained similarly to the one-dimensional case by using the representation formula (16).

Theorem 4. *Let A_0 be piecewise continuous and $A_1,\ A_2$ be continuously differentiable in $\mathcal{I}$. Let furthermore R denote the matrix Riemann function of (8). The system (1) together with (2) is completely controllable in $\mathcal{J}$ if and only if*

$$\begin{aligned} W &= W(s_0,t_0,s_1,t_1) \\ &:= \int_{s_0}^{s_1}\int_{t_0}^{t_1} R(\sigma,\tau,s_1,t_1)B(\sigma,\tau)B^T(\sigma,\tau) \\ &\qquad \times R^T(\sigma,\tau,s_1,t_1)\,\mathrm{d}\sigma\,\mathrm{d}\tau > 0. \end{aligned} \tag{23}$$

Proof. From the representation (16) we conclude that for the control

$$u(\sigma,\tau) := B^T(\sigma,\tau)R^T(\sigma,\tau,s_1,t_1)z, \quad z\in\mathbb{R}^n$$

we have

$$\begin{aligned} x(s_1,t_1) &= R(s_0,t_0,s_1,t_1)x_1(s_0) \\ &\quad + \int_{s_0}^{s_1} R(\sigma,t_0,s_1,t_1) \\ &\qquad \times \big(x_1'(\sigma) - A_2(\sigma,t_0)x_1(\sigma)\big)\,\mathrm{d}\sigma \\ &\quad + \int_{t_0}^{t_1} R(s_0,\tau,s_1,t_1) \\ &\qquad \times \big(x_2'(\tau) - A_1(s_0,\tau)x_2(\tau)\big)\,\mathrm{d}\tau + Wz. \end{aligned}$$

If $W>0$, then with

$$\begin{aligned} z &= W^{-1}\Big(x(s_1,t_1) - R(s_0,t_0,s_1,t_1)x_1(s_0) \\ &\quad - \int_{s_0}^{s_1} R(\sigma,t_0,s_1,t_1)\big(x_1'(\sigma) - A_2(\sigma,t_0)x_1(\sigma)\big)\,\mathrm{d}\sigma \\ &\quad - \int_{t_0}^{t_1} R(s_0,\tau,s_1,t_1)\big(x_2'(\tau) - A_1(s_0,\tau)x_2(\tau)\big)\,\mathrm{d}\tau\Big) \\ &\in \mathbb{R}^n \end{aligned}$$

we see that the control $u = B^TR^Tz$ steers the system from $x(s_0,t_0) = x_1(s_0) = x_2(t_0)$ to $x(s_1,t_1)$ for any given $x(s_1,t_1)$ and any boundary functions $x_1(s), x_2(t)$. Hence the system is completely controllable.

If, on the other hand, the system is supposed to be controllable, then for any given $x_1(s),\ x_2(t)$ and $x(s_1,t_1)\in\mathbb{R}^n$ or, equivalently, for any given $\tilde{x}\in\mathbb{R}^n$ with

$$\begin{aligned} \tilde{x} &= x(s_1,t_1) - R(s_0,t_0,s_1,t_1)x_1(s_0) \\ &\quad - \int_{s_0}^{s_1} R(\sigma,t_0,s_1,t_1)\big(x_1'(\sigma) - A_2(\sigma,t_0)x_1(\sigma)\big)\,\mathrm{d}\sigma \\ &\quad - \int_{t_0}^{t_1} R(s_0,\tau,s_1,t_1)\big(x_2'(\tau) - A_1(s_0,\tau)x_2(\tau)\big)\,\mathrm{d}\tau \end{aligned}$$

there exists an admissible control u steering the system to $x(s_1,t_1)$, which is then equivalent to

$$\tilde{x} = \int_{s_0}^{s_1}\int_{t_0}^{t_1} R(\sigma,\tau,s_1,t_1)B(\sigma,\tau)u(\sigma,\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

Since W is a symmetric and positive semi-definite matrix, all we have to prove is that W is regular or that W has a kernel containing only the zero vector.

If $\tilde{x}\neq 0$ were in the kernel of W (i.e. $W\tilde{x}=0$), then

$$\begin{aligned} &\int_{s_0}^{s_1}\int_{t_0}^{t_1} \tilde{x}^T R(\sigma,\tau,s_1,t_1)B(\sigma,\tau)B^T(\sigma,\tau) \\ &\qquad \times R^T(\sigma,\tau,s_1,t_1)\tilde{x}\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &\qquad\qquad = \int_{s_0}^{s_1}\int_{t_0}^{t_1} |B^TR^T\tilde{x}|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau = 0 \end{aligned}$$

and therefore $B^TR^T\tilde{x} = 0$ a.e.

From controllability and our consideration above we obtained a representation for $\tilde{x}$ using an appropriate control function u. Together with this definition of $\tilde{x}$, we get

$$|\tilde{x}|^2 = \tilde{x}^T\tilde{x} = \int_{s_0}^{s_1}\int_{t_0}^{t_1} \tilde{x}^TRBu\,\mathrm{d}\sigma\,\mathrm{d}\tau = 0,$$

a contradiction. This means $\operatorname{Im} W = \mathbb{R}^n$ and, since in general $W\geq 0$, we have $W>0$. ■

As a first example, we study the controllability of the system (1) in the case of constant coefficients.

Theorem 5. *(*Kalman controllability*) The system*

$$\frac{\partial^2}{\partial s\partial t}x = A_0x + Bu, \quad x(s,0) = x_1(s),$$
$$x(0,t) = x_2(t), \quad x_1(0) = x_2(0),$$

where $A_0\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$ and $x_1,\ x_2$ are piecewise continuously differentiable for $s,t>0$, is completely controllable in $[0,\infty)\times[0,\infty)$, i.e. completely controllable for all $s_1,t_1>0$, if and only if

$$\operatorname{rank}(B, A_0B, A_0^2B, \dots, A_0^{n-1}B) = n. \tag{24}$$

Proof. First, from (7) we determine the Riemann kernel for the equation in Theorem 5, i.e. for a constant coefficient A_0. Iterating equation (7) with $R_0 = I_n$ and $R_{m+1}(s,t,\sigma,\tau) := I_n + \int_\sigma^s \int_\tau^t A_0 R_m(\xi,\eta,\sigma,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$, $m = 0, 1, \dots$ yields

$$R_m = \sum_{j=0}^{m} \frac{1}{(j!)^2} A_0^j (s-\sigma)^j (t-\tau)^j.$$

After showing the convergence, we obtain the Riemann kernel of (7)

$$R(s,t,\sigma,\tau) = \sum_{j=0}^{\infty} \frac{1}{(j!)^2} A_0^j (s-\sigma)^j (t-\tau)^j.$$

We now infer controllability from Theorem 4. The system is not controllable if and only if there exists $(s_1,t_1) \in [0,\infty) \times [0,\infty)$ and $x \in \mathbb{R}^n \setminus \{0\}$ such that $x^T W(0,0,s_1,t_1)x = 0$. Since $x^T RBB^T R^T x \geq 0$, this is equivalent to $B^T R^T x = 0$, i.e.

$$\sum_{j=0}^{\infty} \frac{1}{(j!)^2} (\sigma - s_1)^j (\tau - t_1)^j B^T (A_0^T)^j x = 0.$$

This implies

$$B^T (A_0^T)^j x = 0 \text{ for all } j = 0, 1, 2, \dots.$$

Together with the Cayley-Hamilton theorem, this is equivalent to

$$\operatorname{rank} \begin{pmatrix} B^T \\ B^T A_0^T \\ \vdots \\ B^T (A_0^T)^{n-1} \end{pmatrix} < n.$$

Hence we obtain a contradiction. Since the maximal rank of the above matrix equals n, the theorem is proved. ■

Corollary 1. *Let $A_0 \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $A_1 = \alpha_1 I_n$, $A_2 = \alpha_2 I_n$, $\alpha_1, \alpha_2 \in \mathbb{R}$, $A := \alpha_1\alpha_2 I_n + A_0$. The system*

$$\frac{\partial^2}{\partial s \partial t} x = A_0 x + A_1 \frac{\partial}{\partial s} x + A_2 \frac{\partial}{\partial t} x + Bu,$$

$$x(s,0) = x_1(s), \quad x(0,t) = x_2(t),$$

where x_1, x_2 are piecewise continuously differentiable for $s,t > 0$, is completely controllable in $[0,\infty)\times[0,\infty)$ if and only if

$$\operatorname{rank}(B, AB, A^2B, \dots, A^{n-1}B) = n. \tag{25}$$

Proof. From Remark 2 we infer that there exists a transformation matrix $V(s,t,0,0) = \exp(\alpha_1 t + \alpha_2 s) I_n$ transforming (1) into (20), which is in this case

$$\begin{aligned} \frac{\partial^2}{\partial s\partial t} y - (\alpha_1\alpha_2 I_n + A_0) y &= \frac{\partial^2}{\partial s \partial t} y - Ay \\ &= V^{-1}(s,t)Bu(s,t). \end{aligned} \tag{26}$$

This equation is completely controllable if and only if the original system (1) is completely controllable.

The matrix Riemann function of (26) is again

$$R(s,t,\sigma,\tau) = \sum_{j=0}^{\infty} \frac{1}{(j!)^2} A^j (s-\sigma)^j (t-\tau)^j. \tag{27}$$

For $s_1, t_1 > 0$ we infer from Theorem 4 the complete controllability of (26) if and only if

$$\begin{aligned} &W(0,0,s_1,t_1) \\ &\quad = \int_0^{s_1}\int_0^{t_1} R(\sigma,\tau,s_1,t_1) V^{-1}(\sigma,\tau) \\ &\qquad \times BB^T (V^T)^{-1}(\sigma,\tau) R^T(\sigma,\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &\quad = \int_0^{s_1}\int_0^{t_1} e^{-2(\alpha_1\tau+\alpha_2\sigma)} R(\sigma,\tau,s_1,t_1) \\ &\qquad \times BB^T R^T(\sigma,\tau,s_1,t_1)\,\mathrm{d}\sigma\,\mathrm{d}\tau > 0. \end{aligned}$$

As before, non-controllability is equivalent to the existence of $x \in \mathbb{R}^n \setminus \{0\}$ such that

$$e^{-(\alpha_1\tau+\alpha_2\sigma)} B^T R^T x = 0,$$

which, together with (27), finally yields the desired result. ■

More general controllability criteria in the case of constant coefficients are derived in (Gyurkovics and Jank, 2001; Kaczorek, 1996).

Next we derive general controllability criteria in the case of non-constant coefficients.

Theorem 6. *Let R be the matrix Riemann function of (8) and, furthermore, let $\mathcal{J}$ be an interval such that $R(s_0,t_0,s_1,t_1) \neq 0$. If there exists $y_0 \in \mathbb{R}^{1\times n} \setminus \{0\}$ such that with $y(s,t) = y(s,t,s_1,t_1) := y_0 R(s,t,s_1,t_1)$,*

$$y_0 R(s_0,t_0,s_1,t_1) = y(s_0,t_0,s_1,t_1) \neq 0,$$

and (28)

$$y(s,t)B(s,t) = 0 \quad \textit{for all} \ (s,t) \in \mathcal{J},$$

then the system (1) is not completely controllable in $\mathcal{J}$.

Proof. Notice that with a solution $x \in \mathcal{S}_{n\times 1}$ of (1), from (15) by premultiplying this identity from the left-hand side by $y(s,t)$ and observing that $y(s_1,t_1) = y_0$, we see that

$$
\begin{aligned}
&y_0 x(s_1,t_1)\\
&\quad = y(s_0,t_0,s_1,t_1)x(s_0,t_0)\\
&\qquad + \int_{t_0}^{t_1} y(s_0,\tau,s_1,t_1)\\
&\qquad\quad \times \Big[\frac{\partial}{\partial\tau}x(s_0,\tau) - A_1(s_0,\tau)x(s_0,\tau)\Big]\,\mathrm{d}\tau\\
&\qquad + \int_{s_0}^{s_1} y(\sigma,t_0,s_1,t_1)\\
&\qquad\quad \times \Big[\frac{\partial}{\partial\sigma}x(\sigma,t_0) - A_2(\sigma,t_0)x(\sigma,t_0)\Big]\,\mathrm{d}\sigma\\
&\qquad + \int_{s_0}^{s_1}\int_{t_0}^{t_1} y(\sigma,\tau,s_1,t_1)B(\sigma,\tau)u(\sigma,\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau. \quad (29)
\end{aligned}
$$

If we now assume that the solution x of (1) fulfils the following boundary conditions:

$$x(s_0,\tau) = x_2(\tau) := e^{\int_{t_0}^{\tau} A_1(s_0,\eta)\,\mathrm{d}\eta} x_0,$$

and

$$x(\sigma,t_0) = x_1(\sigma) := e^{\int_{s_0}^{\sigma} A_2(\xi,t_0)\,\mathrm{d}\xi} x_0,$$

where $x_1(s_0) = x_2(t_0) = x_0$, then together with $yB = 0$ we obtain

$$y_0 x(s_1,t_1) = y(s_0,t_0,s_1,t_1)x_0 = y_0 R(s_0,t_0,s_1,t_1)x_0$$

for arbitrary $x_0 \in \mathbb{R}^n$.

Choosing $x_0 \in \mathbb{R}^n$ such that $y(s_0,t_0,s_1,t_1)x_0 \neq 0$ yields a contradiction if we intend to steer the system to $x(s_1,t_1) = 0$. Hence the system is not completely controllable. ■

We say that $y \in \mathcal{S}_{1\times n}(\mathcal{I})$ is a solution of the adjoint differential equation to (8) if it fulfils

$$
\begin{aligned}
&\frac{\partial^2}{\partial s\partial t}y + \frac{\partial}{\partial s}\big(y(s,t)A_1(s,t)\big)\\
&\qquad + \frac{\partial}{\partial t}\big(y(s,t)A_2(s,t)\big) - y(s,t)A_0(s,t) = 0 \quad (30)
\end{aligned}
$$

a.e., and hence if, e.g., $y_0 \in \mathbb{R}^{1\times n}$, then $y(s,t) = y(s,t,\sigma,\tau) := y_0 R(s,t,\sigma,\tau)$ is a solution of the adjoint equation (30). These solutions of the adjoint equation can now be used to obtain sufficient conditions for complete controllability in the case of non-constant coefficients.

Theorem 7. *Let (1) be defined in the interval* $\mathcal{I} = [S_0,S_1]\times[T_0,T_1]$ *and let* R *denote the matrix Riemann function of (1) or (8), respectively. If for all nontrivial solutions* y *of the adjoint equation (30) of the form* $y_0R(s,t,s_0,t_0) = y(s,t)$, $y_0 \in \mathbb{R}^{1\times n}\setminus\{0\}$ *we have*

$$yB \not\equiv 0 \quad \text{on } \mathcal{I}\cap[s_0,\infty)\times[t_0,\infty), \qquad (31)$$

for $s_0 > S_0, t_0 > T_0$*, then there exists* $s_1^* > s_0, t_1^* > t_0$ *such that the system (1) is completely controllable in* $\mathcal{J} = [s_0,s_1^*]\times[t_0,t_1^*]$.

Proof. First we prove that for all $s_0 > S_0$, $t_0 > T_0$, there exists $s_1^* > s_0, t_1^* > t_0$ such that $yB \not\equiv 0$ on $\mathcal{J}$ for all nontrivial solutions y of the adjoint equation (30) that can be represented in the form $y(s,t) = y_0R(s,t,s_0,t_0)$, where $y_0 \in \mathbb{R}^{1\times n}$.

Assume that this is wrong. Then there exist sequences $s_\nu \to S_1$, $t_\nu \to T_1$ as $\nu\to\infty$, and nontrivial solutions $y_\nu(s,t) = y_{0,\nu}R(s,t,s_0,t_0)$ of the adjoint equation with $y_\nu B \equiv 0$ on $[s_0,s_\nu]\times[t_0,t_\nu]$. Without loss of generality we assume $|y_\nu(s_0,t_0)| = 1$, and also (by taking a subsequence if necessary)

$$\lim_{\nu\to\infty} y_\nu(s_0,t_0) = x_0 \in \mathbb{R}^n.$$

Since $y_\nu(s,t) = y_{0,\nu}R(s,t,s_0,t_0)$, we conclude that

$$
\begin{aligned}
\lim_{\nu\to\infty} y_\nu(s,t) &= \big(\lim_{\nu\to\infty} y_{0,\nu}\big)R(s,t,s_0,t_0)\\
&= x_0R(s,t,s_0,t_0) =: y_0(s,t).
\end{aligned}
$$

Hence $y_0(s,t)$ is a nontrivial solution of the adjoint equation, since $x_0 \neq 0$. On the other hand, we have

$$y_0(s,t)B(s,t) = \lim_{\nu\to\infty} y_\nu(s,t)B(s,t) = 0,$$

for $s > s_0, t > t_0$. This contradicts our assumption.

In the next step we prove

$$W(s_0,t_0,s_1^*,t_1^*) > 0,$$

which, by Theorem 4, yields complete controllability.

In general, $W(s_0,t_0,s_1^*,t_1^*) \geq 0$ and we have to show that W is regular. Assume there is some $a \in \mathbb{R}^{1\times n}\setminus\{0\}$ such that $aW = 0$. Hence

$$
\begin{aligned}
aWa^T &= \int_{s_0}^{s_1^*}\int_{t_0}^{t_1^*} aR(\sigma,\tau,s_1^*,t_1^*)B(\sigma,\tau)B^T(\sigma,\tau)\\
&\qquad \times R(\sigma,\tau,s_1^*,t_1^*)a^T\,\mathrm{d}\sigma\,\mathrm{d}\tau\\
&= \int_{s_0}^{s_1^*}\int_{t_0}^{t_1^*} y(\sigma,\tau)B(\sigma,\tau)B^T(\sigma,\tau)\\
&\qquad \times y^T(\sigma,\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau = 0.
\end{aligned}
$$

Therefore

$$aR(\sigma,\tau,s_1^*,t_1^*)B(\sigma,\tau)=0 \quad \text{on} \ [s_0,s_1^*]\times[t_0,t_1^*].$$

This is again a contradiction. Thus, $W>0$ and by Theorem 4 the system is completely controllable on $[s_0,s_1^*]\times[t_0,t_1^*]$. ■

Since Theorem 7 makes use only of particular solutions of the adjoint differential equation (30), there is a stronger sufficient controllability condition applying condition (31) to *all* nontrivial solutions of the adjoint differential equation. Hence we obtain a controllability condition closer to the one-dimensional case.

Corollary 2. *Under the assumptions of Theorem 7 and if condition (31) holds* for all *nontrivial solutions of the adjoint differential equation (30), there exist* $s_1^*>s_0$ *and* $t_1^*>t_0$ *such that the system (1) is completely controllable in* $\mathcal{J}=[s_0,s_1^*]\times[t_0,t_1^*]$.

Next we discuss the observability of the system (1), (2) together with a linear output. We shall introduce a notion of observability analogously to that given in (Sontag, 1998, p. 263).

Definition 2. Let C be piecewise continuous on the interval $\mathcal{I}$, $C(s,t)\in\mathbb{R}^{k\times n}$. Then we define the linear output of the system (1), (2) by

$$y(s,t)=C(s,t)x(s,t), \quad y(s,t)\in\mathbb{R}^k, \tag{32}$$

where $x(s,t)$ is a solution of (1), (2). Suppose that for all $(s_1,t_1)\in\mathcal{I}$ and for all controls $u\in\mathcal{S}_{m\times 1}(\mathcal{I}\cap(-\infty,s_1]\times(-\infty,t_1])$ for any two trajectories $x,\tilde{x}$ of (1) belonging to the same input u, from

$$C(s,t)x(s,t)=C(s,t)\tilde{x}(s,t),$$
$$(s,t)\in\mathcal{I}\cap(-\infty,s_1]\times(-\infty,t_1]$$

it follows necessarily that

$$x(s,t)=\tilde{x}(s,t) \quad \text{in} \ \mathcal{I}\cap(-\infty,s_1]\times(-\infty,t_1].$$

Then the system (1), (2) with the output (32) is said to be *observable* in $\mathcal{I}$.

Remark 3. Write $\hat{x}=x-\tilde{x}$. Then observability is equivalent to the condition that

$$C(s,t)\hat{x}(s,t)=0, \quad (s,t)\in\mathcal{I}\cap(-\infty,s_1]\times(-\infty,t_1]$$

implies

$$\hat{x}(s,t)=0 \quad \text{in} \ \mathcal{I}\cap(-\infty,s_1]\times(-\infty,t_1],$$

where $\hat{x}$ is any solution of the homogeneous system (1), i.e. with $u=0$.

Hence observability is equivalent to the condition that for all $(s_1,t_1)\in\mathcal{I}$ and for all nontrivial solutions $\hat{x}$ of (8) there holds

$$\hat{x}^T C^T \not\equiv 0$$

in $\mathcal{I}\cap(-\infty,s_1]\times(-\infty,t_1]$.

Comparing this last remark with the controllability criterion obtained in Theorem 7 and Corollary 2 yields a necessary criterion for observability.

Theorem 8. *If the system (1), (2) with the output (32) is observable in* $\mathcal{I}$*, then the system of type (30)*

$$\begin{aligned}\frac{\partial^2}{\partial s\partial t}x-A_1^T(-s,-t)\frac{\partial}{\partial s}x-A_2^T(-s,-t)\frac{\partial}{\partial t}x\\ -\Big(A_0^T(-s,-t)+\frac{\partial}{\partial s}A_1^T(-s,-t)+\frac{\partial}{\partial t}A_2^T(-s,-t)\Big)x\\ =C^T(-s,-t)v\end{aligned} \tag{33}$$

is completely controllable in $-\mathcal{I}$.

Proof. From Remark 3 we infer that for any nontrivial solution $\hat{x}$ of (8) there holds $\hat{x}^TC^T\not\equiv 0$ in $\mathcal{I}\cap(-\infty,s_1]\times(-\infty,t_1]$. Then $y(s,t):=\hat{x}(-s,-t)$ is a nontrivial solution of

$$\begin{aligned}\frac{\partial^2}{\partial s\partial t}y+\frac{\partial}{\partial s}\big(yA_1^T(-s,-t)\big)+\frac{\partial}{\partial t}\big(yA_2^T(-s,-t)\big)\\ -y\Big(A_0^T(-s,-t)+\frac{\partial}{\partial s}A_1^T(-s,-t)\\ +\frac{\partial}{\partial t}A_2^T(-s,-t)\Big)=0.\end{aligned}$$

Since this is the adjoint homogeneous differential equation of (33), using Corollary 1 we infer the controllability of (33). ■

4. Initial-Boundary-Value Control

In this section we briefly indicate that for equations of type (1) there is also a possibility of steering the system by its initial-boundary values.

First we define the following operators mapping the set of piecewise continuous functions into itself:

$$\begin{aligned}L_{11}u &:= \int_{s_0}^{s_1} R(\sigma,t_0,s_1,t)u(\sigma)\,\mathrm{d}\sigma,\\ L_{12}u &:= \int_{t_0}^{t} R(s_0,\tau,s_1,t)u(\tau)\,\mathrm{d}\tau,\\ L_{21}u &:= \int_{s_0}^{s} R(\sigma,t_0,s,t_1)u(\sigma)\,\mathrm{d}\sigma,\\ L_{22}u &:= \int_{t_0}^{t_1} R(s_0,\tau,s,t_1)u(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{34}$$

Theorem 9. *Consider the homogeneous system*

$$\frac{\partial^2}{\partial s \partial t} x = A_0(s,t)x + A_1(s,t)\frac{\partial}{\partial s}x + A_2(s,t)\frac{\partial}{\partial t}x_t \quad (35)$$

and let furthermore two piecewise continuous functions $\varphi_1(t),\ \varphi_2(s),\ t_0 \le t \le t_1,\ s_0 \le s \le s_1$ *be given and* $x_0 \in \mathbb{R}^n$*. If the operator*

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, \quad (36)$$

where $L_{11},\ L_{12},\ L_{21},\ L_{22}$*, as defined in (34), is invertible, then there exist initial-boundary values*

$$x_1(\sigma) = x(\sigma, t_0), \quad x_2(\tau) = x(s_0, \tau) \quad (37)$$

with $x_1(s_0) = x_2(t_0) = x_0$*, such that the associated solution of (35) given by (16) fulfils*

$$x(s_1, t) = \varphi_1(t), \quad x(s, t_1) = \varphi_2(s). \quad (38)$$

Proof. From the representation formula (16), with $u = 0$, we see that

$$\begin{aligned} x(s,t) &= R(s_0,t_0,s,t)x_1(s_0) \\ &+ \int_{s_0}^{s} R(\sigma,t_0,s,t)\big(x_1'(\sigma) - A_2(\sigma,t_0)x_1(\sigma)\big)\,\mathrm{d}\sigma \\ &+ \int_{t_0}^{t} R(s_0,\tau,s,t)\big(x_2'(\tau) - A_1(s_0,\tau)x_2(\tau)\big)\,\mathrm{d}\tau. \quad (39) \end{aligned}$$

Now let x_1 and x_2 be determined as the solutions of the following differential equations:

$$\begin{aligned} x_1'(\sigma) - A_2(\sigma,t_0)x_1(\sigma) &= u_1(\sigma), \quad x_1(s_0) = x_0, \\ x_2'(\tau) - A_1(s_0,\tau)x_2(\tau) &= u_2(\tau), \quad x_2(t_0) = x_0, \end{aligned} \quad (40)$$

with $u_1,\ u_2$ to be determined next.

From (40), (39), (34) and using $x(s_1,t) = \varphi_1(t)$, $x(s_1,t) = \varphi_2(t)$, $x(s_0,t_0) = x_0$, we obtain the integral equation

$$\begin{pmatrix} \varphi_1(t) \\ \varphi_2(s) \end{pmatrix} - \begin{pmatrix} R(s_0,t_0,s_1,t)x_0 \\ R(s_0,t_0,s,t_1)x_0 \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \quad (41)$$

If (41) is solvable, then it determines u_1, u_2 in terms of the initial point $x_0 = x(s_0,t_0)$ and the prescribed terminal data $x(s_1,t), x(s,t_1)$. So (40) determines the Goursat data needed to steer the system to the prescribed terminal data. ■

5. Optimal Control

In (Kaczorek, 1995; 1996), among other results, a solution to the minimum energy control problem for (1) was obtained. We present a solution to the optimal control problem for (1) where we relax the condition to meet exactly a predefined endpoint $x(t_1, s_1)$ and, furthermore, impose quadratic costs also for the state. Therefore, in this section we study an optimal control problem associated with (1) and with a quadratic performance criterion. Here we prefer a Hilbert space approach where, beside the spaces already used, we introduce the Hilbert space $\mathcal{H}^k(\mathcal{J}), k \in \mathbb{N}$ of all $\mathbb{R}^k$-valued functions in $\mathcal{J}$, square integrable with the scalar product

$$\begin{aligned} \langle x, y \rangle &= x^T(s_1,t_1)y(s_1,t_1) \\ &\quad + \int_{s_0}^{s_1}\int_{t_0}^{t_1} x^T(\sigma,\tau)y(\sigma,\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau \end{aligned}$$

for all $x, y \in \mathcal{H}^k(\mathcal{J})$.

Definition 3. (i) Let R be the matrix Riemann function of (1). Then

$$\tilde{R} : \mathbb{R}^n \to \mathcal{H}^n(\mathcal{J}), \quad x_0 \mapsto R(s_0,t_0,\cdot,\cdot)x_0.$$

(ii) Let R be the matrix Riemann function of (1) and let B, as in (1), be piecewise continuous on $\mathcal{J}$. Then

$$\begin{aligned} &L : \mathcal{H}^m(\mathcal{J}) \to \mathcal{H}^n(\mathcal{J}), \\ &u \mapsto L(u) = \int_{s_0}^{(\cdot)}\int_{t_0}^{(\cdot)} R(\sigma,\tau,\cdot,\cdot)B(\sigma,\tau)u(\sigma,\tau)\mathrm{d}\sigma\mathrm{d}\tau. \end{aligned}$$

(iii) Let $Q(s,t) \in \mathbb{R}^{n\times n}$ be piecewise continuous in $\mathcal{J}$, $K_1 \in \mathbb{R}^{n\times n}$. Then

$$\begin{aligned} &\tilde{Q} : \mathcal{H}^n(\mathcal{J}) \to \mathcal{H}^n(\mathcal{J}), \\ &x(\cdot,\cdot) \mapsto \left((s,t) \mapsto \begin{cases} Q(s,t)x(s,t) & (s,t) \ne (s_1,t_1) \\ K_1 x(s,t) & (s,t) = (s_1,t_1) \end{cases} \right). \end{aligned}$$

(iv) Let $T(s,t) \in \mathbb{R}^{m\times m}$ be piecewise continuous in $\mathcal{J}$. Then

$$\begin{aligned} &\tilde{T} : \mathcal{H}^m(\mathcal{J}) \to \mathcal{H}^m(\mathcal{J}), \\ &x(\cdot,\cdot) \mapsto \left((s,t) \mapsto \begin{cases} T(s,t)x(s,t) & (s,t) \ne (s_1,t_1) \\ 0 & (s,t) = (s_1,t_1) \end{cases} \right). \end{aligned}$$

Moreover, for the matrix Riemann function R and any piecewise continuously differentiable function $x_1(\sigma) \in \mathbb{R}^n$ in $[s_0, s_1]$ we set

$$\begin{aligned} \Theta_1(\cdot,\cdot) := &\int_{s_0}^{(\cdot)} R(\sigma,t_0,\cdot,\cdot)\big(x_1'(\sigma) \\ &- A_2(\sigma,t_0)x_1(\sigma)\big)\,\mathrm{d}\sigma \in \mathcal{H}^n(\mathcal{J}), \quad (42) \end{aligned}$$

and for any piecewise continuously differentiable function $x_2(\tau) \in \mathbb{R}^n$ in $[t_0, t_1]$ we set

$$\Theta_2(\cdot,\cdot) := \int_{t_0}^{(\cdot)} R(s_0,\tau,\cdot,\cdot)\big(x_2'(\tau) - A_1(s_0,\tau)x_2(\tau)\big)\,\mathrm{d}\tau \in \mathcal{H}^n(\mathcal{J}). \tag{43}$$

With this notation, for any control function u piecewise continuous in $\mathcal{J}$, the representation formula (16) can be rewritten as

$$x(\cdot,\cdot) = \tilde{R}x_0 + \Theta_1 + \Theta_2 + Lu, \quad x_0 \in \mathbb{R}^n. \tag{44}$$

Furthermore, we introduce the following quadratic performance criterion:

$$\begin{aligned} J &:= \frac{1}{2}x^T(s_1,t_1)K_1x(s_1,t_1) \\ &\quad + \frac{1}{2}\int_{s_0}^{s_1}\int_{t_0}^{t_1}\big(x^T(\sigma,\tau)Q(\sigma,\tau)x(\sigma,\tau) \\ &\quad + u^T(\sigma,\tau)T(\sigma,\tau)u(\sigma,\tau)\big)\,\mathrm{d}\sigma\,\mathrm{d}\tau, \end{aligned}$$

$$\begin{aligned} K_1^T &= K_1, \quad Q^T(\sigma,\tau) = Q(\sigma,\tau), \\ T^T(\sigma,\tau) &= T(\sigma,\tau) \ \text{ for all } \ \sigma,\tau \in \mathcal{J}. \end{aligned} \tag{45}$$

Using the definition of the scalar product and Definition 3 (iii), we obtain

$$\begin{aligned} \langle x, \tilde{Q}x\rangle &= x^T(s_1,t_1)K_1x(s_1,t_1) \\ &\quad + \int_{s_0}^{s_1}\int_{t_0}^{t_1} x^T(\sigma,\tau)Q(\sigma,\tau)x(\sigma,\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau. \end{aligned}$$

From (iv) we get

$$\langle u, \tilde{T}u\rangle = \int_{s_0}^{s_1}\int_{t_0}^{t_1} u^T(\sigma,\tau)T(\sigma,\tau)u(\sigma,\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

Hence we obtain for the quadratic performance criterion

$$2J = \langle x, \tilde{Q}x\rangle + \langle u, \tilde{T}u\rangle. \tag{46}$$

Theorem 10. *The functional J in (46) together with the constraint (44) admits a unique minimum among all admissible controls $u \in \mathcal{H}^m(\mathcal{J})$ if and only if*

$$\tilde{T} + L^*\tilde{Q}L > 0 \quad \text{on} \ \mathcal{H}^m(\mathcal{J}). \tag{47}$$

The optimal control is then a piecewise continuous function in $\mathcal{J}$ given as

$$u_{\mathrm{opt}} = -(\tilde{T} + L^*\tilde{Q}L)^{-1}L^*\tilde{Q}(\tilde{R}x_0 + \Theta_1 + \Theta_2), \tag{48}$$

where L^* denotes the adjoint operator to L with respect to the scalar product in $\mathcal{H}^n(\mathcal{J})$.

Proof. Inserting (44) into (46) yields

$$\begin{aligned} 2J &= \langle \tilde{R}x_0+\Theta_1+\Theta_2+Lu, \tilde{Q}(\tilde{R}x_0+\Theta_1+\Theta_2+Lu)\rangle \\ &\quad + \langle u, \tilde{T}u\rangle \\ &= \langle u, \tilde{T}u\rangle + \langle Lu, \tilde{Q}Lu\rangle + \langle Lu, \tilde{Q}(\tilde{R}x_0+\Theta_1+\Theta_2)\rangle \\ &\quad + \langle \tilde{R}x_0+\Theta_1+\Theta_2, \tilde{Q}Lu\rangle + J_0 \\ &= \langle u, (\tilde{T} + L^*\tilde{Q}L)u\rangle \\ &\quad + 2\langle u, L^*\tilde{Q}(\tilde{R}x_0+\Theta_1+\Theta_2)\rangle + J_0, \end{aligned} \tag{49}$$

where J_0 denotes the remaining part of the criterion, which does not depend on u.

This functional now admits a unique minimum in $\mathcal{H}^m(\mathcal{J})$ if and only if (47) holds. In that case the minimum is given by (48) and this is indeed a piecewise continuous function. ■

Notice that (47) is fulfilled if, for instance, $T(\sigma,\tau) > 0$, $Q(\sigma,\tau) \geq 0$ for all $(\sigma,\tau) \in \mathcal{J}$ and $K_1 \geq 0$. This follows immediately from

$$\begin{aligned} \langle u, (\tilde{T} + L^*\tilde{Q}L)u\rangle &= \big((Lu)^*K_1Lu\big)(s_1,t_1) \\ &\quad + \int_{s_0}^{s_1}\int_{t_0}^{t_1}\Big(u^T(\sigma,\tau)T(\sigma,\tau)u(\sigma,\tau) \\ &\quad + \big((Lu)^*QLu\big)(\sigma,\tau)\Big)\,\mathrm{d}\sigma\,\mathrm{d}\tau > 0 \end{aligned}$$

for all $u \in \mathcal{H}^m(\mathcal{J}) \setminus \{0\}$.

Next we calculate the adjoint operator L^* of L. With $w \in \mathcal{H}^n(\mathcal{J})$, $u \in \mathcal{H}^m(\mathcal{J})$ and

$$\langle w, Lu\rangle = \langle u, L^*w\rangle,$$

we obtain

$$\begin{aligned} \langle w, Lu\rangle &= \int_{s_0}^{s_1}\int_{t_0}^{t_1}\Big(w^T(s,t) \\ &\times \int_{s_0}^{s}\int_{t_0}^{t} R(\sigma,\tau,s,t)B(\sigma,\tau)u(\sigma,\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau\Big)\mathrm{d}s\,\mathrm{d}t \\ &+ w^T(s_1,t_1)\int_{s_0}^{s_1}\int_{t_0}^{t_1} R(\sigma,\tau,s_1,t_1)B(\sigma,\tau)u(\sigma,\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau. \end{aligned}$$

Interchanging the order of integration in the first part yields

$$\begin{aligned} &\int_{s_0}^{s_1}\int_{s_0}^{s}\Big(\int_{t_0}^{t_1}\Big[\int_{t_0}^{t} w^T(s,t)R(\sigma,\tau,s,t)B(\sigma,\tau) \\ &\qquad \times u(\sigma,\tau)\,\mathrm{d}\tau\Big]\,\mathrm{d}t\Big)\,\mathrm{d}\sigma\,\mathrm{d}s \\ &= \int_{s_0}^{s_1}\int_{s_0}^{s}\mathrm{d}\sigma\,\mathrm{d}s\Big(\int_{t_0}^{t_1}\mathrm{d}\tau\Big[\int_{\tau}^{t_1} w^T(s,t)R(\sigma,\tau,s,t) \\ &\qquad \times B(\sigma,\tau)u(\sigma,\tau)\,\mathrm{d}t\Big]\Big) \end{aligned}$$

$$= \int_{t_0}^{t_1} \mathrm{d}\tau \int_{\tau}^{t_1} \mathrm{d}t \int_{s_0}^{s_1} \mathrm{d}\sigma \Big(\int_{\sigma}^{s_1} w^T(s,t) R(\sigma,\tau,s,t) \times B(\sigma,\tau) u(\sigma,\tau)\, \mathrm{d}s \Big)$$

$$= \int_{s_0}^{s_1}\int_{t_0}^{t_1} \mathrm{d}\sigma\, \mathrm{d}\tau \Big[\Big(\int_{\sigma}^{s_1}\int_{\tau}^{t_1} w^T(s,t) R(\sigma,\tau,s,t)\, \mathrm{d}s\, \mathrm{d}t \Big) \times B(\sigma,\tau) u(\sigma,\tau) \Big]$$

$$= \Big(\int_{s_0}^{s_1}\int_{t_0}^{t_1} \mathrm{d}\sigma\, \mathrm{d}\tau \Big[\Big(\int_{\sigma}^{s_1}\int_{\tau}^{t_1} w^T(s,t) R(\sigma,\tau,s,t)\, \mathrm{d}s\, \mathrm{d}t \Big) \times B(\sigma,\tau) u(\sigma,\tau) \Big] \Big)^T$$

$$= \int_{s_0}^{s_1}\int_{t_0}^{t_1} u^T(\sigma,\tau) B^T(\sigma,\tau) \Big(\int_{\sigma}^{s_1}\int_{\tau}^{t_1} R^T(\sigma,\tau,s,t) \times w(s,t)\, \mathrm{d}s\, \mathrm{d}t \Big)\, \mathrm{d}\sigma\, \mathrm{d}\tau$$

$$= \Big\langle u, B^T(\cdot,\cdot) \int_{(\cdot)}^{s_1}\int_{(\cdot)}^{t_1} R^T(\cdot,\cdot,s,t) w(s,t)\, \mathrm{d}s\, \mathrm{d}t \Big\rangle.$$

Hence, adding the transpose of the second part we obtain for $w \in \mathcal{H}^n(\mathcal{J})$

$$(L^* w)(\sigma,\tau) = B^T(\sigma,\tau) \Big(\int_{\sigma}^{s_1}\int_{\tau}^{t_1} R^T(\sigma,\tau,s,t) w(s,t)\, \mathrm{d}s\, \mathrm{d}t + R^T(\sigma,\tau,s_1,t_1) w(s_1,t_1) \Big). \quad (50)$$

Corollary 3. *Under the assumptions of Theorem 10 and with T being piecewise continuous on $\mathcal{J}$, $T(s,t) \in \mathbb{R}^{m\times m}$, $T^T(s,t) = T(s,t)$, if $T^{-1}(s,t)$ exists for all $(s,t) \in \mathcal{J}$, then the optimal control is given by*

$$u_{\mathrm{opt}}(\sigma,\tau) = -T^{-1}(\sigma,\tau) B^T(\sigma,\tau) \tilde{\Phi}(x)(\sigma,\tau),$$

where x is the solution of the integral equation

$$x(s,t) = y_0(s,t) - \int_{s_0}^{s}\int_{t_0}^{t} R(\sigma,\tau,s,t) B(\sigma,\tau) T^{-1}(\sigma,\tau) \times B^T(\sigma,\tau) \tilde{\Phi}(x)(\sigma,\tau)\, \mathrm{d}\sigma\, \mathrm{d}\tau, \quad (51)$$

with

$$y_0(s,t) = R(s_0,t_0,s,t) x_0 + \int_{s_0}^{s} R(\sigma,t_0,s,t) \big(x_1'(\sigma) - A_2(\sigma,t_0) x_1(\sigma) \big)\, \mathrm{d}\sigma + \int_{t_0}^{t} R(s_0,\tau,s,t) \big(x_2'(\tau) - A_1(s_0,\tau) x_2(\tau) \big)\, \mathrm{d}\tau,$$

and

$$\tilde{\Phi} : \mathcal{H}^n(\mathcal{J}) \to \mathcal{H}^n(\mathcal{J}),$$

$$x \mapsto \Phi(x)(\cdot,\cdot) = \int_{(\cdot)}^{s_1}\int_{(\cdot)}^{t_1} R^T(\cdot,\cdot,\xi,\eta) Q(\xi,\eta) x(\xi,\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta + R^T(\cdot,\cdot,s_1,t_1) K_1 x(s_1,t_1).$$

Proof. From (44) we get $x - Lu = \tilde{R}x_0 + \Theta_1 + \Theta_2$ and this, together with (48), yields $\tilde{T} u_{\mathrm{opt}} = -L^* \tilde{Q} x$. Notice that $\tilde{T}$ is not invertible on all of $\mathcal{H}^m(\mathcal{J})$, since, e.g., with

$$\mathring{u}(s,t) := \begin{cases} 0 & \text{for} \quad (s,t) \neq (s_1,t_1), \\ 1 & \text{for} \quad (s_1,t_1), \end{cases}$$

we obtain $\tilde{T}\, \mathring{u} = 0$, whereas $\|u\| = 1$.

Conversely, if u is piecewise continuous and if, moreover, $T^{-1}(s,t)$ exists for all $(s,t) \in \mathcal{J}$, then $\tilde{T}u = 0$ implies $Tu(s,t) = 0$ for $(s,t) \neq (s_1,t_1)$, and this holds if and only if $u = 0$ on $\mathcal{J}$. Hence the restriction of $\tilde{T}$ to the set of piecewise continuous functions is invertible and its inverse is T^{-1}. Let T^{-1} exist for all $(\sigma,\tau) \in \mathcal{J}$. Then for the optimal control we obtain

$$u_{\mathrm{opt}} = -T^{-1} L^* \tilde{Q} x. \quad (52)$$

Furthermore, using Definition 3 (iii) and (50), we obtain

$$u_{\mathrm{opt}}(\sigma,\tau) = -T^{-1}(\sigma,\tau) B^T(\sigma,\tau) \tilde{\Phi}(x)(\sigma,\tau), \quad (53)$$

where x is a solution of (1), (2) or (44), i.e. of (51). Notice that in a similar way as in Section 2 it can be proved that the integral equation (51) has a unique solution. It remains an open question if the operator $\tilde{\Phi}$ can be represented in feedback form, hence if u_{opt} can be written as $u_{\mathrm{opt}} = -T^{-1} B^T K x$.

6. Conclusions

For the system (1), (2) we obtain criteria for controllability for general, non-constant coefficients. Along with the well-known necessary and sufficient conditions, where one demands the positive definiteness of the “Gramian” matrix (23), we also obtain a sufficient controllability criterion by using solutions of the adjoint system.

Observability is defined in the usual way, i.e. that a given input and a known linear output determine uniquely the state. In Theorem 8 we then obtain a necessary condition for observability in terms of the controllability of an associated system.

We also derive conditions to control the homogenous systems by prescribed “Goursat” data. Finally, using a Hilbert-space approach, we solve the optimal control problem with a quadratic performance criterion.

All these results are based on a representation formula for solutions of (1), (2) using the associated matrix Riemmann function. For completeness, and also for the reader's convenience, in Section 2 we provide the reader with all the necessary tools concerning the matrix Riemann function.

References

Arnol'd V.I. (1980): *Gewöhnliche Differentialgleichungen*. — Berlin: Springer.

Bergmann J., Burgmeier P. and Weiher P. (1989): *Explicit controllability criteria for Goursat problems*. — Optim., Vol. 20, No. 3, pp. 281–290.

Gantmacher F.R. (1986): *Matrizentheorie*. — Berlin: Springer.

Gyurkovics É and Jank G. (2001): *Instant controllability for Goursat problems*. — Int. Rep., Cadernos de Matemática, Universidade de Aveiro, Portugal, CM01/I-02.

Jain A.K. and Jain J.R. (1977): *Partial differential equations and finite difference methods in image processing, Part I—Image Representation*. — J. Optim. Theory Appl., Vol. 23, pp. 65–91.

Jain A.K. and Jain J.R. (1978): *Partial differential equations and finite difference methods in image processing, Part II—Image Restoration*. — I.E.E.E. Trans. Automat. Contr., Vol. 23, No. 5, pp. 817–833.

Kaczorek T. (1995): *Local controllability and minimum energy control of 2-D linear time invariant systems with variable coefficients*. — Multidim. Syst. Signal Process., Vol. 6, pp. 69–75.

Kaczorek T. (1996): *Local controllability and minimum energy control of continuous 2-D linear time invariant systems*. — Bull. Polish Acad. Sci. Tech. Sci., Vol. 44, No. 1, pp. 67–74.

Pulvirenti G. and Santagati G. (1975): *Sulla controllabilita completa in luogu determinatu ed in luogu variabile por un processo di controllo con parametri distributi*. — Boll. U.M.I., IV. Ser. 11, Fasc. 3, pp. 316–328.

Sontag E.D. (1998): *Mathematical Control Theory*. — New York: Springer.

Vekua I.N. (1967): *New Methods for Solving Elliptic Equations*. — Amsterdam: North-Holland.

Wolfersdorf v. L. (1978): *Zum Maximum-Prinzip von Pontrjagin für eine Klasse partieller Differentialgleichungssysteme 1. Ordnung*. — ZAMM, Vol. 58, pp. 261–269.

Received: 18 December 2000
Revised: 15 June 2001
Re-revised: 6 November 2001