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GENERATION OF GRAY CODES THROUGH THE ROUGH IDENTITY–SUMMAND GRAPH OF FILTERS OF A ROUGH BI–HEYTING ALGEBRA

BASHYAM PRABA [a,*], LOURDUSAMY PACKIAMMAL ANTO FREEDA [a]

[a]Department of Mathematics
Sri Sivasubramaniya Nadar College of Engineering
Rajiv Gandhi Salai (OMR), Kalavakkam – 603 110 Chennai, Tamil nadu, India
e-mail: prabab@ssn.edu.in

This paper introduces the concept of filters in a rough bi-Heyting algebra. The rough bi-Heyting algebra defined through the rough semiring offers interesting properties. Filters on this rough bi-Heyting algebra are to be described in terms of the R-upset. Then a one-to-one correspondence between the filters, the principle ideal and R-upsets is established. Various filters are characterized on this rough bi-Heyting algebra. For each filter, a rough identity-summand graph is constructed. This rough identity-summand graph is proved to be a complete bipartite graph in certain cases involving pairs of elements. When more than two elements are involved, a rough identity-summand graph exists and generates multiple complete bipartite graphs. The number of distinct complete bipartite graphs generated from this graph is defined to be an RBP number. The union of these distinct complete bipartite graphs forms a subgraph of the rough identity-summand graph. Additionally, this study demonstrates how two transition sequences obtained from the distinct complete bipartite graphs of the rough identity-summand graph can be utilized to generate Gray codes, making a substantial contribution.

Keywords: Heyting algebra, filter, identity-summand graph, complete bipartite graph, Gray code.

1. Introduction

Rough set theory introduced by Pawlak (1982) has been successfully applied in many fields. The idea behind this theory is to approximate the information that is not accurately described. The most important rough set applications received extensive attention in various fields. In rough set theory, for an approximation space $I = (U, R)$, lower and upper approximations are defined for the subsets of U. This is achieved by introducing a partition on U, forming the equivalence classes by an arbitrary equivalence relation R. To expand the scope of rough set theory in various applications, defining the equivalence relation on U is extended to an arbitrary equivalence relation. This is a more general and broader way of considering the relationship between the objects in U. This leads to relaxing the constraints over the elements in the universe U imposed by R. In this paper, the arbitrary equivalence relation R defined on U induces the equivalence classes. Then for every $X \subseteq U$, the rough set is defined through its lower and upper approximations. The set of rough sets formed is denoted by T and was proved to be a rough lattice by Praba and Mohan (2013). Then the semiring structure on T was described under the operations Praba Δ and Praba ∇ by Praba *et al*. (2015) and later its characterization was given by Manimaran *et al*. (2017). The ideals of rough semiring (T, Δ, ∇) are defined by Chandrasekaran *et al*. (2017), and then the principal ideals concerning Praba Δ and Praba ∇ are extended by Praba *et al*. (2025). This rough semiring (T, Δ, ∇) was taken as an underlying structure in defining rough bi-Heyting algebra $(T, \Delta, \nabla, ^*, ^+ \rightarrow, \leftarrow, RS(\emptyset), RS(U))$ in (Praba and Freeda, 2022). The rough bi-Heyting algebra in (Praba and Freeda, 2022) is defined by characterizing each weaker complement on a rough semiring (T, Δ, ∇). The construction of the rough bi-Heyting algebra from the pseudocomplement and relative pseudocomplement along with their duals has many associations with lattices and Boolean algebras (Halmos and Givant, 2009).

On the other hand, the graph theoretical concepts

*Corresponding author

of West (2001) and Bondy and Murthy (1976) widened their research area in algebraic structures such as rings, ideals, and lattices. The association of graphs with rings and the generalization of their structures have become an increased research topic in various disciplines. The application of semirings which are a powerful tool in studying various languages in theoretical computer science, further extended to graph theoretical background. Combining graph and algebraic structures was initiated on rings by defining a zero-divisor graph. A zero-divisor graph involves both graph and ring properties.

Various graph structures that are associated with the algebraic structures include identity-summand graphs as well. An identity-summand graph is a graph whose vertices are non-identity elements and two distinct vertices are adjacent if their sum is 1 (Atani *et al.*, 2015b). The study of co-ideals of a commutative semiring $(R, +, \cdot)$ (Atani *et al.*, 2015b; Atani *et al.*, 2014; Atani *et al.*, 2015a) verifies the co-ideal properties using $*$ and $+$. Then the primary properties and basic structures of the identity-summand graph denoted by $\Gamma(R)$ are discussed by Atani *et al.* (2014). Later the results on identity-summand graph $\Gamma_I(R)$ being a bipartite graph and their planar properties are provided by Atani *et al.* (2015b). Then, the total graph for the co-ideals of a commutative semiring were established by Atani *et al.* (2015a). The filter (co-ideals) based identity-summand graph of a lattice L (Ebrahimi Atani *et al.*, 2023; Atani *et al.*, 2018) verifies the filter and graph-theoretic properties for $\vee$ and $\wedge$. In (Atani *et al.*, 2018), the properties of filter F and the basic structure of $\Gamma_F(L)$ are given. Then the verification of planarity of $\Gamma_F(L)$ for any filter F of a lattice is provided. Finally, Ebrahimi Atani *et al.* (2023), characterized the planarity of $\Gamma_F(L)$.

Gray codes are a unique type of binary coding sequence sometimes referred to as reflected binary codes in which there is a single bit difference between neighboring values. It is named after Frank Gray, who got a patent for the binary reflected code in 1953. The characteristic of the Gray code sequence which has only one bit of difference between neighboring numbers is very important in applications where it is necessary to minimize errors due to multiple-bit changes. A list of binary words, each of length n, with only one-bit position differentiating subsequent words forms the Gray code of length n. The Gray code is said to be cyclic if both the initial and final words indicate this characteristic; otherwise, it is not cyclic. The transition sequence is made up of the transition numbers, which are the points at which subsequent code words differ. Vertices labeled from 1 to n form the induced graph produced by the Gray code. In the transition sequence of the Gray code, two vertices are connected if and only if their corresponding bit positions are successive transitions.

The graph structure captures the relationships between bit places in the Gray code. Suparta (2017) investigates bipartite graphs induced by Gray codes and proves that certain kinds of bipartite graphs induced by Gray codes are complete bipartite graphs. The construction of an n-bit Gray code that induces the complete graph K_n is the problem addressed by Wilmer and Ernst address (2002). They propose a building technique to generate an n-bit Gray code, $G(n)$, such that the complete graph K_n is induced by the transition sequence $S(n)$ of this Gray code. The main finding in (Suparta and Van Zanten, 2008) is that the complete graph K_n is induced by the generated Gray code $G(n)$. This indicates that the edges in the graph $\mathfrak{g}_{(G(n))}$ correspond to the edges in K_n. Also, the transition sequence of a Gray code is defined as $S(n) = S(n-1), n, S(n-1), n \geq 2$.

In this paper, the connection between the above-mentioned concepts will be in the realm of rough of bi-Heyting algebra. A mathematical structure of bi-Heyting algebra combines the aspects of both Heyting algebras and Boolean algebras in nature. The generalization of a Boolean algebra by two weaker complements, namely pseudocomplement and dual pseudocomplement, a gives rise to a bi-Heyting algebra. Bi-Heyting algebras satisfy both intuitionistic and classical logical operations except the law of excluded middle which holds for classical logic and does not hold for intuitionistic logic. The study by Yao (1998) offers insight into the many approaches used to investigate and characterize rough sets in the literature by highlighting the differences between the two methodologies: constructive, which is based on binary relations, and algebraic, which is based on axiomatization. To think about finite growing sequences over Boolean algebras, SanJuan (2008) offers an algebraic formalism. This formalism is mainly focused on generalizations of rough set notions, with a particular focus on how well they can be applied to representing document relevance in an information retrieval system. As a generalization of symmetric Heyting algebras of order n and an extension of T-rough Heyting algebras, Gallardo *et al.* (2023) introduce a new class of algebras called T-rough symmetric Heyting algebras.

This article presents and investigates filters of rough bi-Heyting algebras through rough semiring (T, Δ, ∇), adding to the theoretical foundations of rough sets. The idea is to understand how these filters behave within this rough bi-Heyting algebras. For any $X \subseteq U$, the filter $F_X(T)$ is a subset of T and the R-upset $(RS(X))$ is proved to be a filter $F_X(T)$. Therefore, for defining the filters of a rough bi-Heyting algebra, the binary operations Praba Δ and Praba ∇ are introduced over $F_X(T)$ to validate the filter conditions. Then the characterization of filters is given in terms of the R-upset. The principle ideal for Δ defined by Chandrasekaran *et al.* (2017) correlates with the filter $F_X(T)$.

Various filters such as a proper filter and a prime filter are defined, and their characterizations are also provided. The idea of associating filters with graph structures provides a way to represent knowledge hierarchies and relationships in the approximation space. Also, the graph structures derived from filters are used to represent and visualize complex relationships among entities in a knowledge base. Filters play a crucial role in various algebraic and logical systems. The extensive study of identity-summand graphs of lattices (Ebrahimi Atani *et al*., 2023; Atani *et al*., 2018) and identity-summand graphs of commutative semirings (Atani *et al*., 2015b; Atani *et al*., 2014; Atani *et al*., 2015a) helped to define $G(F_X(T))$ for each filter of a rough bi-Heyting algebra $(T, \Delta, \nabla, ^*, ^+, \rightarrow, \leftarrow, RS(\emptyset), RS(U))$. The existence of $G(F_X(T))$ for the proper filters and prime filters of T are verified in this study.

In general, this means the existence of $G(F_X(T))$ for the filter $F_X(T)$ when $X = \{z_1, z_2, \dots, z_\rho\}$, where $z_1, z_2, \dots, z_\rho$ are x_i (or) X_j for $i, j \in \{1, 2, \dots, m, m+1, \dots, n\}$ and $2 \le \rho \le m, n$ are extended.

The rough identity-summand graph $G(F_X(T))$ obtained for any $X \subseteq U$, generates various complete bipartite graphs. However, the results on distinct complete bipartite graphs generated from $G(F_X(T))$ whose union is the subgraph of $G(F_X(T))$ are discussed in detail. For any X, the vertex cardinality of the distinct complete bipartite graph of $G(F_X(T))$ is considered in finding two transition sequences that generate Gray codes. The significance of defining the rough identity-summand graph is highlighted in this paper by deriving Gray codes of length k. Although filters are established through various algebraic structures, their characterization and behavior within the rough bi-Heyting algebra particularly through the rough semiring (T, Δ, ∇) remain under-explored.

This research intends to deepen the understanding of the structure filter and its application, which has the potential to represent complex knowledge hierarchies that have not been thoroughly examined so far. The identity-summand graph of lattices (Ebrahimi Atani *et al*., 2023; Atani *et al*., 2018) and the identity-summand graph of a commutative semiring (Atani *et al*., 2015b; Atani *et al*., 2014; Atani *et al*., 2015a) for the co-ideals extend the graph structure from a complete 2-partite graph to a complete r-partite graph and characterize their planarity. In contrast, this paper uniquely focuses on generating complete bipartite graphs from $G(F_X(T))$. Furthermore, it explores the distinct complete bipartite graphs whose union forms the subgraph of $G(F_X(T))$, which is a perspective not addressed in the existing literature.

Various works provide insight into our proposed method. In (Liu *et al*., 2021), complete bipartite graphs analyze the coherence and stability of the networks, which is discussed under the Laplacian matrix. Liu and Pan (2016) offer insights into how to balance connectivity and bipartiteness, which has applications in network design, electrical networks, and theoretical graph studies. Also, the paper focuses on graphs with a fixed number of vertices and vertex bipartiteness. Liu *et al*. (2024) derive precise expressions for consensus algorithms and investigate network properties like the Kirchhoff index. Applications of Gray codes go beyond particular fields, making them useful resources for information processing and optimization. Gray codes can communicate knowledge by encoding connections between various concepts (or) entities. They are useful for expressing relationships between two sets of elements (e.g., genes and functions, words and meanings) in a complete bipartite network. Additionally, there is a research gap on how these network structures generate Gray codes and their implication in optimization techniques. By offering a thorough analysis of filters in rough bi-Heyting algebras, their graph-theoretic representations, and their potential applications across various domains, this paper aims to fill these gaps.

Finally, this paper is organized as follows: In Section 2, some notation and basic concepts required are reviewed. In Section 3, we define filters of a rough bi-Heyting algebra and provide their characterization. Also, various filters of a rough bi-Heyting algebra, for any $X \subseteq U$ are characterized. In Section 4, we define the rough identity-summand graph $G(F_X(T))$ for the filters of a rough bi-Heyting algebra and discuss the nature of $G(F_X(T))$. The generation of Gray codes through the complete bipartite graph is discussed in Section 5. In Section 6, some conclusions and future work are given.

2. Preliminaries

A structure $I = (U, R)$ where U is a nonempty finite set of objects, called the universe, and R is an arbitrary equivalence relation on U, is called an approximation space. A partition induced by the relation R consists of an equivalence class denoted by $[x]_R$, which is a subset of U. The lower and upper approximations defined for the subset X of U are

$$R_-(X) = \{x \in U \mid [x]_R \subseteq X\},$$
$$R^-(X) = \{x \in U \mid [x]_R \cap X \neq \emptyset\}.$$

Definition 1. (*Pawlak, 1982*) If X is an arbitrary subset of U, then the *rough set* $RS(X)$ is an ordered pair $(R_-(X), R^-(X))$. The set of *rough sets* is denoted by T and defined by $T = \{RS(X) \mid X \subseteq U\}$.

Definition 2. (*Praba et al., 2015*) For the approximation space $I = (U, R)$, let $U = \bigcup_{r=1}^{n} X_r$ be the union of equivalence classes formed using the relation R and $T = \{RS(X)/X \subseteq U\}$ be the set of rough sets. Choose an element x_r, a representative element from an equivalence class X_r, whose cardinality is greater than 1. Then the pivot set $B = \{x_r \in X_r \mid |X_r| > 1\}$.

Definition 3. (*Praba and Freeda, 2022*) A rough bi-Heyting algebra $(T, \Delta, \nabla, ^*, ^+, \rightarrow, \leftarrow, RS(\emptyset), RS(U))$ is a rough semiring (T, Δ, ∇) with a rough Heyting algebra and a dual rough Heyting algebra.

Definition 4. (*Halmos and Givant, 2009*) An *atom* of a Boolean algebra is an element that has no nontrivial proper subelements, when q is an atom if $q \neq 0$ and if there are only two elements p such that $p \leq q$, namely 0 and q.

Definition 5. (*West et al., 2001*) The null graph is the graph whose vertex set and edge set are empty.

3. Characterization of filters of a rough bi-Heyting algebra

This section discusses an approximation space $I = (U, R)$ and its corresponding rough bi-Heyting algebra (Praba and Freeda, 2022).

Definition 6. For any $RS(X) \in T$, the *rough upset* of $RS(X)$ is a subset of T defined by

$$\text{R-upset}\,(RS(X)) = \{RS(Y) \in T \mid RS(X) \leq RS(Y)\}.$$

Definition 7. For any $RS(X) \in T$, the *rough downset* of $RS(X)$ is a subset of T defined by

$$\text{R-downset}\,(RS(X)) = \{RS(Y) \in T \mid RS(Y) \leq RS(X)\}.$$

Definition 8. For any $X \subseteq U$, the set $S \subseteq T$ is a *filter* of a rough bi-Heyting algebra if

(i) S is closed under ∇,

(ii) for $RS(Y) \in S$ and $RS(Z) \in T$, $RS(Y \Delta Z) \in S$.

Theorem 1. *R-upset* $(RS(X))$ *is a filter.*

Proof.
(i) Let $RS(Y_1), RS(Y_2) \in$ R-upset$(RS(X))$. Then

$$RS(X) \leq RS(Y_1), \tag{1}$$

$$RS(X) \leq RS(Y_2). \tag{2}$$

Now we prove $RS(Y_1 \nabla Y_2) \in$ R-upset$(RS(X))$. From (1) and (2) we get

$$RS(X \nabla X) \leq RS(Y_1 \nabla Y_2),$$
$$RS(X) \leq RS(Y_1 \nabla Y_2),$$

which yields $RS(Y_1 \nabla Y_2) \in$ R-upset$(RS(X))$.

Therefore, R-upset$(RS(X))$ is closed under ∇.

(ii) Let $RS(Y) \in$ R-upset$(RS(X))$ and $RS(Z) \in T$. Consider $RS(Y) \in$ R-upset$(RS(X))$. It follows that

$$RS(X) \leq RS(Y).$$

Now we prove $RS(Y \Delta Z) \in$ R-upset$(RS(X))$. It is clear that $RS(X) \leq RS(Y)$ and $RS(Y) \leq RS(Y \Delta Z)$, which implies

$$RS(X) \leq RS(Y \Delta Z).$$

Thus,

$$RS(Y \Delta Z) \in \text{R-upset}(RS(X))$$

Therefore, R-upset$(RS(X))$ is a filter. ■

Definition 9. For any $X \subseteq U$, define $F_X(T)$ by $F_X(T) = \{RS(Y) \mid Y = X \Delta V \Delta W, V \in P(E \setminus E_X), W \in P(B \setminus B_X)\}$, where E is the set of equivalence classes in U and E_X be the set of equivalence classes in X, B is the pivot set of representative elements of equivalence classes and B_X is the pivot set of representative elements in X of equivalence classes, whose cardinality is greater than 1.

Theorem 2. $F_X(T)$ *is a filter.*

Proof. Let $RS(X_1), RS(X_2) \in F_X(T)$. Then by definition

$$RS(X_1), RS(X_2) \in \{RS(Y) \mid Y = X \Delta V \Delta W, V \in P(E \setminus E_X), W \in P(B \setminus B_X)\}.$$

Therefore, when $RS(X_1) \leq RS(X_2)$,

$$RS(X_1 \nabla X_2) = RS(X_1) \in F_X(T)$$

and, if $RS(X_2) \leq RS(X_1)$,

$$RS(X_1 \nabla X_2) = RS(X_2) \in F_X(T).$$

Hence

$$RS(X_1 \nabla X_2) \in F_X(T).$$

$F_X(T)$ is closed under ∇.

Now, let $RS(X_1) \in F_X(T)$ and $RS(X_2) \notin F_X(T)$. We see that

$$RS(X_2) \in T \setminus F_X(T)$$

This gives

$$RS(X_2) \in \{RS(A) \mid A \in P(E \setminus R^-(X)) \Delta P(B \setminus B_X)\} \subseteq T \setminus F_X(T).$$

Consequently,

$$RS(X_1 \Delta X_2) \in \{RS(Z) \mid Z \in X \Delta P(E \setminus E_X) \Delta P(B \setminus B_X)\}.$$

Thus

$$RS(X_1 \Delta X_2) \in F_X(T).$$

Therefore, $F_X(T)$ is a filter. ■

Theorem 3. *R-upset*$(RS(X)) = F_X(T)$.

Proof. First we prove that $F_X(T) \subseteq$ R-upset$(RS(X))$. Let $RS(Y) \in F_X(T)$. Then $Y = X\Delta V\Delta W$, for $V \in P(E \setminus E_X)$ and $W \in P(B \setminus B_X)$. To prove $RS(Y) \in$ R-upset$(RS(X))$ means to prove $RS(X) \leq RS(Y)$. Since $Y = X\Delta V\Delta W$ and $RS(X) \leq RS(X\Delta V\Delta W)$, we get

$$RS(X) \leq RS(Y)$$

and then

$$RS(Y) \in \text{R-upset}(RS(X)).$$

Therefore,

$$F_X(T) \subseteq \text{R-upset}(RS(X)). \tag{3}$$

Now we show that R-upset$(RS(X)) \subseteq F_X(T)$. Let $RS(Y) \in$ R-upset$(RS(X))$. Then $RS(X) \leq RS(Y)$. To prove $RS(Y) \in F_X(T)$, it is enough to prove $Y = X\Delta V\Delta W$ for $V \in P(E \setminus E_X)$ and $W \in P(B \setminus B_X)$. From the hypothesis $RS(X) \leq RS(Y)$ we get

$$\begin{aligned} &RS(X)\Delta RS(Y) = RS(X\Delta Y) = RS(Y) \\ &\quad = RS(X\Delta(E_Y \setminus E_X \cap E_Y)\Delta(B_Y \setminus B_X \cap B_Y)). \end{aligned}$$

This forces

$$RS(Y) \in F_X(T).$$

Therefore,

$$\text{R-upset}(RS(X)) \subseteq F_X(T) \tag{4}$$

Thus, from (3) and (4), R-upset$(RS(X)) = F_X(T)$ ■

Remark 1. When $X = B$, $F_B(T) = \{RS(X) \in T \mid X = B\Delta N, N \in P(E)\}$, where E is the set of equivalence classes in U and the pivot set $B = \{x_r \in X_r \mid |X_r| > 1\}$.

Remark 2. R-upset$(RS(B)) = F_B(T)$.

Definition 10. For any subset X of U, the *principal ideal* generated by $RS(X)$ in T (with respect to Δ) is given by $RS(X)\Delta T = \{RS(Y) \mid Y \in X\Delta P(E\setminus E_X)\Delta P(\overline{P_X})\}$, where $\overline{P_X}$ is the complement of the pivot set P_X.

Theorem 4. *For any subset X of U, $F_X(T) = RS(X)\Delta T$.*

Proof. Let $RS(Y) \in F_X(T)$. Then

$$Y = X\Delta V\Delta W,$$

for $V \in P(E \setminus E_X)$ and

$$W \in P(B \setminus B_X), \quad Y = X\Delta(V\Delta W)$$

for $V \cup W \in P((E \setminus E_X) \cup (B \setminus B_X))$, with

$$RS(Y) = RS(X)\Delta RS(V\Delta W) \in RS(X)\Delta T,$$

$$F_X(T) \subseteq RS(X)\Delta T. \tag{5}$$

Now, let $RS(Y) \in RS(X)\Delta T$. Then $Y \in X\Delta P(E \setminus E_X)\Delta P(\overline{P_X})$ and we see that

$$Y = X\Delta V_1\Delta W_1 \text{ for } V_1 \in P(E \setminus E_X),$$

$$W_1 \in P(\overline{P_X}), \quad RS(Y) \in F_X(T).$$

Accordingly,

$$RS(X)\Delta T \subseteq F_X(T). \tag{6}$$

Hence, from (5) and (6), $F_X(T) = RS(X)\Delta T$. ■

Definition 11. A filter $F_X(T)$ is said to be to be *proper* if $F_X(T) \neq T$.

Theorem 5. (Characterization theorem for proper filters) *$F_X(T)$ is a proper filter iff $RS(X) \neq RS(\emptyset)$.*

Proof. Assume $F_X(T)$ is a proper filter. Then $F_X(T) \subset T$ implies

$$RS(X) \neq RS(\emptyset).$$

Conversely, assume that $RS(X) \neq RS(\emptyset)$. To prove that $F_X(T)$ is a proper filter, we must have $F_X(T) \subset T$. Suppose that $F_X(T) = T$. We get

$$RS(\emptyset) \in F_X(T),$$

which yields

$$RS(X) = RS(\emptyset),$$

a contradiction. Hence $F_X(T)$ is a proper filter. ■

Definition 12. A filter $F_X(T)$ of a rough bi-Heyting algebra is said to be *prime* if

(i) $F_X(T)$ is a proper filter,

(ii) $RS(Y\Delta Z) \in F_X(T)$ implies either $RS(Y) \in F_X(T)$ (or) $RS(Z) \in F_X(T)$.

Theorem 6. (Characterization theorem for prime filters) *$F_X(T)$ is a prime filter.*

Proof. Let $RS(Y) \in F_X(T)$. Then

$$\begin{aligned} F_X(T) = \{&RS(Y)|Y = X\Delta V\Delta W \\ &\text{for } V \in P(E \setminus E_X), W \in P(B \setminus B_X)\}. \end{aligned}$$

When $X = \emptyset$, we get $F_X(T) = T$, which implies that $F_X(T)$ is not proper (by Definition 11).

Suppose that when $X \neq \emptyset$, there exists $RS(Y) \in F_X(T)$ and let $Y = Y_1\Delta Y_2$. We get

$$RS(Y_1\Delta Y_2) \in F_X(T),$$

which implies

$$RS(Y_1)\Delta RS(Y_2) \in F_X(T).$$

Hence

$$RS(Y_1) \in F_X(T) \text{ (or) } RS(Y_2) \in F_X(T).$$

Therefore, $F_X(T)$ is a prime filter. ■

4. Rough identity-summand graph of filters of a rough bi-Heyting algebra

In Section 3, the filters of the rough bi-Heyting algebra are defined. The identity-summand graph of lattices (Ebrahimi Atani *et al.*, 2023; Atani *et al.*, 2018) and the commutative semiring from (Atani *et al.*, 2015b; Atani *et al.*, 2014; Atani *et al.*, 2015a) are also integrated to define the rough identity-summand graph $G(F_X(T))$ for the filters of a rough bi-Heyting algebra. The results on the graph $G(F_X(T))$ obtained for the various filters of T are discussed in this section.

Definition 13. Let

$$(T, \Delta, \nabla, ^*, ^+, \rightarrow, \leftarrow, RS(\emptyset), RS(U))$$

be a rough bi-Heyting algebra and $X \subseteq U$. A *rough identity-summand graph* of filter $F_X(T)$, denoted by $G(F_X(T))$, is a graph whose vertex set is $V(F_X(T)) = \{RS(Y) \in T \setminus F_X(T)|$ for some $RS(Z) \in T \setminus F_X(T), RS(Y)\Delta RS(Z) = RS(Y\Delta Z) \in F_X(T)\}$ and the edge between the vertices of $RS(Y)$ and $RS(Z)$ exists if and only if $RS(Y\Delta Z) \in F_X(T)$.

Definition 14. An element $RS(X) \in T$ of a rough bi-Heyting algebra is said to be to be an *atom* if there exists $RS(Y) \in T$ and, if $RS(Y) \leq RS(X)$, then $RS(Y) = RS(\emptyset)$.

Remark 3. An atom of a rough bi-Heyting algebra is defined by

$$\begin{aligned} A(T) = \{&RS(\{x_1\}), RS(\{x_2\}), \dots, \\ &RS(\{x_m\}), RS(X_{m+1}), \\ &RS(X_{m+2}), \dots, RS(X_n)\}. \end{aligned}$$

Theorem 7. *The rough identity-summand graph $G(F_X(T))$ is a null graph if $RS(X)$ is an atom.*

Proof. Let $RS(X)$ be an atom. Then $RS(X) \in A(T)$. To prove that $G(F_X(T))$ is a null graph, assume that $RS(Y_1), RS(Y_2) \in V(F_X(T))$. We have

$$\begin{aligned} &RS(Y_1), RS(Y_2) \in T \setminus F_X(T) \\ &\Rightarrow RS(Y_1), RS(Y_2) \in T \setminus \{RS(Y) \mid Y \\ &\quad = X\Delta V\Delta W, V \in P(E \setminus E_X), W \in P(B \setminus B_X)\} \\ &\quad = \{RS(Z) | Z \in P(E \setminus R^-(X))\Delta P(B \setminus B_X)\} \\ &\Rightarrow RS(Y_1), RS(Y_2) \in \{RS(Z)| \\ &\quad Z \in P(E \setminus R^-(X))\Delta P(B \setminus B_X)\} \\ &\Rightarrow RS(Y_1\Delta Y_2) \notin F_X(T) \\ &\Rightarrow V(F_X(T)) = \emptyset. \end{aligned}$$

Therefore, $G(F_X(T))$ is a null graph. ∎

Remark 4. For Theorem 7, the converse is not true. Indeed, if $G(F_X(T))$ is a null graph, then $RS(X)$ need not be an atom.

Example 1. To verify Remark 4, consider $I = (U, R)$ as an approximation space, where the universal set $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and an arbitrary equivalence relation R induces the equivalence classes $X_1 = \{x_1, x_3\}$, $X_2 = \{x_2, x_4, x_6\}$ and $X_3 = \{x_5\}$ on U.

Let $X = \{x_1, x_3\} \subseteq U$. Then the filter for $X = \{x_1, x_3\}$ is

$$\begin{aligned} &F_{\{x_1,x_3\}}(T) \\ &\quad = \{RS(X_1), RS(X_1 \cup \{x_2\}), \\ &\qquad RS(X_1 \cup X_3), RS(X_1 \cup X_2), \\ &\qquad RS(X_1 \cup \{x_2\} \cup X_3), RS(U)\} \end{aligned}$$

and

$$\begin{aligned} &T \setminus F_{\{x_1,x_3\}}(T) \\ &\quad = \{RS(\emptyset), RS(\{x_1\}), RS(\{x_2\}), \\ &\qquad RS(\{x_1\} \cup \{x_2\}), RS(X_2), RS(X_3), \\ &\qquad RS(\{x_1\} \cup X_2), RS(\{x_1\} \cup X_3), \\ &\qquad RS(\{x_2\} \cup X_3), RS(X_2 \cup X_3), \\ &\qquad RS(\{x_1\} \cup \{x_2\} \cup X_3), \\ &\qquad RS(\{x_1\} \cup X_2 \cup X_3)\}. \end{aligned}$$

There is no $RS(Y), RS(Z) \in T \setminus F_{\{x_1,x_3\}}(T)$ such that $RS(Y\Delta Z) \in F_{\{x_1,x_3\}}(T)$. This forces

$$V(F_{\{x_1,x_3\}}(T)) = \emptyset.$$

Thus $G(F_{\{x_1,x_3\}}(T))$ is a null graph, but $RS(X_1)$ is not an atom. ♦

Theorem 8. *The rough identity-summand graph $G(F_X(T))$ is a null graph if $X = x_i$ (or) X_j for $i = 1, \dots m$ and $j = 1, \dots, n$.*

Proof. We consider two cases: when $X = x_i$ (or) X_j for $i = 1, \dots m$ and $j = 1, \dots n$.

Case 1. Suppose $X = x_i$ for $i = 1, \dots, m$. Then $E_X = \emptyset$ and $B_X = \{x_i\}$. Assume $RS(Y) \in V(F_{\{x_i\}}(T))$. Then

$$\begin{aligned} &RS(Y) \in T \setminus F_{\{x_i\}}(T) \\ &\Rightarrow RS(Y) \in T \setminus \{RS(Z) | Z \in \{x_i\} \\ &\quad \Delta P(E)\Delta P(B \setminus \{x_i\})\} \\ &= \{RS(A) | A \in P(E \setminus \{X_i\})\Delta P(B \setminus \{x_i\})\} \\ &\Rightarrow RS(Y) \in \{RS(A)| A \\ &\quad \in P(E \setminus \{X_i\})\Delta P(B \setminus \{x_i\})\}. \end{aligned}$$

Let $RS(Y) = RS(Y_1\Delta Y_2)$, where $Y_1 \in P(E \setminus \{X_i\})$ and $Y_2 \in P(B \setminus \{x_i\})$. Now let some $RS(Z) \in T \setminus F_{\{x_i\}}(T)$. Then $RS(Z) = RS(Z_1\Delta Z_2)$, where $Z_1 \in P(E \setminus \{X_i\})$ and $Z_2 \in P(B \setminus \{x_i\})$. Therefore,

$$\begin{aligned} RS(Y)\Delta RS(Z) &= RS(Y_1\Delta Y_2)\Delta RS(Z_1\Delta Z_2) \\ &= RS(Y_1\Delta Z_1)\Delta RS(Y_2\Delta Z_2). \end{aligned}$$

Since $Y_1, Z_1 \in P(E \setminus \{X_i\})$ and $Y_2, Z_2 \in P(B \setminus \{x_i\})$, we get

$$\begin{aligned} Y_1 \Delta Z_1 &\in P(E \setminus \{X_i\}) \\ &\Rightarrow Y_2 \Delta Z_2 \in P(B \setminus \{x_i\}) \\ &\Rightarrow Y \Delta Z \in P(E \setminus \{X_i\}) \Delta P(B \setminus \{x_i\}) \\ &\Rightarrow RS(Y \Delta Z) \notin F_{\{x_i\}}(T) \\ &\Rightarrow RS(Y) \notin V(F_{\{x_i\}}(T)), \end{aligned}$$

$$V(F_{\{x_i\}}(T)) = \emptyset,$$

which means that $G(F_{\{x_i\}}(T))$ is a null graph.

Case 2. Suppose $X = X_j$ for $j = 1, \ldots, n$. Then $B_X = \emptyset$ and $E_X = \{X_j\}$. Assume that $RS(Y) \in V(F_{\{X_j\}}(T))$. We get

$$\begin{aligned} &RS(Y) \in T \setminus F_{\{X_j\}}(T) \\ &\Rightarrow RS(Y) \in T \setminus \{RS(Z) | Z \in \{X_j\} \\ &\quad \Delta P(E \setminus \{X_j\}) \Delta P(B)\} \\ &= \{RS(A) | A \in P(E \setminus \{X_j\}) \Delta P(B)\} \\ &\Rightarrow RS(Y) \in \{RS(A) | A \in P(E \setminus \{X_j\}) \Delta P(B)\}. \end{aligned}$$

Let $RS(Y) = RS(Y_1 \Delta Y_2)$, where $Y_1 \in P(E \setminus \{X_j\})$ and $Y_2 \in P(B)$. Now for some $RS(Z) \in T \setminus F_{\{X_j\}}(T)$ we have $RS(Z) = RS(Z_1 \Delta Z_2)$, where $Z_1 \in P(E \setminus \{X_j\})$ and $Z_2 \in P(B)$. Thus

$$\begin{aligned} RS(Y) \Delta RS(Z) &= RS(Y_1 \Delta Y_2) \Delta RS(Z_1 \Delta Z_2) \\ &= RS(Y_1 \Delta Z_1) \Delta RS(Y_2 \Delta Z_2). \end{aligned}$$

Since $Y_1, Z_1 \in P(E \setminus \{X_j\})$ and $Y_2, Z_2 \in P(B)$, we have

$$\begin{aligned} Y_1 \Delta Z_1 &\in P(E \setminus \{X_j\}) \\ &\Rightarrow Y_2 \Delta Z_2 \in P(B) \\ &\Rightarrow Y \Delta Z \in P(E \setminus \{X_j\}) \Delta P(B) \\ &\Rightarrow RS(Y \Delta Z) \notin F_{\{X_j\}}(T) \\ &\Rightarrow RS(Y) \notin V(F_{\{X_j\}}(T)). \end{aligned}$$

Therefore,

$$V(F_{\{X_j\}}(T)) = \emptyset,$$

which implies that $G(F_{\{X_j\}}(T))$ is a null graph. ■

Remark 5. The converse holds for Theorem 8.

Example 2. To verify Remark 6, let $X = \{x_5\} \subseteq U$ from Example 1. Then the filter for $X = \{x_5\}$ is

$$\begin{aligned} F_{\{x_5\}}(T) = \{&RS(X_3), RS(\{x_1\} \cup X_3), \\ &RS(\{x_2\} \cup X_3), RS(X_1 \cup X_3), \\ &RS(X_2 \cup X_3), RS(\{x_1\} \cup \{x_2\} \cup X_3), \\ &RS(X_1 \cup \{x_2\} \cup X_3), \\ &RS(\{x_1\} \cup X_2 \cup X_3), RS(U)\}, \end{aligned}$$

$F_{\{x_5\}}(T)$ is a prime filter, and

$$\begin{aligned} &T \setminus F_{\{x_5\}}(T) \\ &\quad = \{RS(\emptyset), RS(\{x_1\}), RS(\{x_2\}), \\ &\qquad RS(\{x_1\} \cup \{x_2\}), RS(X_1), RS(X_2), \\ &\qquad RS(X_1 \cup \{x_2\}), RS(\{x_1\} \cup X_2), \\ &\qquad RS(X_1 \cup X_2)\} \end{aligned}$$

As it is clear, that there are no two $RS(Y), RS(Z) \in T \setminus F_{\{x_5\}}(T)$ such that $RS(Y \Delta Z) \in F_{\{x_5\}}(T)$, we get $V(F_{\{x_5\}}(T)) = \emptyset$ which means that $G(F_{\{x_5\}}(T))$ is a null graph, where $F_{\{x_5\}}(T)$ is a prime filter and also $RS(\{x_5\})$ is an atom. ♦

Theorem 9. *The rough identity-summand graph $G(F_X(T))$ generates complete bipartite graphs when $X = \{z_1, z_2, \ldots, z_\rho\}$, where $z_1, \ldots, z_\rho$ are x_i (or) X_j for $i, j \in \{1, \ldots, m, m+1, \ldots, n\}$, $2 \le \rho \le m, n$.*

Proof. From the definition of $G(F_X(T))$, $V(F_X(T)) \subseteq T \setminus F_X(T)$.

Case 1. Let $X = \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$ for $i_1, i_2, \ldots, i_r \in \{1, 2, \ldots m\}$. Observe that when $p = 1$, the vertex set partitioning of $V(F_X(T))$ into $V_{\{x_{i_1}\}}(F_X(T))$ and $V_{\{x_{i_2}, x_{i_3}, \ldots, x_{i_r}\}}(F_X(T))$ is given by

$$\begin{aligned} &V_{\{x_{i_1}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{x_{i_1}\} \Delta P(E \setminus \{X_{i_2}, X_{i_3}, \ldots, X_{i_r}\}) \\ &\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\})\} \end{aligned}$$

and

$$\begin{aligned} &V_{\{x_{i_2}, x_{i_3}, \ldots, x_{i_r}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{x_{i_2}, x_{i_3}, \ldots, x_{i_r}\} \\ &\qquad \Delta P(E \setminus \{X_{i_1}\})) \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\})\}. \end{aligned}$$

Thus

$$V_{\{x_{i_1}\}}(F_X(T)) \cap V_{\{x_{i_2}, x_{i_3}, \ldots, x_{i_r}\}}(F_X(T)) = \emptyset.$$

Let

$$\begin{aligned} &V_{\{x_{i_1}, x_{i_2} x_{i_3} \ldots x_{i_r}\}}(F_X(T)) \\ &\quad = V_{\{x_{i_1}\}}(F_X(T)) \cup V_{\{x_{i_2}, x_{i_3}, \ldots, x_{i_r}\}}(F_X(T)). \end{aligned}$$

We want to prove that

$$V_{\{x_{i_1}, x_{i_2} x_{i_3} \ldots x_{i_r}\}}(F_X(T)) \subseteq V(F_X(T)).$$

Let $RS(Y) \in V_{\{x_{i_1}, x_{i_2} x_{i_3} \ldots x_{i_r}\}}(F_X(T))$. Then

$$RS(Y) \in V_{\{x_{i_1}\}}(F_X(T))$$

(or)

$$RS(Y) \in V_{\{x_{i_2}, x_{i_3}, \ldots, x_{i_r}\}}(F_X(T))$$

implies $RS(Y) \in T \setminus F_X(T)$

Suppose there exists some $RS(Z) \in T \setminus F_X(T)$ and $RS(Y\Delta Z) \in F_X(T)$ such that $RS(Y) \in V(F_X(T))$. It follows that

$$V_{\{x_{i_1}\}}(F_X(T)) \subseteq V(F_X(T))$$

and

$$V_{\{x_{i_2},x_{i_3},\dots,x_{i_r}\}}(F_X(T)) \subseteq V(F_X(T)),$$

which yields

$$V_{\{x_{i_1}\}}(F_X(T)) \cup V_{\{x_{i_2},x_{i_3},\dots,x_{i_r}\}}(F_X(T)) \subseteq V(F_X(T)),$$

This implies

$$V_{\{x_{i_1},x_{i_2}x_{i_3}\dots x_{i_r}\}}(F_X(T)) \subseteq V(F_X(T)).$$

Now let $RS(Y) \in V_{\{x_{i_1}\}}(F_X(T))$ and $RS(Z) \in V_{\{x_{i_2},x_{i_3},\dots,x_{i_r}\}}(F_X(T))$. Then $RS(Y\Delta Z) \in F_X(T)$. This implies there is an edge connecting $RS(Y)$ and $RS(Z)$ in $G(F_X(T))$.

Observe that, for every $RS(Y) \in V_{\{x_{i_1}\}}(F_X(T))$, where $Y = \{x_{i_1}\}\Delta K_1 \Delta K_2$, for $K_1 \in P(E \setminus \{X_{i_2}, X_{i_3}, \dots, X_{i_r}\}), K_2 \in P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})$ and $RS(Z) \in V_{\{x_{i_2},x_{i_3},\dots,x_{i_r}\}}(F_X(T))$, where we have $Z = \{x_{i_2}, x_{i_3}, \dots, x_{i_r}\}\Delta G_1 \Delta G_2$ for $G_1 \in P(E \setminus \{X_{i_1}\}), G_2 \in P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})$, an edge exists between $RS(Y)$ and $RS(Z)$. This complete bipartite graph is denoted by $B(F_X(x_{i_1}, x_{i_2}x_{i_3}\dots x_{i_r}))$ and similar complete bipartite graphs obtained are $B(F_X(x_{i_2}, x_{i_1}x_{i_3}\dots x_{i_r})), \dots, B(F_X(x_{i_r}, x_{i_1}x_{i_2}\dots x_{i_{r-1}}))$. Thus ${}^r\mathrm{C}_1$ complete bipartite graphs are obtained when $p = 1$.

Suppose $p = 2$. Then using an argument similar to $p = 1$, the number of complete bipartite graphs obtained are $({}^2\mathrm{C}_2 + {}^2\mathrm{C}_1)^r\mathrm{C}_2$ and they are denoted by

$$B(F_X(x_{i_1}x_{i_2}, x_{i_3}x_{i_4}, \dots, x_{i_r})),$$

$$B(F_X(x_{i_1}x_{i_3}, x_{i_2}x_{i_4}, , \dots, x_{i_r})),$$

$$\vdots$$

$$B(F_X(x_{i_{r-1}}x_{i_r}, x_{i_1}x_{i_2}, \dots, x_{i_{r-2}})),$$

$$\vdots$$

$$B(F_X(x_{i_1}x_{i_2}, x_{i_1}x_{i_3}x_{i_4}, \dots, x_{i_r})),$$

$$B(F_X(x_{i_1}x_{i_2}, x_{i_2}x_{i_3}x_{i_4}, \dots, x_{i_r})),$$

$$\vdots$$

$$B(F_X(x_{i_{r-1}}x_{i_r}, x_{i_1}x_{i_2}x_{i_3}, \dots, x_{i_{r-2}}x_{i_r})).$$

This procedure is repeated for $p = 3, 4, 5, \dots$ and so on and the number complete bipartite graphs formed are $({}^3\mathrm{C}_3 + {}^3\mathrm{C}_2 + {}^3\mathrm{C}_1)^r\mathrm{C}_3, ({}^4\mathrm{C}_4 + {}^4\mathrm{C}_3 + {}^4\mathrm{C}_2 + {}^4\mathrm{C}_1)^r\mathrm{C}_4, ({}^5\mathrm{C}_5 + {}^5\mathrm{C}_4 + {}^5\mathrm{C}_3 + {}^5\mathrm{C}_2 + {}^5\mathrm{C}_1)^r\mathrm{C}_5, \dots$, respectively.

Finally, when $p = r - 1$, similarly partitioning the vertex set and using arguments from $p = 1$, the complete bipartite graphs obtained is ${}^{(r-1)}\mathrm{C}_1[{}^r\mathrm{C}_{(r-1)}/2]$.

Therefore, when $X = \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\}$, $G(F_X(T))$ generates a total of ${}^r\mathrm{C}_1 + ({}^2\mathrm{C}_2 + {}^2\mathrm{C}_1)^r\mathrm{C}_2 + \dots + {}^{(r-1)}\mathrm{C}_1[{}^r\mathrm{C}_{(r-1)}/2]$ complete bipartite graphs.

Case 2. Let $X = \{X_{s_1}, X_{s_2}, \dots, X_{s_t}\}$ for $s_1, s_2, \dots, s_t \in \{1, 2, \dots, m\}$. When $p = 1$, the vertex set $V(F_X(T))$ is partitioned into $V_{\{X_{s_1}\}}(F_X(T))$ and $V_{\{X_{s_2},X_{s_3},\dots,X_{s_t}\}}(F_X(T))$, where

$$\begin{aligned} &V_{\{X_{s_1}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{X_{s_1}\}\Delta P(E \setminus \{X_{s_1}, X_{s_2}, \\ &\qquad X_{s_3}, \dots, X_{s_t}\})\Delta P(B \setminus \{x_{s_1}\})\} \end{aligned}$$

and

$$\begin{aligned} &V_{\{X_{s_2},X_{s_3},\dots,X_{s_t}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{X_{s_2}, X_{s_3}, \dots, X_{s_t}\} \\ &\qquad \Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \dots, X_{s_t}\})) \\ &\qquad \Delta P(B \setminus \{x_{s_2}, x_{s_3}, \dots, x_{s_t}\})\}. \end{aligned}$$

Thus, by the arguments in Case 1, the complete bipartite graph obtained is $B(F_X(X_{s_1}, X_{s_2}X_{s_3}, \dots, X_{s_t}))$. Similarly partitioning the vertex set $V(F_X(T))$, the complete bipartite graphs obtained are

$$B(F_X(X_{s_2}, X_{s_1}X_{s_3}, \dots, X_{s_t})),$$

$$\vdots$$

$$B(F_X(X_{s_t}, X_{s_1}X_{s_2}, \dots, X_{s_{t-1}})).$$

The complete bipartite graphs obtained for $p = 1$ is ${}^t\mathrm{C}_1$ and so on.

Accordingly, when $X = \{X_{s_1}, X_{s_2}, \dots, X_{s_t}\}$, $G(F_X(T))$ generates a total of ${}^t\mathrm{C}_1 + ({}^2\mathrm{C}_2 + {}^2\mathrm{C}_1)^t\mathrm{C}_2 + \dots + {}^{(t-1)}\mathrm{C}_1[{}^t\mathrm{C}_{(t-1)}/2]$ complete bipartite graphs.

Case 3. Let $X = \{X_{j_1}, X_{j_2}, \dots, X_{j_k}\}$ for $j_1, j_2, \dots, j_k \in \{m+1, m+2, \dots, n\}$. When $p = 1$, the vertex set $V(F_X(T))$ is of the form $V_{\{X_{j_1}\}}(F_X(T))$ and $V_{\{X_{j_2},X_{j_3},\dots,X_{j_k}\}}(F_X(T))$, where

$$\begin{aligned} &V_{\{X_{j_1}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{X_{j_1}\}\Delta P(E \setminus \{X_{j_1}, \\ &\qquad X_{j_2}, X_{j_3}, \dots, X_{j_k}\})\Delta P(B)\}, \end{aligned}$$

and

$$V_{\{X_{j_2},X_{j_3},\dots,X_{j_k}\}}(F_X(T)) = \{RS(Z)|Z \in \{X_{j_2}, X_{j_3}, \dots, X_{j_k}\} \Delta P(E \setminus \{X_{j_1}, X_{j_2}, X_{j_3}, \dots, X_{j_k}\})) \Delta P(B)\}.$$

Then by the arguments in Case 1, the complete bipartite graph obtained is $B(F_X(X_{j_1}, X_{j_2}X_{j_3}, \dots, X_{j_k}))$. Similarly partitioning the vertex set $V(F_X(T))$ for $p = 1$, the complete bipartite graphs obtained are

$$B(F_X(X_{j_2}, X_{j_1}X_{j_3} \dots X_{j_k})),$$

$$\vdots$$

$$B(F_X(X_{j_k}, X_{j_1}X_{j_2} \dots X_{j_{k-1}})).$$

Thus the complete bipartite graphs obtained for $p = 1$ is ${}^k\mathrm{C}_1$ and so on.

Hence, when $X = \{X_{j_1}, X_{j_2}, \dots, X_{j_k}\}$, $G(F_X(T))$ generates total of ${}^k\mathrm{C}_1 + ({}^2\mathrm{C}_2 + {}^2\mathrm{C}_1)^k\mathrm{C}_2 + \cdots + {}^{(k-1)}\mathrm{C}_1[{}^k\mathrm{C}_{(k-1)}/2]$ complete bipartite graphs.

Case 4. Let

$$X = \{x_{i_1}, x_{i_2}, \dots, x_{i_r}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\},$$

where $i_1, i_2, \dots, i_r \in \{1, 2, \dots, m\}$ and $j_1, j_2, \dots, j_k \in \{m+1, m+2, \dots, n\}$. When $p = 1$, the vertex set $V(F_X(T))$ is partitioned into

$$V_{\{x_{i_1}\}}(F_X(T))$$

and

$$V_{\{x_{i_2},\dots,x_{i_r},X_{j_1},X_{j_2},\dots,X_{j_k}\}}(F_X(T)),$$

where

$$V_{\{x_{i_1}\}}(F_X(T)) = \{RS(Z)|Z \in \{x_{i_1}\}\Delta P(E \setminus \{X_{i_2}, X_{i_3}, \dots, X_{i_r}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}) \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\}$$

and

$$V_{\{x_{i_2},\dots,x_{i_r},X_{j_1},X_{j_2},\dots,X_{j_k}\}}(F_X(T)) = \{RS(Z)|Z \in \{x_{i_2}, \dots, x_{i_r}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\} \Delta P(E \setminus \{X_{i_1}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\})) \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\}.$$

Then using the argument of Case 1, the complete bipartite graph obtained is $B(F_X(x_{i_1}, x_{i_2}, \dots, x_{i_r}X_{j_1}X_{j_2} \dots X_{j_k}))$. We see that similarly partitioning the vertex set $V(F_X(T))$ for $p = 1$, the complete bipartite graphs obtained are

$$B(F_X(x_{i_2}, x_{i_1}, \dots, x_{i_r}X_{j_1}X_{j_2} \dots X_{j_k})),$$

$$\vdots$$

$$B(F_X(x_{i_r}, x_{i_1}x_{i_2}, \dots, x_{i_{r-1}}X_{j_1}X_{j_2} \dots X_{j_k})).$$

Thus the complete bipartite graphs exist when $p = 1$ and their number is ${}^r\mathrm{C}_1$.

Also for $p = 1$, the ${}^k\mathrm{C}_1$ ways vertex set $V(F_X(T))$ is divided into

$$V_{\{X_{j_1}\}}(F_X(T))$$

and

$$V_{\{x_{i_1},x_{i_2},\dots,x_{i_r},X_{j_2},X_{j_3},\dots,X_{j_k}\}}(F_X(T)),$$

where

$$V_{\{X_{j_1}\}}(F_X(T)) = \{RS(Z)|Z \in \{X_{j_1}\}\Delta P(E \setminus \{X_{i_1}, X_{i_2}, \dots, X_{i_r}, X_{j_1}, X_{j_2}, X_{j_3}, \dots, X_{j_k}\}) \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\}$$

and

$$V_{\{x_{i_1},x_{i_2},\dots,x_{i_r},X_{j_2},X_{j_3},\dots,X_{j_k}\}}(F_X(T)) = \{RS(Z)|Z \in \{x_{i_1}, x_{i_2}, \dots, x_{i_r}, X_{j_2}, X_{j_3}, \dots, X_{j_k}\} \Delta P(E \setminus \{X_{j_1}, X_{j_2}, X_{j_3}, \dots, X_{j_k}\})) \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\}.$$

Then using the arguments, the complete bipartite graph obtained is $B(F_X(X_{j_1}, x_{i_1}, \dots, x_{i_r}X_{j_2}X_{j_3}, \dots, X_{j_k}))$. Similarly partitioning the vertex set $V(F_X(T))$ for $p = 1$, the complete bipartite graphs obtained are $B(F_X(X_{j_2}, x_{i_1} \dots x_{i_r}X_{j_1}X_{j_3} \dots X_{j_k})), \dots, B(F_X(X_{j_k}, x_{i_1}x_{i_2} \dots x_{i_r}X_{j_1}X_{j_2} \dots X_{j_{k-1}}))$. Thus the complete bipartite graphs obtained for $p = 1$ is ${}^k\mathrm{C}_1$. Hence for $p = 1$, a ${}^{(r+k)}\mathrm{C}_1$ complete bipartite graph exists.

When $p = 2$, a $({}^2\mathrm{C}_2 + {}^2\mathrm{C}_1)^r\mathrm{C}_2 + ({}^2\mathrm{C}_2 + {}^2\mathrm{C}_1)^k\mathrm{C}_2 + ({}^2\mathrm{C}_2 + {}^2\mathrm{C}_1)^r\mathrm{C}_1{}^k\mathrm{C}_1$ complete bipartite graph exists and so on.

Therefore, when

$$X = \{x_{i_1}, x_{i_2}, \dots, x_{i_r}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\},$$

$G(F_X(T))$ generates a total of ${}^{(r+k)}\mathrm{C}_1 + ({}^2\mathrm{C}_2 + {}^2\mathrm{C}_1)^{(r+k)}\mathrm{C}_2 + \cdots + {}^{(r+k)-1}\mathrm{C}_1[{}^{(r+k)}\mathrm{C}_{(r+k)-1}/2]$ complete bipartite graphs.

Case 5. Let

$$X = \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}, X_{s_1}, X_{s_2}, \ldots, X_{s_t}\},$$

where $\{i_1, i_2, \ldots, i_r, s_1, s_2, \ldots, s_t\} \in \{1, 2, \ldots, m\}$. When $p = 1$, the vertex set $V(F_X(T))$ is partitioned into

$$V_{\{x_{i_1}\}}(F_X(T))$$

and

$$V_{\{x_{i_2},\ldots,x_{i_r},X_{s_1},X_{s_2},\ldots,X_{s_t}\}}(F_X(T)),$$

where

$$\begin{aligned} &V_{\{x_{i_1}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{x_{i_1}\} \Delta P(E \setminus \{X_{i_2}, X_{i_3}, \ldots, \\ &\qquad X_{i_r}, X_{s_1}, X_{s_2}, \ldots, X_{s_t}\}) \\ &\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\})\} \end{aligned}$$

and

$$\begin{aligned} &V_{\{x_{i_2},\ldots,x_{i_r},X_{s_1},X_{s_2},\ldots,X_{s_t}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{x_{i_2}, \ldots, x_{i_r}, X_{s_1}, X_{s_2}, \ldots, X_{s_t}\} \\ &\qquad \Delta P(E \setminus \{X_{i_1}, X_{s_1}, X_{s_2}, \ldots, X_{s_t}\})) \\ &\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}, x_{s_1}, x_{s_2}, \ldots, x_{s_t}\})\}. \end{aligned}$$

Then using the arguments from Case 1, the complete bipartite graph obtained is $B(F_X(x_{i_1}, x_{i_2} \ldots x_{i_r} X_{s_1} X_{s_2}, \ldots, X_{s_t}))$. Similarly partitioning the vertex set $V(F_X(T))$ for $p = 1$, the complete bipartite graphs obtained are $B(F_X(x_{i_2}, x_{i_1} \ldots x_{i_r} X_{s_1} X_{s_2} \ldots X_{s_t})), \ldots, B(F_X(x_{i_r}, x_{i_1} x_{i_2} \ldots x_{i_{r-1}} X_{s_1} X_{s_2} \ldots X_{s_t}))$. Thus ${}^r\mathrm{C}_1$ complete bipartite graphs exist for $p = 1$.

Also for $p = 1$, the vertex set $V(F_X(T))$ is divided into

$$V_{\{X_{s_1}\}}(F_X(T))$$

and

$$V_{\{x_{i_1},\ldots,x_{i_r},X_{s_2},X_{s_3},\ldots,X_{s_t}\}}(F_X(T)),$$

where

$$\begin{aligned} &V_{\{X_{s_1}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{X_{s_1}\} \Delta P(E \setminus \{X_{i_1}, X_{i_2}, \ldots, \\ &\qquad X_{i_r}, X_{s_1}, X_{s_2}, \ldots, X_{s_t}\}) \\ &\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}, x_{s_1}\})\} \end{aligned}$$

and

$$\begin{aligned} &V_{\{x_{i_1},\ldots,x_{i_r},X_{s_2},\ldots,X_{s_t}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}, X_{s_2}, X_{s_3}, \ldots, X_{s_t}\} \\ &\qquad \Delta P(E \setminus \{X_{s_1}, X_{s_2}, \ldots, X_{s_t}\})) \\ &\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}, x_{s_2}, x_{s_3}, \ldots, x_{s_t}\})\}. \end{aligned}$$

Then, using the arguments the from Case 1, the complete bipartite graph obtained is $B(F_X(X_{s_1}, x_{i_1} \ldots x_{i_r} X_{s_2} X_{s_3} \ldots X_{s_t}))$. Similarly partitioning the vertex set $V(F_X(T))$ for $p = 1$, the complete bipartite graphs obtained are $B(F_X(X_{s_2}, x_{i_1} \ldots x_{i_r} X_{s_1} X_{s_3} \ldots X_{s_t})), \ldots, B(F_X(X_{s_t}, x_{i_1} x_{i_2} \ldots x_{i_{r-1}} X_{s_1} X_{s_2} \ldots X_{s_{t-1}}))$. Thus the number of complete bipartite graphs obtained for $p = 1$ is ${}^t\mathrm{C}_1$. Hence when $p = 1$, ${}^{(r+t)}\mathrm{C}_1$ complete bipartite graphs exist.

When $p = 2$, $({}^2\mathrm{C}_2 + {}^2\mathrm{C}_1)^r\mathrm{C}_2 + ({}^2\mathrm{C}_2 + {}^2\mathrm{C}_1)^t\mathrm{C}_2 + ({}^2\mathrm{C}_2 + {}^2\mathrm{C}_1)^r\mathrm{C}_1{}^t\mathrm{C}_1$ complete bipartite graphs are obtained and so on.

Hence, when

$$X = \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}, X_{s_1}, X_{s_2}, \ldots, X_{s_t}\},$$

$G(F_X(T))$ generates a total of ${}^{(r+t)}\mathrm{C}_1 + ({}^2\mathrm{C}_2 + {}^2\mathrm{C}_1)^{(r+t)}\mathrm{C}_2 + \cdots + {}^{(r+t)-1}\mathrm{C}_1[{}^{(r+t)}\mathrm{C}_{(r+t)-1}/2]$ complete bipartite graphs.

Case 6. Let

$$X = \{X_{s_1}, X_{s_2}, \ldots, X_{s_t}, X_{j_1}, X_{j_2}, \ldots, X_{j_k}\}$$

where we have $s_1, s_2, \ldots, s_t \in \{1, 2, \ldots, m\}$, and $j_1, j_2, \ldots, j_k \in \{m+1, m+2, \ldots, n\}$. When $p = 1$, the vertex set $V(F_X(T))$ is partitioned into

$$V_{\{X_{s_1}\}}(F_X(T))$$

and

$$V_{\{X_{s_2},\ldots,X_{s_t},X_{j_1},X_{j_2},\ldots,X_{j_k}\}}(F_X(T))$$

where

$$\begin{aligned} &V_{\{X_{s_1}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{X_{s_1}\} \Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \ldots, \\ &\qquad X_{s_t}, X_{j_1}, X_{j_2}, \ldots, X_{j_k}\}) \Delta P(B \setminus \{x_{s_1}\})\} \end{aligned}$$

and

$$\begin{aligned} &V_{\{X_{s_2},\ldots,X_{s_t},X_{j_1},X_{j_2},\ldots,X_{j_k}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{X_{s_2}, \ldots, X_{s_t}, X_{j_1}, X_{j_2}, \ldots, X_{j_k}\} \\ &\qquad \Delta P(E \setminus \{X_{s_1}, X_{s_2}, \ldots, X_{s_t},) X_{j_1}, X_{j_2}, \ldots, X_{j_k}\}) \\ &\qquad \Delta P(B \setminus \{x_{s_2}, x_{s_3}, \ldots, x_{s_t}\})\}. \end{aligned}$$

Then by using the arguments of Case 1, the complete bipartite graph obtained is $B(F_X(X_{s_1}, X_{s_2} \ldots X_{s_t} X_{j_1} X_{j_2} \ldots X_{j_k}))$. Similarly partitioning the vertex set $V(F_X(T))$ for $p = 1$, the remaining complete bipartite graphs obtained are

$$B(F_X(X_{s_2}, X_{s_1} X_{s_3} \ldots X_{s_t} X_{j_1} X_{j_2} \ldots X_{j_k})),$$

$$\vdots$$

$$B(F_X(X_{s_t}, X_{s_1}X_{s_2}\dots X_{s_{t-1}}X_{j_1}X_{j_2}\dots X_{j_k})).$$

Thus the complete bipartite graphs for $p = 1$ is ${}^{t}\mathrm{C}_1$.

Also for $p = 1$, the vertex set $V(F_X(T))$ is divided into

$$V_{\{X_{j_1}\}}(F_X(T))$$

and

$$V_{\{X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_2}, X_{j_3}, \dots, X_{j_k}\}}(F_X(T)),$$

where

$$\begin{aligned} &V_{\{X_{j_1}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{X_{j_1}\}\Delta P(E \setminus \{X_{s_1}, X_{s_2}, \dots, \\ &\qquad X_{s_t}, X_{j_1}, X_{j_2}, X_{j_3}, \dots, X_{j_k}\})\Delta P(B)\} \end{aligned}$$

and

$$\begin{aligned} &V_{\{X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_2}, X_{j_3}, \dots, X_{j_k}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{X_{s_1}, X_{s_2}, \dots, X_{s_t}, \\ &\qquad X_{j_2}, X_{j_3}, \dots, X_{j_k}\} \\ &\qquad \Delta P(E \setminus \{X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \\ &\qquad X_{j_3}, \dots, X_{j_k}\}))\Delta P(B)\}. \end{aligned}$$

Then, by using arguments of Case 1, the complete bipartite graph obtained is $B(F_X(X_{j_1}, X_{s_1}\dots X_{s_t}X_{j_2}\dots X_{j_k}))$. Similarly partitioning the vertex set $V(F_X(T))$ for $p = 1$, the remaining complete bipartite graphs obtained are

$$B(F_X(X_{j_2}, X_{s_1}X_{s_2}\dots X_{s_t}X_{j_1}X_{j_3}\dots X_{j_k})),$$

$$\vdots$$

$$B(F_X(X_{j_k}, X_{s_1}X_{s_2}\dots X_{s_t}X_{j_1}X_{j_2}\dots X_{j_{k-1}})).$$

Thus the number of complete bipartite graphs for $p = 1$ is ${}^{k}\mathrm{C}_1$. Hence for $p = 1$, the number of complete bipartite graphs obtained is ${}^{(t+k)}\mathrm{C}_1$ and so on.

Therefore, when

$$X = \{X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\},$$

$G(F_X(T))$ generates a total of ${}^{(t+k)}\mathrm{C}_1 + ({}^{2}\mathrm{C}_2 + {}^{2}\mathrm{C}_1)^{(t+k)}\mathrm{C}_2 + \dots + {}^{(t+k)-1}\mathrm{C}_1[{}^{(t+k)}\mathrm{C}_{(t+k)-1}/2]$ complete bipartite graphs.

Case 7. Let

$$\begin{aligned} X = \{&x_{i_1}, x_{i_2}, \dots, x_{i_r}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, \\ &X_{j_1}, X_{j_2}, \dots, X_{j_k}\}, \end{aligned}$$

where $i_1, i_2, \dots, i_r, s_1, s_2, \dots, s_t \in \{1, 2, \dots m\}$, and $j_1, j_2, \dots, j_k \in \{m+1, m+2, \dots, n\}$.

For $p = 1$, the vertex set $V(F_X(T))$ is partitioned into

$$V_{\{x_{i_1}\}}(F_X(T))$$

and

$$V_{\{x_{i_2}, \dots, x_{i_r}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}}(F_X(T)),$$

where

$$\begin{aligned} &V_{\{x_{i_1}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{x_{i_1}\}\Delta P(E \setminus \{X_{i_2}, X_{i_3}, \dots, \\ &\qquad X_{i_r}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, \\ &\qquad X_{j_2}, \dots, X_{j_k}\}) \\ &\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\} \end{aligned}$$

and

$$\begin{aligned} &V_{\{x_{i_2}, \dots, x_{i_r}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{x_{i_2}, \dots, x_{i_r}, \\ &\qquad X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\} \\ &\qquad \Delta P(E \setminus \{X_{i_1}, X_{s_1}, X_{s_2}, \dots, \\ &\qquad X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\})) \\ &\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}, x_{s_1}, x_{s_2}, \dots, x_{s_t}\})\}. \end{aligned}$$

Then it can be proved that

$$\begin{aligned} &V_{\{x_{i_1}, x_{i_2}\dots x_{i_r}X_{s_1}X_{s_2}\dots X_{s_t}X_{j_1}X_{j_2}\dots X_{j_k}\}}(F_X(T)) \\ &\qquad \subseteq V(F_X(T)). \end{aligned}$$

By using the similar arguments from Case 1, the complete bipartite graph obtained is $B(F_X(x_{i_1}, x_{i_2}\dots x_{i_r}X_{s_1}X_{s_2}\dots X_{s_t}X_{j_1}X_{j_2}\dots X_{j_k}))$.

Similarly partitioning the vertex set $V(F_X(T))$ for $p = 1$, the complete bipartite graphs obtained are

$$\begin{aligned} B(F_X(&x_{i_2}, x_{i_1}, \dots, x_{i_r}X_{s_1}X_{s_2}, \dots, \\ &X_{s_t}X_{j_1}X_{j_2}, \dots, X_{j_k})), \dots, \end{aligned}$$

$$\vdots$$

$$\begin{aligned} B(F_X(&x_{i_r}, x_{i_1}, \dots, x_{i_{r-1}}X_{s_1}X_{s_2}, \dots, \\ &X_{s_t}X_{j_1}X_{j_2}, \dots, X_{j_k})). \end{aligned}$$

Thus the number of complete bipartite graphs for $p = 1$ is ${}^{r}\mathrm{C}_1$.

Also for $p = 1$, the ${}^{t}\mathrm{C}_1$ and ${}^{k}\mathrm{C}_1$ complete bipartite graphs are obtained. Therefore, the number of complete bipartite obtained for $p = 1$ is ${}^{(r+t+k)}\mathrm{C}_1$ and so on.

Hence, when

$$\begin{aligned} X = \{&x_{i_1}, x_{i_2}, \dots, x_{i_r}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, \\ &X_{j_1}, X_{j_2}, \dots, X_{j_k}\}, \end{aligned}$$

$G(F_X(T))$ generates a total of

$$\begin{aligned} &{}^{(r+t+k)}\mathrm{C}_1 + ({}^{2}\mathrm{C}_2 + {}^{2}\mathrm{C}_1)^{(r+t+k)}\mathrm{C}_2 \\ &\quad + \dots + {}^{(r+t+k)-1}\mathrm{C}_1\left[\frac{{}^{(r+t+k)}\mathrm{C}_{(r+t+k)-1}}{2}\right] \end{aligned}$$

complete bipartite graphs. ∎

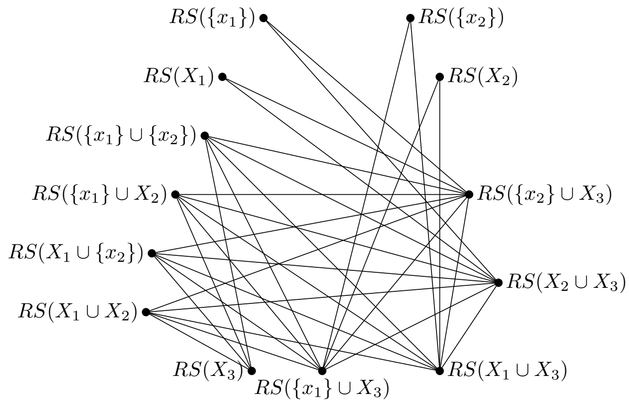

Fig. 1. $G(F_{\{x_1,x_2,x_5\}}(T))$.

Remark 6.

1. In the cases discussed in Theorem-9, the union of all complete bipartite graphs generated will be equal to $G(F_X(T))$.

2. The number of such complete bipartite graphs can be reduced by identifying isomorphic graphs. For example, when $p = 1$, the vertex set of $G(F_X(T))$ is partitioned into generating ${}^r\mathrm{C}_1 = r$ complete bipartite graphs. Also, when $p = r-1$, ${}^r\mathrm{C}_{r-1} = r$ complete bipartite graphs are generated. This leads to generating distinct complete bipartite graphs from $G(F_X(T))$. It is important to note that the union of these complete bipartite graphs will be a subgraph of $G(F_X(T))$.

Lemma 1. *The rough identity-summand graph $G(F_X(T))$ is a complete bipartite graph if $X = \{z_1, z_2\}$, where z_1, z_2 are x_i (or) X_j for $i, j \in \{1, 2, \dots, m, m+1, m+2, \dots n\}$.*

Lemma 2. *The rough identity-summand graph $G(F_X(T))$ is a $\sum_{r=0}^{n-2} {}^{(n-2)}\mathrm{C}_r 2^{n-(r+1)}$-regular complete bipartite graph if $X = \{z_1, z_2\}$, where X is $\{x_i, x_j\}$ (or) $\{X_i, X_j\}$ for $i, j \in \{1, 2, \dots, m\}$.*

Lemma 3. *The rough identity-summand graph $G(F_X(T))$ is a $\sum_{t=2}^{m} {}^{(m-2)}\mathrm{C}_{m-t} \sum_{k=0}^{n-m} {}^{(n-m)}\mathrm{C}_k 2^{m-\{m-(t-1)\}}$ regular complete bipartite graph if $X = \{z_1, z_2\}$, where X is $\{x_i, x_j\}$ (or) $\{X_i, X_j\}$ for $i, j \in \{1, 2, \dots, m, m+1, m+2, \dots, n\}$.*

Definition 15. The number of distinct complete bipartite graphs generated from $G(F_X(T))$ is called the *rough bipartite number*. The rough bipartite number (or) RBP number of $G(F_X(T))$ is denoted by $RBP(G(F_X(T)))$.

Example 3. From Example 1, the rough identity-summand graph exists for the filter $F_X(T)$ when $X = \{x_1, x_2, x_5\}$ is given in Fig. 1

The total number of complete bipartite graphs generated from $G(F_{\{x_1,x_2,x_5\}}(T))$ is 6. But the number of distinct complete bipartite graphs is 3. The union of these distinct complete bipartite graphs forms a subgraph of $G(F_{\{x_1,x_2,x_5\}}(T))$ and is given in Fig. 2. This subgraph contains all the vertices of $G(F_X(T))$. This is a proper subgraph that has no edge between $RS(\{x_1\} \cup \{x_2\})$ and $RS(\{x_2\} \cup X_3)$, but has an edge in $G(F_{\{x_1,x_2,x_5\}}(T))$. The subgraph of $G(F_X(T))$ is considered in generating the Gray code which will be discussed in the following section. ♦

5. Gray code generation through a rough identity–summand graph

A Gray code of length n is a list of 2^n binary numbers of n bis in which two consecutive numbers differ by only one bit. The transition sequence represents the changes in the bit positions between two consecutive binary numbers represented by a decimal number sequence of length $2^n - 1$. The applications of Gray codes are found in various areas such as error correction, rotary encoder sensors, the conversion of analog to digital representations, and network addressing in communication devices. The advantage of the Gray code with the subsequent numbers, differing by one bit is crucial in applications by minimizing the error caused by the multiple changes. In this section, the idea of generating Gray codes is extended through a rough identity-summand graph $G(F_X(T))$. This will begin by considering an approximation space $I = (U, R)$, where U is the collection of n-bit binary numbers say $U = \{00\dots0, 10\dots0, 01\dots0, \dots, 11\dots1\}$. Note that the cardinality of U is 2^n. Consider the partition of U by $\mathfrak{C} = \{X_1, X_2, \dots, X_{n+1}\}$, where X_i is the set of n-bit sequences containing i zeros for $1 \le i \le n$ and X_{n+1} is the n-bit sequence of all 1's. Using this partition, we can have a rough bi-Heyting algebra $(T, \Delta, \nabla, ^*, ^+, \to, \leftarrow, RS(\emptyset), RS(U))$. From Section 4, for any subset X of U, the rough identity-summand graph $G(F_X(T))$ is obtained for the filter $F_X(T)$. Also by the results from Section 4, the subgraph of the rough identity-summand graph $G(F_X(T))$ can be expressed as the union of distinct complete bipartite graphs.

The main application is to generate two Gray codes of length $k, k \ge n$ using $G(F_X(T))$. This is achieved by obtaining two transition sequences of length $2^k - 1$, where $k = |V(F_X(T))|$. Any set X of Gray codes in U is defined using

$$RS(X) = (R_-(X), R^-(X)).$$

In any given n-bit sequence α, $N_0^n(\alpha)$ denotes the number of zeros in α. For $X \subseteq U$, let $N_0^n(X) = \sum_{\alpha \in X} N_0^n(\alpha)$ (mod 2). Now the initial lower approximation transition sequence $S_L(0) = N_0^n(R_-(X))$ and the initial

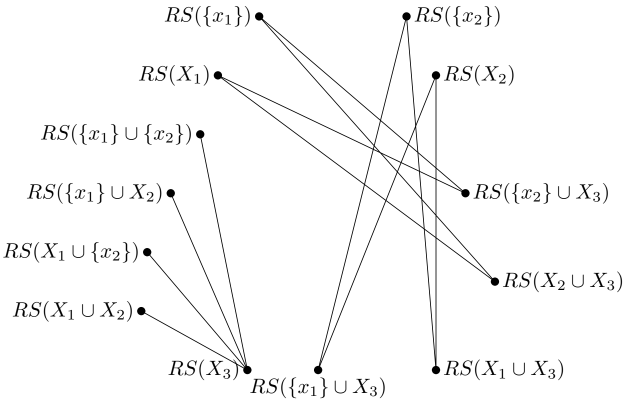

Fig. 2. Subgraph of $G(F_{\{x_1,x_2,x_5\}}(T))$.

upper approximation transition sequence $S_U(0) = N_0^n(R^-(X))$. Using these initial transition sequences, two Gray codes of length k can be generated. The lower approximation transition sequence is defined by

$$S_L(k) = S_L(k-1), k, S_L(k-1)$$

and the upper approximation transition sequence is defined by

$$S_U(k) = S_U(k-1), k, S_U(k-1).$$

For example, consider the set of 3-bit binary number sequences as the universal set U. The partition on U is $\mathfrak{E} = \{X_1, X_2, X_3, X_4\}$. If $X = \{x_1, x_2, X_3, X_4\}$, the distinct complete bipartite graphs generated from the rough identity summand graph $G(F_X(T))$ are given to be

$$B(F_X(x_1, x_2X_3X_4)),$$

$$B(F_X(x_2, x_1X_3X_4)),$$

$$B(F_X(X_3, x_1x_2X_4)),$$

$$B(F_X(X_4, x_1x_2X_3)),$$

$$B(F_X(x_1X_3, x_2X_4)),$$

$$B(F_X(x_1X_4, x_2X_3)),$$

$$B(F_X(x_1x_2, X_3X_4)).$$

For $X = \{x_1, x_2, X_3, X_4\}$, one of the complete bipartite graphs of $G(F_{\{x_1,x_2,X_3,X_4\}}(T))$ is considered to be $B(F_X(x_1x_2, X_3X_4))$ and

$$|V(B(F_X(x_1x_2, X_3X_4)))| = 5$$

The initial lower and upper transition sequences are $S_L(1) = 1$ and $S_U(0) = 0$.

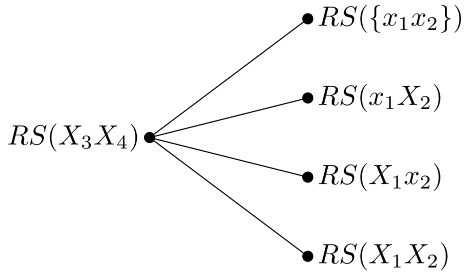

Fig. 3. $B(F_X(x_1x_2, X_3X_4))$.

Hence the lower approximation transition sequence is

$$\begin{aligned}S_L(5) = \{&1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,5,\\ &1,2,1,3,1,2,1,4,1,2,1,3,1,2,1\}\end{aligned}$$

and the upper approximation transition sequence is

$$\begin{aligned}S_U(4) = \{&0,1,0,2,0,1,0,3,0,1,0,2,0,1,0,4,\\ &0,1,0,2,0,1,0,3,0,1,0,2,0,1,0\}.\end{aligned}$$

This will generate a 5-bit Gray code sequence

$$\begin{aligned}G(5) = \big\{&00000, 00001, 00011, 00010, 00110,\\ &00111, 00101, 00100, 01100, 01101,\\ &01111, 01110, 01010, 01011, 01001,\\ &01000, 11000, 11001, 11011, 11010,\\ &11110, 11111, 11101, 11100, 10100,\\ &10101, 10111, 10110, 10010, 10011,\\ &10001, 10000\big\}.\end{aligned}$$

For the remaining complete bipartite graphs of $G(F_{\{x_1,x_2,X_3,X_4\}}(T))$, 14 Gray codes of varied length can be obtained.

6. Conclusion

In this paper, various filters of a rough bi-Heyting algebra $(T, \Delta, \nabla, ^*, ^+, \rightarrow, \leftarrow, RS(\emptyset), RS(U))$ are formalized. Filters of this rough bi-Heyting algebra are defined through the R-upset of elements of T. The filters of a rough bi-Heyting algebra are characterized and later its generalization is addressed to establish the relevant graph structures. For any $X \subseteq U$, a rough identity-summand graph $G(F_X(T))$ is constructed for the filter $F_X(T)$ of a rough bi-Heyting algebra. Also $G(F_X(T))$ is proved to generate complete bipartite graphs. Then the results providing the number of distinct complete bipartite graphs generated from $G(F_X(T))$ are given. The RBP number of $G(F_X(T))$ for any X is defined and the $RBP(G(F_X(T)))$ is also obtained.

The generation of the Gray code of length k for the complete bipartite graphs of $G(F_X(T))$ is achieved by two transition sequences of length 2^k-1. Generating Gray codes for the rough identity-summand graph presents numerous challenges in figuring out the most suitable way to partition the vertex set and analyzing the graph's regularity. The study of the Gray code generation from the rough identity-summand graph $G(F_X(T))$ in network analysis and optimization, for further advancement can be viewed as a new direction for future work. The computational efficiency of constructing complete bipartite graphs from $G(F_X(T))$ and generating Gray codes are yet to be explored. This could be considered as a limitation when scaling to larger datasets (or) more complex graph structures.

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Bashyam Praba is a professor in the Department of Mathematics, Sri Sivasubramaniya Nadar College of Engineering, Chennai, India. Her research areas include rough set theory, soft set theory, and algebraic graph theory. She has over 35 years of research and teaching experience and has published over 100 papers in various indexed journals and conference proceedings. She has authored 8 books on discrete mathematics, probability and random processes.

Lourdusamy Packiammal Anto Freeda is a PhD student in the Department of Mathematics, Sri Sivasubramaniya Nadar College of Engineering, Chennai, India. She received her MS degree, in mathematics from Stella Maris College, Chennai, India. Her research include rough set theory and algebraic graph theory.

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