COMPARISON OF THE STABILITY BOUNDARY AND THE FREQUENCY RESPONSE STABILITY CONDITION IN LEARNING AND REPETITIVE CONTROL

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In iterative learning control (ILC) and in repetitive control (RC) one is interested in convergence to zero tracking error as the repetitions of the command or the periods in the command progress. A condition based on steady state frequency response modeling is often used, but it does not represent the true stability boundary for convergence. In this paper we show how this useful condition differs from the true stability boundary in ILC and RC, and show that in applications of RC the distinction between these conditions is of no practical significance. In ILC satisfying this frequency condition is important for good learning transients, even though the true stability boundary is very different.

Keywords: iterative learning control, repetitive control, stability, monotonic convergence

1. Introduction

Iterative learning control (ILC) refers to methods of iteratively adjusting the command to a closed loop control system, to converge on that command which produces zero tracking error following a desired trajectory. The system is restarted from the same initial condition each time a command is given. The very closely related repetitive control (RC) applies to systems with a periodic desired output or with a constant desired output, and there are periodic disturbances. The command to the closed loop system is adjusted from one period to the next in order to converge to zero tracking error. The year 1984 saw a sudden flurry of activity in these fields, occurring independently and simultaneously on four different continents, motivated by robotics. Robots on assembly lines perform the same operation many times a day, and it seems natural to have the robot learn to eliminate its tracking error by paying attention to its experience executing the trajectory. ILC papers that appeared that year with this motivation include (Arimoto *et al.*, 1984; Casalino *et al.*, 1984; Craig, 1984), and also submitted that year was the RC paper (Middleton *et al.*, 1985). Uchiyama (1978) served as a precursor with this same motivation; Edwards (1974), Owens (1977), and Edwards and Owens (1982) are other precursors, treating multipass processes, and motivated by problems in coal mining. Early repetitive control publications include (Inoue *et al.*, 1981; Omata *et al.*, 1984; Hara *et al.*, 1985a, 1985b; Nakano *et al.*, 1986, Tomizuka *et al.*, 1989).

In both ILC and RC it is common to aim to satisfy a frequency response based stability condition, normally heuristically derived to suggest convergence based on decay of the steady state frequency response components of the error. Various papers in the literature address the problems from a frequency response point of view, e.g. De Luca *et al.* (1992). Elci *et al.* (1994) and Longman (2000) show that a condition indicating monotonic decay of the steady state frequency response components with repetitions is a sufficient condition for asymptotic stability of ILC, independent of whether any part of the finite time trajectory can be considered as steady state. However, it is suggested that the real use of this condition is as a technique for generating good transients during the learning process. Huang and Longman (1996) suggest that the discrepancy between this frequency response based condition and the true stability boundary is very large for ILC, but for RC the condition will normally be very close to the true stability boundary. It is the purpose of this paper to investigate the distinction between the frequency response based condition and the true stability boundary, for the set of all first order systems, all second order systems, and all third order systems with no zero.

This is done for the most basic form of ILC and RC, integral control based learning. In the case of ILC this adjusts the command at time step k of repetition j according to $u_{j+1}(kT) = u_j(kT) + K_l e_j((k+1)T)$, where K_l is the learning gain, T is the time step interval, and e_j is the tracking error (desired output minus actual). The one step time shift in the error accounts for the usual one step delay between a change in the input to a change in the output in a digital system. In repetitive control the change in repetition number j is replaced by a time shift of p time steps corresponding to one period. Written in terms of z-transforms this becomes $z^p U(z) = U(z) + K_l z E(z)$. The associated repetitive control system has a very standard looking block diagram with unity feedback as shown in Fig. 1. The command input is the desired output as is usual with feedback control. The controller box contains the repetitive control law with the transfer function $K_l z/(z^p - 1)$. What is unusual is that the plant transfer function block contains the *closed* loop feedback control system, and the repetitive controller is adjusting the command to this feedback control system. The authors always consider this formulation, but we comment that some of the literature has the repetitive control action adjusting the manipulated variable within the feedback control system, rather than the simpler adjustment of the command to the control system.

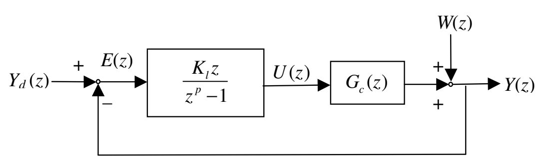

Fig. 1. Block diagram of a repetitive controller modifying the command to a feedback control system.

2. A Frequency Response Based Stability Condition—An Approximate Monotonic Decay Condition

The above-mentioned frequency response condition for both iterative learning control and repetitive control asks that

$$\left|1 - K_l e^{i\omega T} G_c(e^{i\omega T})\right| < 1 \tag{1}$$

for all frequencies ω up to Nyquist, where T is the sample time interval, and $G_c(z)$ is the z-transfer function of the associated closed loop control system. Longman (2000), Elci *et al.* (1994), Huang and Longman (1996) prove that this condition is a sufficient condition for convergence to zero tracking error for both ILC and RC. But it is suggested that the real importance of the condition (1) is as an approximate condition for assuring monotonic decay of the tracking error with repetitions or periods. To see this for ILC, suppose that the output of the closed loop control system is $Y(z) = G_c(z)U(z) + W(z)$ where W represents any disturbance that appears every time the command is given. Write this for repetitions $j+1$ and j, and take the difference. Express the result in terms of a difference of errors, and use the learning control law described in the previous section to produce the input in terms of the error. The result can be written as $E_{j+1}(z) = [1 - K_l z G_c(z)]E_j(z)$ (note that the initial condition on E is zero for ILC). By satisfying (1), it is guaranteed that the steady state frequency response components of the error will decay monotonically with repetitions. However, since the trajectory is a finite time trajectory, technically it is never in steady state. Nevertheless, when the trajectory is significantly longer than a few time constants of the system, this condition makes a good condition to satisfy in order to ensure good learning transients, by creating monotonic decay of the error for a substantial part of the finite time trajectory.

To see this for RC, find the transfer function from the periodic desired trajectory $Y_d(z)$ to the associated error, for the block diagram described in the previous section. This produces $[z^p - 1 + K_l z G_c(z)]E(z) = (z^p - 1)[Y_d(z) - W(z)]$. The right-hand side is zero due to the periodicity with period p of the desired trajectory and the disturbance. This makes a homogeneous difference equation whose transients determine the convergence of the error. Rewrite this equation as $z^p E(z) = [1 - K_l z G_c(z)]E(z)$ and note that the multiplication by z^p is a shift one period forward in time. This suggests that if (1) is satisfied, there will again be monotonic decay of each frequency component of the error from one period to the next. This time there is a quasi-static assumption made, in order to have steady state frequency response thinking apply.

If one chooses to satisfy (1) in order to obtain good transients of the learning process, it is of interest to know how much more restrictive satisfying (1) is, than simply satisfying the if and only if condition for stability. In this paper we show how these differ for ILC and RC for first, second, and third order systems. We also show how to find the true stability boundary, and see that in the case of RC it is much more difficult than using condition (1).

3. True Stability Boundary for ILC and RC

In ILC the true stability boundary when using integral control based learning with learning gain K_l, is given by

$$0 < (CB)K_l < 2, \tag{2}$$

where B and C are from the discrete-time state-space model $x((k+1)T) = Ax(kT) + Bu(kT)$, $y(kT) =$

$Cx(kT)$ of the closed-loop single-input, single-output (SISO) system (see, e.g., Phan and Longman, 1988).

In repetitive control, the most natural way to express the true stability boundary is in terms of the Nyquist stability criterion. The repetitive control loop contains the closed-loop feedback control system $G_c(z)$ and the repetitive control law $K_l z/(z^p - 1)$ together with unity feedback, so that the characteristic polynomial can be written in the form $1 + K_l G_r(z) = 0$ with $G_r(z) = zG_c(z)/(z^p - 1)$. It will be convenient for us to separate the DC gain K_c of the feedback control system from the rest of its transfer function according to $G_c(z) = K_c G(z)$, define the product of the DC gain with the learning gain as $K = K_c K_l$, and then this serves as our gain parameter. The characteristic equation becomes $1 + KG(z) = 0$. The direct application of the discrete time Nyquist criterion is inconvenient because of the p roots on the unit circle. With a sample rate of 1000 Hz and a one-second trajectory there are 1000 roots on the unit circle, and the Nyquist contour must go around each of them. Following Huang and Longman (1996), we apply the method of Pierre (1989) to handle this difficulty. Rewrite the characteristic polynomial in the form

$$-K + Q(z) = 0, \quad Q(z) = -1/G(z). \tag{3}$$

The troublesome poles on the unit circle become zeros of Q, simplifying the plotting. Plot $Q(e^{i\theta})$ for θ going from 0 to 180 deg, deleting any points for which Q is singular (one does not have to go around these points). For any gain K as a point on the real axis of the Q plane, where $\operatorname{Im} Q(e^{i\theta}) \neq 0$, we have

$$Z = (-W/180^\circ) + P + (n'/2), \tag{4}$$

where W is the angle swept by the vector pointing from point $(K, 0)$ to the moving point $Q(e^{i\theta})$ for θ going from 0 to 180 deg with singularities deleted (clockwise is counted as positive), P is the total number of poles of Q outside the unit circle, finite poles plus poles at infinity, Z is the number of zeros of $-K + Q(z) = 0$ that are outside the unit circle, and n' is the number of poles of Q on the unit circle.

In using (4), one normally knows the values for P and n', W is determined from the plot, and hence Z is known. The system is stable for all K that produce Z equal zero. Note that this stability condition, which represents the true stability boundary, depends on the number of time steps p in a period, whereas the previous approximate monotonic decay condition (1) does not.

4. Stability of First Order Systems

Stability Conditions: Now let us examine the distinction between stability condition (1) and the true stability boundary (Eqn. (2) for ILC and (4) for RC) for all possible first order systems. Start with a continuous time transfer function $G_s(s) = K_c a/(s + a)$ where K_c is the DC gain. When fed by a zero order hold, this converts according to the rule $G_c(z) = (1 - z^{-1})Z[G_s(s)/s]$ where the Z indicates taking the z-transform of the function represented in the square bracket. Then the $G_c(z)$ for equation (1) is

$$G_c(z) = K_c\big(1 - e^{-aT}\big)/\big(z - e^{-aT}\big). \tag{5}$$

The $G(z)$ for Eqns. (3) and (4) is given by

$$G(z) = \big(1 - e^{-aT}\big)z/\left[\big(z - e^{-aT}\big)\big(z^p - 1\big)\right] \tag{6}$$

and condition (2) becomes

$$0 < K_l K_c\big(1 - e^{-aT}\big) < 2. \tag{7}$$

There are three parameters whose values may affect these stability conditions: the gain $K = K_l K_c$, which is the product of the learning gain with the DC gain of the system, the value of aT related to the time constant of the system and the sample time, and the number of time steps p in the desired trajectory or period.

Concerning Limits on the Parameters: In order for a discrete time control system to function well, one should have the sample rate such that there are several time steps in a time constant of the system. In this case the time constant is $1/a$, so a generous upper limit on the value of aT is unity. When we get to second order systems $s^2 + 2\zeta\omega_0 s + \omega_0^2$, we need at least one sample per time constant when the roots are real, and when the roots are complex we again ask for one sample per time constant for the real part of the root, and at least two sample times per period for the oscillatory part of the root (this limits the values of $\omega_0 T$ to a maximum of $\pi/\sqrt{1-\zeta^2}$).

Approximate Monotonic Decay Condition: The range of gain K satisfying condition (1) can be found by plotting $K_l e^{i\theta} G_c(e^{i\theta})$ for θ going from 0 to 180 deg, and seeing how large K can be before the curve goes outside the unit circle centered at $+1$. This happens first when $\theta = 0$, and produces the inequality $0 < K < 2$. Condition (1) is always independent of p, but in this case it is also independent of parameter aT as well. As stated above, this is a sufficient condition for stability for both ILC and RC.

True Stability Boundary for Learning Control: The discrete time state variable representation of (5) has matrices A, B and C given by e^{-aT}, $K_c(1 - e^{-aT})$ and 1, respectively. Then the stability boundary is given by $0 < K < 2/(1 - e^{-aT})$. This is independent of p, and the stable range on K tends to infinity as the sample time tends to zero. The boundary is shown in Fig. 2. It is always larger than the sufficient condition (1) as it must

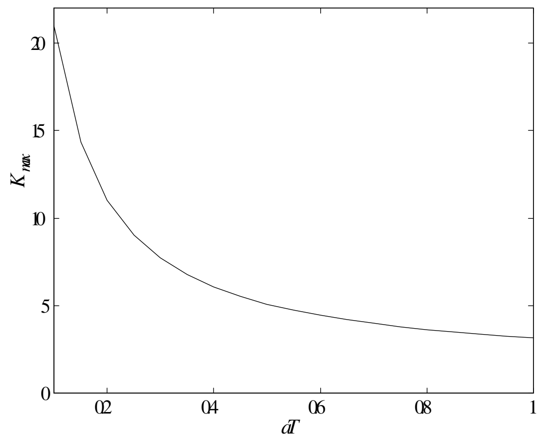

Fig. 2. The stability boundary of learning control applied to the first order system.

be, and it becomes arbitrarily larger as the sample time approaches zero.

True Stability Boundary for Repetitive Control: For conciseness, denote sine by s and cosine by c. Then the $Q(e^{i\theta})$ from (6) can be written as

$$Q\big(e^{i\theta}\big) = \big(1 - e^{-aT}\big)^{-1} \big[1 - c(p\theta) + e^{-aT} \big(c\big((p-1)\theta\big) - c\theta\big) + i\big(e^{-aT}\big)\big(s\big((p-1)\theta\big) + s\theta\big) - is(p\theta)\big]. \quad (8)$$

A typical plot of $Q(e^{i\theta})$ is given in Fig. 3, $p = 10$ and $aT = 0.88$. Applying the modified Nyquist plot rules above to any point $(K, 0)$ between 0 and the first time the plot crosses the positive real axis produces, $W = 5 \times 360°$, $P = 10$, $n' = 0$ with $Z = 0$. Hence, all gains K between zero and this first crossing of the positive real axis correspond to stability. For this first order system, it happens that the first loop is always the one determining stability, but for the second and third order systems discussed below, this is not necessarily the case. Then, the procedure for determining the maximum stable gain $K_{\max}$ as the parameters of the system are run through their range of values is as follows:

(i) Set the imaginary part of $Q(e^{i\theta})$ equal to zero. For the case of Eqn. (8) this can be rewritten as

$$e^{-aT} c\big((p-2)\theta/2\big) - c(p\theta/2) = 0. \quad (9)$$

Then solve this numerically to get the p solutions for θ.

(ii) Substitute each solution for θ into $Q(e^{i\theta})$ to find the associated value of K according to (3). The minimum of these values is the gain for the stability boundary, $K_{\max}$.

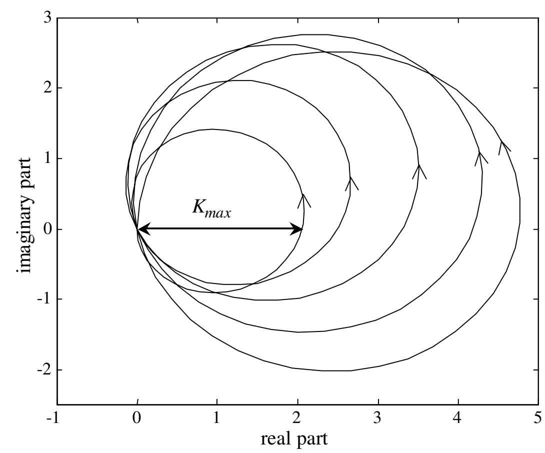

Fig. 3. Plot of $Q(e^{i\theta})$, with θ from 0 to $180°$, of the first order system when $p = 10$ and $aT = 0.88$.

Figure 4 shows the results of this procedure for the first order system, giving $K_{\max}$ for aT in the range of reasonable values from 0 to 1, and for various values of the number of time steps in a period, p. As must be the case, the stability boundary is always above the value $K = 2$ given by the monotonic decay condition (1). However, this difference is only substantial when the number of time steps in a period is quite small, e.g., for a p of 10. In typical digital control systems with sample rates like 100 or 1000 Hz, any reasonable length period for the desired motion will have a p sufficiently large that the distinction between the true stability boundary in repetitive control, Eqn. (4), and the approximate monotonic decay condition (1) becomes insignificant. Hence, in most practical situations, satisfying the condition (1) is close to the requirement, even though it does not correspond to the true stability boundary. The difference between (1) and (4) is insignificant for typical length trajectories. And use of (1) in designing repetitive controllers is much easier than using (4).

5. Stability of Second Order Systems

Now consider the set of all stable strictly proper second order systems. The transfer function in continuous time is $G_s(s) = K_c\omega_0^2(ds + 1)/(s^2 + 2\zeta\omega_0 s + \omega_0^2)$, and we consider that it is fed by a zero order hold, and then converted to the associated z-transfer function. This time the parameters that can influence stability are K, $\omega_0 T$, ζ, d/T, and p.

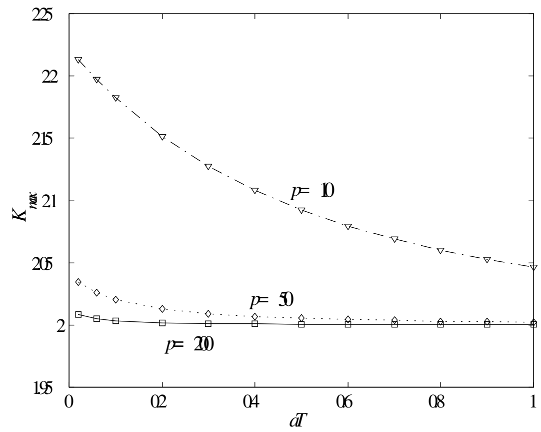

Fig. 4. True stability boundary of repetitive control applied to the first order system.

For underdamped systems $(0 < \zeta < 1)$:

$$
\begin{aligned}
G_c(z) &= K_c(A_1 z + B_1)/\big(z^2 - 2e^{-\alpha_1}c\beta_1 z + e^{-2\alpha_1}\big),\\
A_1 &= 1 - e^{-\alpha_1}c\beta_1 - (\alpha_1/\beta_1)e^{-\alpha_1}s\beta_1\\
&\quad + \big(\gamma^2\delta/\beta_1\big)s\beta_1,\\
B_1 &= e^{-2\alpha_1} + (\alpha_1/\beta_1)e^{-\alpha_1}s\beta_1 - e^{-\alpha_1}c\beta_1\\
&\quad - \big(\gamma^2\delta/\beta_1\big)s\beta_1,\\
\alpha_1 &= \zeta\omega_0 T, \quad \beta_1 = \omega_0 T\sqrt{1-\zeta^2},\\
\gamma &= \omega_0 T, \quad \delta = d/T.
\end{aligned}
\tag{10}
$$

For overdamped systems $(\zeta > 1)$:

$$
\begin{aligned}
G_c(z) &= K_c(A_2 z + B_2)/\left[(\alpha_2 - \beta_2)\left(e^{\alpha_2+\beta_2}z^2\right.\right.\\
&\quad \left.\left. -\big(e^{\alpha_2} + e^{\beta_2}\big)z + 1\right)\right],\\
A_2 &= (\beta_2\delta - 1)\alpha_2 e^{\alpha_2} + (1 - \alpha_2\delta)\beta_2 e^{\beta_2}\\
&\quad + (\alpha_2 - \beta_2)e^{\alpha_2+\beta_2},\\
B_2 &= (\beta_2\delta - 1)\alpha_2 e^{\beta_2} + (1 - \alpha_2\delta)\beta_2 e^{\alpha_2} + \alpha_2 - \beta_2,\\
\alpha_2 &= \omega_0 T\left(\zeta + \sqrt{\zeta^2 - 1}\right),\\
\beta_2 &= \omega_0 T\left(\zeta - \sqrt{\zeta^2 - 1}\right).
\end{aligned}
\tag{11}
$$

For critically damped systems $(\zeta = 1)$:

$$
\begin{aligned}
G_c(z) &= K_c(A_3 z + B_3)/\big(e^{2\gamma}z^2 - 2e^{\gamma}z + 1\big),\\
A_3 &= \big(\gamma^2\delta - \gamma - 1\big)e^{\gamma} + e^{2\gamma},\\
B_3 &= 1 + \big(\gamma - \gamma^2\delta - 1\big)e^{\gamma}.
\end{aligned}
\tag{12}
$$

These $G_c(z)$ are used in (1) for the approximate monotonic decay condition. The conversion of $G_c(z)$ to $G(z)$ for use in (4) is analogous to Eqns. (5) and (6).

True Stability Boundary for Learning Control: For the $G_c(z)$ above for the underdamped, overdamped and critically damped cases, we can convert the associated second order scalar difference equation into a state variable form, and substitute into the ILC stability condition (2) to obtain, respectively,

$$0 < K < 2/A_1,$$

$$0 < K < 2(\alpha_2 - \beta_2)e^{\alpha_2+\beta_2}/A_2, \tag{13}$$

$$0 < K < 2\big/\big(A_3 e^{-2\omega_0 T}\big). \tag{14}$$

The results are shown in Fig. 5 for the case $d = 0$, where again the range on K goes to infinity as the sample time T goes to zero. As a result, a figure is inserted to show the curves away from the zero singularity.

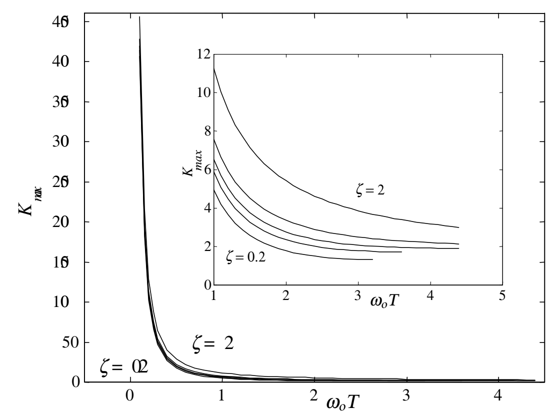

Fig. 5. Stability boundary of learning control applied to the second order system when $d/T = 0$ and $\zeta = 0.2, 0.5, 0.7, 1$, and 2 from left to right.

Approximate Monotonic Decay Condition: It is not possible to obtain for second order systems a simple analytical expression for the range of K as was done for the first order case. Figures 6 and 7 give the results for the cases of $d/T = 0$ and $d/T = 1$, respectively. Again K is limited by 2, the diameter of the unit circle that the plot of $K_l z G_c(z)$ should stay within. But this time it is not only along the real axis that one might start leaving this circle when the gain is too large, but when there is a resonant peak it can easily leave at some frequency other than zero. Hence, as ζ decreases, the range of K is made smaller and smaller. All results are, of course, independent of p, and the monotonic decay condition is vastly different than the true stability boundary in the case of ILC. In Figs. 6 and 7 as well as the figures that follow, the rectangle, triangle, circle, triangle on its point, and diamond correspond to damping rations ζ of 2, 1, 0.7, 0.5, and 0.2, respectively.

True Stability Boundary for Repetitive Control: Figures 8 and 9 give the true stability boundary using

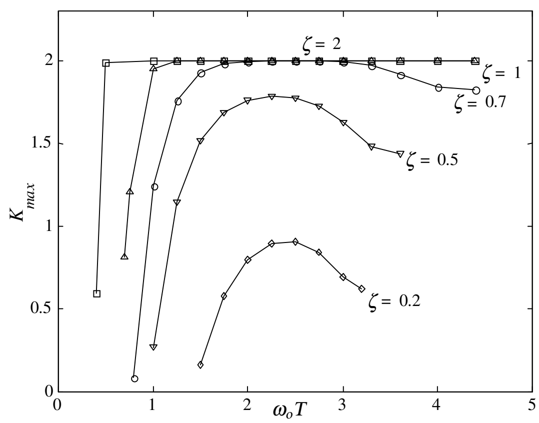

Fig. 6. Monotonic decay boundary of repetitive control applied to the second order system when $d/T = 0$.

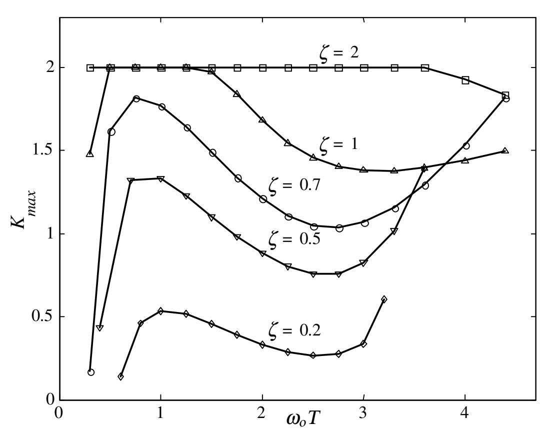

Fig. 7. Monotonic decay boundary of repetitive control applied to the second order system when $d/T = 1$.

Eqn. (4). In these figures as well as Figs. 12 and 13 below, there are three lines plotted for each, for $p = 10$, $p = 50$, and $p = 200$. On the left side of the plots and also on the top of the plots one can often distinguish these different curves, and then the curve for $p = 10$ is the left most curve or the top most curve, and $p = 50$ and $p = 200$ progress to the right or downward.

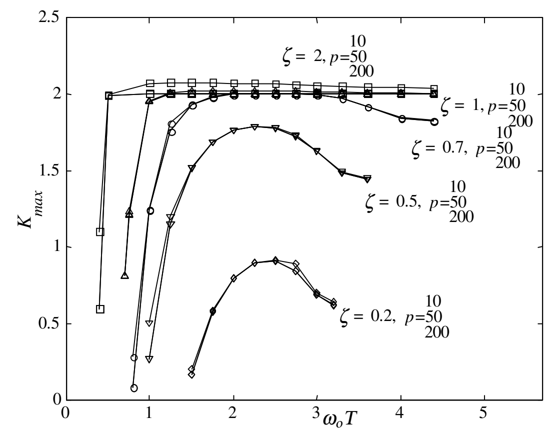

Fig. 8. True stability boundary of repetitive control applied to the second order system when $d/T = 0$.

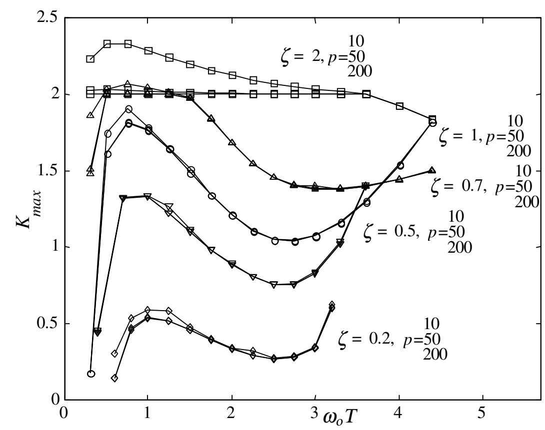

Fig. 9. True stability boundary of repetitive control applied to the second order system when $d/T = 1$.

6. Stability of Third Order Systems

The same procedure is applied to third order systems of the form

$$G_s(s) = K_c \left(\frac{d}{s+d}\right)\left(\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}\right), \quad (15)$$

which represents a general third order system except that it is restricted to having no zero in continuous time. After discretizing, the parameters are K, $\omega_0 T$, dT, ζ, and p.

The equations involved are much more complicated than those of the second order system. The discrete time version of this transfer function fed by a zero order hold takes the form

$$G_{c3}(z) = K_c \frac{Az^2 + Bz + D}{Ez^3 + Fz^2 + Gz + H}, \quad (16)$$

where the coefficients are given as follows, depending on whether the second order term is underdamped, critically damped, or overdamped (the case of three repeated real roots is not considered).

For underdamped systems $(0 < \zeta < 1)$:

$$\begin{aligned} A = {} & \left(2\gamma_{111}e^{\alpha+\delta} - \gamma_{012}e^{\alpha+\delta}\right)c\beta \\ & + \left(\gamma_{201}e^{\alpha+\delta} - \gamma_{102}e^{\alpha+\delta} - \gamma_{021}e^{\alpha+\delta}\right)s\beta \\ & - (\gamma_{030} + \gamma_{210})e^{2\alpha} + (\gamma_{030} + \gamma_{012} - \gamma_{111})e^{2\alpha+\delta}, \end{aligned}$$

$$
\begin{aligned}
B = {} & \big((2\gamma_{030} + 2\gamma_{210} - 2\gamma_{111} + \gamma_{012})e^{\alpha} \\
& - (2\gamma_{030} + 2\gamma_{210} + 2\gamma_{111} - \gamma_{012})e^{\alpha+\delta}\big)c\beta \\
& + \big((\gamma_{102} + \gamma_{021} - \gamma_{201})e^{\alpha} \\
& + (\gamma_{102} + \gamma_{021} - \gamma_{201})e^{\alpha+\delta}\big)s\beta \\
& + (2\gamma_{111} - \gamma_{012})e^{2\alpha} + (\gamma_{012} - 2\gamma_{111})e^{\delta}, \qquad (17)\\
D = {} & (\gamma_{012} - 2\gamma_{111})e^{\alpha}c\beta + (\gamma_{201} - \gamma_{102} - \gamma_{021})e^{\alpha}s\beta \\
& + (\gamma_{030} + \gamma_{012})e^{\delta} + (2\gamma_{111} - \gamma_{030} - \gamma_{210} - \gamma_{012}),\\
E = {} & \beta\big((\alpha-\delta)^2 + \beta^2\big)e^{2\alpha+\delta},\\
F = {} & -\beta e^{2\alpha}\big((\alpha-\delta)^2 + \beta^2\big) \\
& - 2\beta\big((\alpha-\beta)^2 + \delta^2\big)e^{\alpha+\delta}c\beta,\\
G = {} & \beta\big((\alpha-\delta)^2 + \beta^2\big)e^{\alpha} + 2\beta\big((\alpha-\delta)^2 + \beta^2\big)c\beta,\\
H = {} & -\beta\big((\alpha-\delta)^2 + \beta^2\big),\\
\alpha = {} & \zeta\omega_0 T, \quad \beta = \omega_0 T\sqrt{1-\zeta^2},\\
& \delta = dT \quad \text{and} \quad \gamma_{ijk} = \alpha^i \beta^j \delta^k.
\end{aligned}
$$

For overdamped systems $(\zeta > 1)$:

$$
\begin{aligned}
A = {} & (\gamma_{210} - \gamma_{120})e^{\alpha+\beta} + (\gamma_{102} - \gamma_{201})e^{\alpha+\delta} \\
& + (\gamma_{021} - \gamma_{012})e^{\beta+\delta} - (\alpha-\beta) \\
& \times (\alpha-\delta)(\beta-\delta)e^{\alpha+\beta+\delta},\\
B = {} & (\gamma_{210} - \gamma_{120})\big(e^{\delta} - 1\big)\big(e^{\alpha} + e^{\beta}\big) \\
& + (\gamma_{102} - \gamma_{201})\big(e^{\beta} - 1\big)\big(e^{\alpha} + e^{\delta}\big) \\
& + (\gamma_{021} - \gamma_{012})\big(e^{\alpha} - 1\big)\big(e^{\beta} + e^{\delta}\big),\\
D = {} & (\gamma_{012} - \gamma_{021})\big(e^{\alpha} - 1\big) + (\gamma_{201} - \gamma_{102}) \qquad (18)\\
& \times \big(e^{\beta} - 1\big) + (\gamma_{120} - \gamma_{210})\big(e^{\delta} - 1\big),\\
E = {} & (\alpha-\beta)(\alpha-\delta)(\beta-\delta)e^{\alpha+\beta+\delta},\\
F = {} & (\alpha-\beta)(\alpha-\delta)(\beta-\delta) \\
& \times \big(e^{\alpha+\beta} + e^{\alpha+\delta} + e^{\beta+\delta}\big),\\
G = {} & -(\alpha-\beta)(\alpha-\delta)(\beta-\delta)\big(e^{\alpha} + e^{\beta} + e^{\delta}\big),\\
H = {} & (\alpha-\beta)(\alpha-\delta)(\beta-\delta),\\
\alpha = {} & \omega_0 T\left(\zeta + \sqrt{\zeta^2 - 1}\right),\\
\beta = {} & \omega_0 T\left(\zeta - \sqrt{\zeta^2 - 1}\right),\\
& \delta = dT \quad \text{and} \quad \gamma_{ijk} = \alpha^i \beta^j \delta^k.
\end{aligned}
$$

For critically damped systems $(\zeta = 1)$:

$$
\begin{aligned}
A = {} & \big(e^{\delta}\big((\alpha-\delta)^2 e^{\alpha} - \delta(\delta + \alpha(\delta - \alpha - 2))\big) \\
& - \alpha^2 e^{\alpha}\big)\, e^{\alpha},\\
B = {} & -e^{\alpha}\big(\delta - 2 + (\delta+2)e^{\delta}\big)\alpha^2 \\
& - \big(e^{\alpha} - 1\big)\big(e^{\alpha} + e^{\delta}\big)\delta^2 \\
& + \alpha\left(2\big(e^{\alpha} - 1\big)\big(e^{\alpha} + e^{\delta}\big) + \delta e^{\alpha}\big(e^{\delta} + 1\big)\right)\delta,\\
D = {} & e^{\delta}\alpha^2 - (\alpha-\delta)^2 + \delta e^{\alpha}\big((\alpha-\delta)\alpha - 2\alpha + \delta\big), \qquad (19)\\
E = {} & e^{2\alpha+\delta}(\alpha-\delta)^2,\\
F = {} & -\big(e^{2\alpha} + 2e^{\alpha+\delta}\big)(\alpha-\delta)\big)^2,\\
G = {} & \big(2e^{\alpha} + e^{\delta}\big)(\alpha-\delta)^2,\\
H = {} & -(\alpha-\delta)^2,\\
\alpha = {} & \omega_0 T \quad \text{and} \quad \delta = dT.
\end{aligned}
$$

Figures 10 and 11 give the monotonic decay condition (1) results, and Figs. 12 and 13 give the true stability boundary. The implications of these plots is similar to the second order case. All points plotted in Figs. 12 and 13 have both of the zeros that are introduced by the discretization within the unit circle. Going beyond the plotted points on the left causes one of the zeros to go outside the unit circle, and $K_{\max}$ goes to zero. Such a zero corresponds to a pole in Q, and going outside the unit circle changes P in Eqn. (4) without changing anything else, and hence the system becomes unstable. The values of these variables in applications will normally be off the lower end of the plot in this manner. An example is the third order model of the command to response for each link of a Robotics Research Corporation robot as dis-

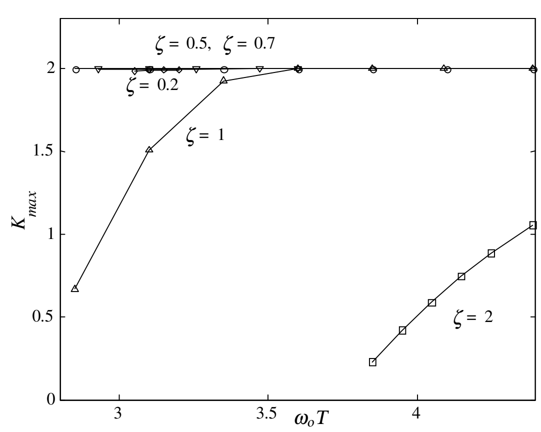

Fig. 10. Monotonic decay boundary of repetitive control applied to the third order system when $dT = 0.5$.

cussed in Elci *et al.* (1994). For this robot the sample rate was 400 Hz for the feedback control loops and substantially slower when considering the inverse kinematics updates from the upper level controller, and $d = 8.8$ so that

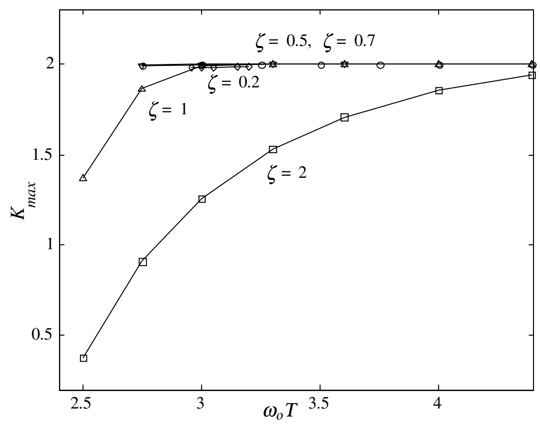

Fig. 11. Monotonic decay boundary of repetitive control applied to the third order system when $dT = 1$.

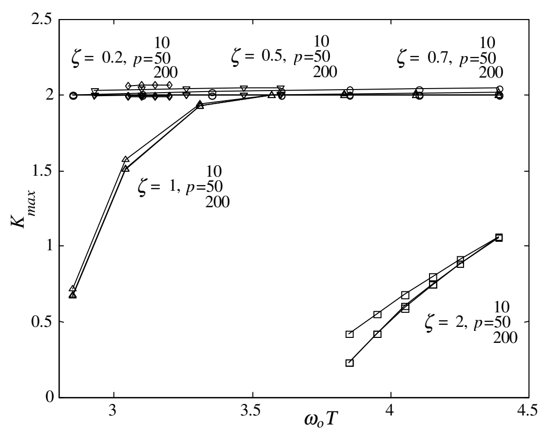

Fig. 12. True stability boundary of repetitive control applied to the third order system when $dT = 0.5$.

the product can be near 1 as in Fig. 13, and $\omega_0 T = 0.37$. This is far off the left side of the plot after the maximum learning gain for stability has become zero. To handle such situations, one can employ a compensator and a zero phase low pass filter as is done in (Elci *et al.*, 1994; Longman, 2000).

7. Conclusions

This paper has shown that the approximate monotonic decay condition (1), a sufficient condition for stability, is sufficiently close to the stability boundary for repetitive control that in practical applications one should aim to satisfy

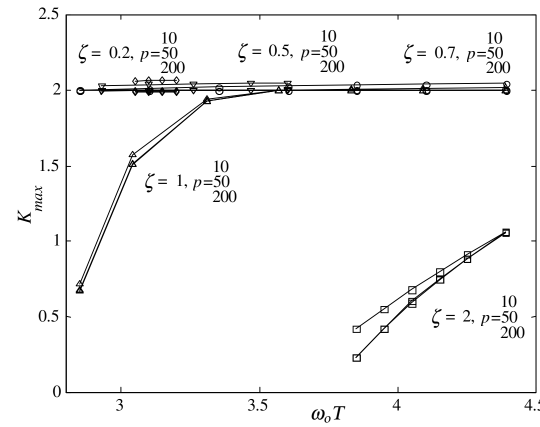

Fig. 13. True stability boundary of repetitive control applied to the third order system when $dT = 1$.

it and ignore the true stability boundary condition. This approximate condition is a sufficient condition for stability, and differs from the true stability boundary substantially only for very small p, and for first order systems. Otherwise the difference becomes negligible. Determining the true stability boundary is difficult in repetitive control because of the large number of roots in the characteristic equation, equal to the order of the system plus p, the number of time steps in a period, and this can easily give thousands of roots. Hence, methods such as the Jury test and the Routh stability criterion with a bilinear transformation are useless. The standard application of the Nyquist criterion would require using extra small contour arcs going around the p poles on the unit circle, and again this is normally unmanageable. Here we make use of the method of Pierre (1989) to get around this problem, but it can still be somewhat difficult. On the other hand, the approximate stability boundary is independent of the number of time steps p in a period, and this makes the testing of this condition quite easy. It is also a desirable property to have guaranteed stability regardless of the period of the desired trajectory, or of the periodic disturbance being cancelled.

For ILC there is a very big difference between the approximate monotonic decay condition (1) and the true stability boundary. It is easy to satisfy the true stability boundary condition that is almost independent of the system dynamics. The range of stabilizing learning gains tends to infinity as the sample time tends to zero. But it is hard to satisfy the approximate monotonic decay condition (1), which depends heavily on the system dynamics. Nevertheless, it is very important in ILC to ensure good learning transients, and as suggested in Elci *et al.* (1994), satisfying (1) is perhaps the simplest approach for doing this.

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