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# EXACT AND APPROXIMATE SOLUTIONS OF A FRACTIONAL DIFFUSION PROBLEM WITH FIXED SPACE MEMORY LENGTH

MALGORZATA KLIMEK [a], TOMASZ BLASZCZYK [a,*]

[a]Department of Mathematics
Czestochowa University of Technology
42-200 Czestochowa, Poland
e-mail: {tomasz.blaszczyk,malgorzata.klimek}@pcz.pl

We study a fractional differential diffusion equation, where the spatial derivative is expressed by the fractional differential operator with a fixed space memory length. The exact solution of the considered problem is presented, taking into account the homogeneous Dirichlet boundary conditions. Additionally, since the solution is in the form of a trigonometric series, we also present approximate solutions in the form of the truncated series. The accuracy of the approximation is controlled by the derived bound of a approximation error.

**Keywords:** fractional diffusion problem, series solution, error estimation, fixed memory length.

## 1. Introduction

Diffusion is the spontaneous spreading and permeation of particles from regions of higher concentration to regions of lower concentration, constituting a fundamental process in various natural phenomena. Classical diffusion appears when the described process has a linear relationship between the mean square displacement and time. On the other hand, for many processes the linear relation does not occur. In complex media, anomalous diffusion occurs due to the presence of heterogeneities, obstacles, or other factors that affect the motion of particles. Then we deal with an anomalous diffusion, which is characterized by a nonlinear relationship between the mean square displacement and time. In this case, traditional linear diffusion models fail, while mathematical models based on fractional differential equations give us promising results.

In contrast to their classical (integer order) counterparts, fractional diffusion equations reflect the nonlocal character of anomalous diffusion phenomena. Anomalous diffusion phenomena are extensively observed in the fields of physics, chemistry and biology (Chechkin *et al.*, 2017; Elkott *et al.*, 2023; Magin, 2006; Magin *et al.*, 2008; Metzler and Klafter, 2000; Tsallis and Lenzi, 2002; Zaslavsky, 2000). For a more detailed analysis of the current state of knowledge on anomalous diffusion, we refer the reader to the review papers by dos Santos (2019) and Evangelista and Lenzi (2018).

This shift in the modeling approach has certain consequences. Namely, the use of fractional derivatives makes the diffusion equation even more difficult to solve. Analytical solutions for partial fractional differential equations are rarely available (Alaroud *et al.*, 2024; Bekir *et al.*, 2015; Das, 2009; Echchaffani *et al.*, 2024; Malinowska *et al.*, 2023). Therefore, various numerical methods for fractional problems have been proposed: an iterative method (Wang and Du, 2013), finite differences (Ciesielski and Leszczynski, 2006; Gu *et al.*, 2021; Lu and Fan, 2025; Meerschaert and Tadjeran, 2004; Tian *et al.*, 2015), the spectral collocation method (Kilbas *et al.*, 2006; Yang *et al.*, 2023) or the finite element method (Zhuang *et al.*, 2016).

A fractional diffusion equation could be considered as a time (when the first-order time derivative is replaced by a fractional derivative), a space (when the second-order spatial derivative is replaced by a fractional derivative) or a time-space (when both the derivatives are modified) fractional diffusion equation. A fractional diffusion equation can contain different types of fractional derivatives. Fractional calculus offers many types of derivatives such as Riemann–Liouville, Caputo,

*Corresponding author

Grünwald–Letnikov, Riesz, or Hadamard (Kilbas *et al.*, 2006; Podlubny, 1999).

In this paper, we analyze a space fractional diffusion problem described by an equation containing a composition of fractional derivatives with a fixed spatial memory length. In the last few years, fractional operators with the fixed memory length have gained attention across various scientific disciplines (Stempin *et al.*, 2023; Sumelka *et al.*, 2015; Sumelka *et al.*, 2020; Voyiadjis *et al.*, 2023). They were introduced in papers (Blaszczyk *et al.*, 2021; Ledesma *et al.*, 2022; Ledesma *et al.*, 2023; Wei *et al.*, 2017) in both left- and right-sided versions. Then, in the paper (Klimek and Blaszczyk, 2024), we studied the oscillator problem within the framework of fractional calculus with fixed memory length. The derived results are applied in the present study of the partial differential space-fractional equation, extending the classical diffusion problem under homogeneous boundary conditions. These types of equations usually describe complex phenomena, where many specific parameters are required. The inclusion of numerous parameters in the model complicates the formal analysis of the problem under consideration. Therefore, the analyzed problem is considered in a nondimensional form. This approach also allows us to capture the universal behavior of the obtained solution during the process of finding and analyzing it.

The article is structured as follows: Section 2 includes all the necessary definitions and properties of fractional operators with fixed memory length. We also recall the results on the eigenfunctions and eigenvalues of the corresponding fractional oscillator problem with homogeneous Dirichlet boundary conditions. Next, Section 3 is divided into two main parts. First, we formulate the fractional diffusion problem based on fractional derivatives with the fixed space memory length. In Section 3.1, we derive its exact solution and study its properties in detail. Section 3.2 is devoted to approximation results and discussions on approximation accuracy. In this part, we also include an analysis of the solutions' behaviour for various examples of the choice of the initial function and problem parameters. The paper is closed with a short Conclusions part and Appendix containing results on estimation of the eigenvalues.

## 2. Preliminaries

Let us start with a short review of the definitions and properties of fractional operators with the fixed memory length. The fractional integrals and derivatives with fixed memory length are given in the definition below.

**Definition 1.** Let $\alpha \in (0,1)$, $L > 0$. The left-sided fractional integral and derivative with fixed memory length are respectively defined as follows:

$$ {}_{x-L}I_x^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_{x-L}^{x} (x-s)^{\alpha-1} f(s)\,\mathrm{d}s = \frac{1}{\Gamma(\alpha)} \int_{x-L}^{x} |x-s|^{\alpha-1} f(s)\,\mathrm{d}s, \tag{1} $$

$$ {}_{x-L}D_x^{\alpha} f(x) := \frac{\mathrm{d}}{\mathrm{d}x} {}_{x-L}I_x^{1-\alpha} f(x). \tag{2} $$

The corresponding right-sided operators are given by

$$ {}_{x}I_{x+L}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{x+L} (s-x)^{\alpha-1} f(s)\,\mathrm{d}s = \frac{1}{\Gamma(\alpha)} \int_{x}^{x+L} |s-x|^{\alpha-1} f(s)\,\mathrm{d}s, \tag{3} $$

$$ {}_{x}D_{x+L}^{\alpha} f(x) := -\frac{\mathrm{d}}{\mathrm{d}x} {}_{x}I_{x+L}^{1-\alpha} f(x). \tag{4} $$

It is an interesting feature of the defined operators that their left- and right-sided versions are connected by the action of the following reflection operators:

$$ Qf(x) := f(L-x), \tag{5} $$

$$ \tilde{Q}f(x) := f(-x). \tag{6} $$

The relations between the left and right fractional operators with fixed memory length are described in the proposition below.

**Proposition 1.** *Let $\alpha \in (0,1)$, $L > 0$ and reflection operators $Q, \tilde{Q}$ be defined by (5) and (6). Then hold*

$$ \begin{aligned} Q\ {}_{x-L}I_x^{\alpha} Qf(x) &= {}_{x}I_{x+L}^{\alpha} f(x), \\ Q\ {}_{x-L}D_x^{\alpha} Qf(x) &= {}_{x}D_{x+L}^{\alpha} f(x), \end{aligned} \tag{7} $$

$$ \begin{aligned} \tilde{Q}\ {}_{x-L}I_x^{\alpha} \tilde{Q}f(x) &= {}_{x}I_{x+L}^{\alpha} f(x), \\ \tilde{Q}\ {}_{x-L}D_x^{\alpha} \tilde{Q}f(x) &= {}_{x}D_{x+L}^{\alpha} f(x). \end{aligned} \tag{8} $$

*Proof.* Let us note that formulas (7) were explicitly calculated by Klimek and Blaszczyk (2024). We shall prove the first part of formula (8) for integrals:

$$ \begin{aligned} &\tilde{Q}\ {}_{x-L}I_x^{\alpha} \tilde{Q}f(x) \\ &= \frac{1}{\Gamma(\alpha)} \int_{-x-L}^{-x} |-x-s|^{\alpha-1} f(-s)\,\mathrm{d}s \\ &= -\frac{1}{\Gamma(\alpha)} \int_{x+L}^{x} |-x+u|^{\alpha-1} f(u)\,\mathrm{d}u \\ &= \frac{1}{\Gamma(\alpha)} \int_{x}^{x+L} |u-x|^{\alpha-1} f(u)\,\mathrm{d}u \\ &= {}_{x}I_{x+L}^{\alpha} f(x). \end{aligned} $$

From this result we get the second part of (8) for derivatives

$$ \tilde{Q}\ {}_{x-L}D_x^{\alpha} \tilde{Q}f(x) = \tilde{Q}\ \frac{\mathrm{d}}{\mathrm{d}x} \tilde{Q}\tilde{Q}_{x-L}I_x^{1-\alpha} \tilde{Q}f(x) $$

$$-\frac{\mathrm{d}}{\mathrm{d}x}\, {}_xI^{1-\alpha}_{x+L} f(x) = {}_xD^{\alpha}_{x+L} f(x).$$

■

The next interesting property of fractional operators is the following formula connecting the left and right fractional integrals valid for periodic or antiperiodic functions. The proof can be found in the paper (Klimek and Blaszczyk, 2024).

**Proposition 2.** *Assume that functions $f$ and $g$ are defined on $\mathbb{R}$, they are locally integrable and simultaneously are both periodic,*

$$f(x \pm 2ML) = f(x), \quad g(x \pm 2ML) = g(x), \tag{9}$$

*or simultaneously antiperiodic,*

$$f(x \pm 2ML) = -f(x), \quad g(x \pm 2ML) = -g(x), \tag{10}$$

*with an arbitrary constant $M \in \mathbb{N}$. Then*

$$\int_{-ML}^{ML} f(x)\ \ {}_{x-L}I^{\alpha}_{x} g(x)\,\mathrm{d}x = \int_{-ML}^{ML} g(x)\ \ {}_xI^{\alpha}_{x+L} f(x)\,\mathrm{d}x. \tag{11}$$

Applying formula (11), we derived the integration by parts formula (explicit calculations can be found in (Klimek and Blaszczyk, 2024)).

**Proposition 3.** *Let the assumptions of Proposition 2 be fulfilled and derivatives ${}_{x-L}D^{\alpha}_{x} f$ and ${}_{x-L}D^{\alpha}_{x} g$ be continuous functions in $[-ML, ML]$. Then*

$$\int_{-ML}^{ML} f(x)\ \ {}_{x-L}D^{\alpha}_{x} g(x)\,\mathrm{d}x = \int_{-ML}^{ML} g(x)\ \ {}_xD^{\alpha}_{x+L} f(x)\,\mathrm{d}x + f(x)\ {}_{x-L}I^{1-\alpha}_{x} g(x)\Big|_{x=-ML}^{ML}. \tag{12}$$

Taking into account the fact that for a periodic function $g$ its integral ${}_{x-L}I^{1-\alpha}_{x} g$ is also a periodic one, we arrive at the following simple version of formula (12):

$$\int_{-ML}^{ML} f(x)\ \ {}_{x-L}D^{\alpha}_{x} g(x)\,\mathrm{d}x = \int_{-ML}^{ML} g(x)\ \ {}_xD^{\alpha}_{x+L} f(x)\,\mathrm{d}x. \tag{13}$$

In the sequel, we shall study the partial fractional differential equation involving the symmetric fractional differential operator constructed using the left and right spatial derivatives, i.e., the oscillator operator with fixed memory length,

$$\mathcal{L} := \frac{1}{2}\left({}_xD^{\alpha}_{x+L}\ \ {}_{x-L}D^{\alpha}_{x} + {}_{x-L}D^{\alpha}_{x}\ \ {}_xD^{\alpha}_{x+L}\right). \tag{14}$$

In our previous work (Klimek and Blaszczyk, 2024), we investigated the solutions through the eigenfunctions of the oscillator equation:

$$\mathcal{L}\ (C_1 \sin(\lambda x) + C_2 \cos(\lambda x)) = \rho(\lambda)(C_1 \sin(\lambda x) + C_2 \cos(\lambda x)), \tag{15}$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary constants and the eigenvalues, connected to parameters $\lambda \in \mathbb{R}$, are given by

$$\rho(\lambda) := (A_{L,\alpha}(\lambda))^2 + (B_{L,\alpha}(\lambda))^2. \tag{16}$$

Functions $A_{L,\alpha}(\lambda)$ and $B_{L,\alpha}(\lambda)$ are given by the explicit formulas

$$\begin{aligned} A_{L,\alpha}(\lambda) &= -\lambda^2 L^{2-\alpha} E_{2,3-\alpha}(-\lambda^2 L^2), \\ B_{L,\alpha}(\lambda) &= \lambda L^{1-\alpha} E_{2,2-\alpha}(-\lambda^2 L^2) \end{aligned} \tag{17}$$

with Mittag-Leffler functions determined for $z \in \mathbb{C}$ by the series

$$E_{\gamma,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + \beta)}. \tag{18}$$

We observe that, in the study of the equation presented above, we have $\frac{2\pi}{\lambda}$-periodic solutions determined for real numbers, with parameters $\alpha \in (0, 1)$, $\lambda \in \mathbb{R}$, $C_1, C_2 \in \mathbb{R}$. When we supplement the oscillator equation with homogeneous Dirichlet or Neumann boundary conditions, we obtain the discrete sets of eigenfunctions establishing orthogonal function bases in the space $L^2(-ML, ML)$. In this paper, we shall study the diffusion type problem with Dirichlet boundary conditions in interval $[-ML, ML]$ with natural number parameter $M \in \mathbb{N}$ determining the finite domain, where we consider the equation

$$\mathcal{L}Y_{\Lambda}(x) = \rho(\Lambda) Y_{\Lambda}(x), \quad x \in [-ML, ML] \tag{19}$$

with solutions connected to eigenvalues $\rho(\Lambda)$ obeying the homogeneous Dirichlet boundary conditions in the form

$$Y_{\Lambda}(-ML) = Y_{\Lambda}(ML) = 0. \tag{20}$$

**Remark 1.** We observe that the even and odd solutions of the above problem can be studied separately due to the commutation relation

$$\tilde{Q}\mathcal{L} = \mathcal{L}\tilde{Q} \tag{21}$$

and the resulting equations

$$\mathcal{L}\frac{1}{2}(1 \pm \tilde{Q})Y_\Lambda(x) = \rho(\Lambda)\frac{1}{2}(1 \pm \tilde{Q})Y_\Lambda(x),$$
$$\frac{1}{2}(1 \pm \tilde{Q})Y_\Lambda(-ML) = \frac{1}{2}(1 \pm \tilde{Q})Y_\Lambda(ML) = 0. \quad (22)$$

The above homogeneous boundary conditions lead to the following two discrete sets of eigenfunctions. The first of them comprises odd functions, corresponds to condition $C_2 = 0$ in (15) and is indexed by parameters $\Lambda_k, \ k \in \mathbb{N}$,

$$\Lambda_k = \frac{k\pi}{ML}, \quad Y_{\Lambda_k}(x) = \sin\left(\frac{k\pi x}{ML}\right). \quad (23)$$

All the eigenfunctions from the first set are orthogonal:

$$\int_{-ML}^{ML} Y_{\Lambda_k}(x)\, Y_{\Lambda_m}(x)\, \mathrm{d}x = 0, \quad \Lambda_k \neq \Lambda_m \quad (24)$$

and $2ML$-periodic.

The second subset of eigenfunctions, obtained for constant $C_1 = 0$ in (15) includes even functions and is indexed by $\tilde{\Lambda}_k, \ k \in \mathbb{N}_0$

$$\tilde{\Lambda}_k = \frac{(k+\frac{1}{2})\pi}{ML},$$
$$Y_{\tilde{\Lambda}_k}(x) = \cos\left(\frac{(k+\frac{1}{2})\pi x}{ML}\right). \quad (25)$$

It is easy to check that the eigenfunctions from the second subset are $2ML$-antiperiodic, cf. (10),

$$Y_{\tilde{\Lambda}_k}(x \pm 2ML) = -Y_{\tilde{\Lambda}_k}(x),$$

and orthogonal

$$\int_{-ML}^{ML} Y_{\tilde{\Lambda}_k}(x)\, Y_{\tilde{\Lambda}_m}(x)\, \mathrm{d}x = 0, \quad \tilde{\Lambda}_k \neq \tilde{\Lambda}_m. \quad (26)$$

Finally, we note that the orthogonality relation for eigenfunctions from the two subsets is also valid,

$$\int_{-ML}^{ML} Y_{\Lambda_k}(x)\, Y_{\tilde{\Lambda}_m}(x)\, \mathrm{d}x = 0. \quad (27)$$

It results from the fact that functions $Y_{\Lambda_k}$ are odd in interval $[-ML, ML]$ whereas $Y_{\tilde{\Lambda}_m}$ are even.

## 3. Space-fractional diffusion problem with fixed space memory length

We shall consider the following diffusion problem, where we defined the $\mathcal{L}$-operator in (14) by using the left and right fractional spatial derivatives with fixed memory length:

$$\frac{\partial u(x,t)}{\partial t} + \mathcal{L}u(x,t) = 0,$$
$$(x,t) \in (-ML, ML) \times (0, \infty). \quad (28)$$

The solutions are subjected to the homogeneous Dirichlet boundary conditions

$$u(-ML, t) = u(ML, t) = 0, \quad t \in [0, \infty) \quad (29)$$

and fulfill the following initial condition:

$$u(x, 0) = g(x), \quad x \in [-ML, ML]. \quad (30)$$

Let us point out that, due to Remark 1, we can separately study the even and odd solutions generated respectively by the even and odd initial functions.

**3.1. Exact solutions of the diffusion problem subjected to the homogeneous Dirichlet boundary conditions.** In the solution of the problem formulated above, we apply the fact that eigenfunctions of the $\mathcal{L}$-operator form an orthogonal function basis in $L^2(-ML, ML)$. Therefore, we construct the solution in the form of the series

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t) Y_{\Lambda_k}(x) + \sum_{m=0}^{\infty} b_m(t) Y_{\tilde{\Lambda}_m}(x) \quad (31)$$

and check when (28) is fulfilled,

$$\sum_{k=1}^{\infty} a'_k(t) Y_{\Lambda_k}(x) + \sum_{m=0}^{\infty} b'_m(t) Y_{\tilde{\Lambda}_m}(x)$$
$$= -\sum_{k=1}^{\infty} a_k(t)\rho(\Lambda_k) Y_{\Lambda_k}(x)$$
$$- \sum_{m=0}^{\infty} b_m(t)\rho(\tilde{\Lambda}_m) Y_{\tilde{\Lambda}_m}(x). \quad (32)$$

Applying the orthogonality properties (24), (26) and (27), we arrive at the set of differential equations for coefficient functions $a_k, \ b_m$,

$$a'_k(t) = -\rho(\Lambda_k) \ a_k(t), \quad k \in \mathbb{N}, \quad (33)$$
$$b'_m(t) = -\rho(\tilde{\Lambda}_m) \ b_m(t), \quad m \in \mathbb{N}_0 \quad (34)$$

which can be easily solved by providing the exact form of coefficients dependent on the time variable,

$$a_k(t) = A_k \exp\left(-\rho(\Lambda_k)\, t\right), \quad k \in \mathbb{N}, \quad (35)$$
$$b_m(t) = B_m \exp\left(-\rho(\tilde{\Lambda}_m)\, t\right), \quad m \in \mathbb{N}_0 \quad (36)$$

with constants $A_k,\ \ B_m$ determined by the initial condition

$$u(x,0)=\sum_{k=1}A_kY_{\Lambda_k}(x)+\sum_{m=0}B_mY_{\tilde{\Lambda}_m}(x)=g(x). \tag{37}$$

These are as follows:

$$\begin{aligned}A_k&=\frac{1}{ML}\int_{-ML}^{ML}g(x)\sin\left(\frac{k\pi x}{ML}\right)\,\mathrm{d}x\\&=\frac{1}{ML}\langle g,Y_{\Lambda_k}\rangle,\end{aligned} \tag{38}$$

$$\begin{aligned}B_m&=\frac{1}{ML}\int_{-ML}^{ML}g(x)\cos\left(\frac{(m+\frac{1}{2})\pi x}{ML}\right)\,\mathrm{d}x\\&=\frac{1}{ML}\langle g,Y_{\tilde{\Lambda}_m}\rangle.\end{aligned} \tag{39}$$

Let us point out that the solution (31) is now given as a formal series

$$\begin{aligned}u(x,t)=\frac{1}{ML}\Bigg(&\sum_{k=1}^{\infty}e^{-\rho(\Lambda_k)t}\langle g,Y_{\Lambda_k}\rangle Y_{\Lambda_k}(x)\\&+\sum_{m=0}^{\infty}e^{-\rho(\tilde{\Lambda}_m)t}\langle g,Y_{\tilde{\Lambda}_m}\rangle Y_{\tilde{\Lambda}_m}(x)\Bigg)\end{aligned} \tag{40}$$

and we shall prove its convergence and analyze the properties of the solution in the sequel. It appears that they depend on the choice of the initial function $g$. Following Remark 1, we shall separately check the convergence and approximation results for the even and odd solutions. Let us begin with initial function $g$, which is an even and continuous function.

**Theorem 1.** *Let function $g\in C[-ML,ML]$ be an even function obeying homogeneous Dirichlet boundary conditions. Then the solution $u$ given by (40) is an absolutely and uniformly convergent series in any compact subset of $[-ML,ML]\times[0,\infty)$, thereby a continuous function of two variables. It also is of class $C^\infty(G)$, where $G$ is any open subset of $[-ML,ML]\times(0,\infty)$.*

*Proof.* For an even version of the solution (40), the coefficients $A_k$ vanish

$$A_k=0,\quad k\in\mathbb{N},$$

while for coefficients $B_m,\ \ m\in\mathbb{N}_0$ the relation (39) holds. In the considered case, the even solution is

$$u(x,t)=\frac{1}{ML}\sum_{m=0}^{\infty}e^{-\rho(\tilde{\Lambda}_m)t}\langle g,Y_{\tilde{\Lambda}_m}\rangle Y_{\tilde{\Lambda}_m}(x). \tag{41}$$

We estimate the absolute value of sum series $u$ at any point of set $(-ML,ML)\times(0,\infty)$ applying Proposition A1 and determining lower bounds for eigenvalues $\rho(\tilde{\Lambda}_m),\ \ m\in\mathbb{N}$,

$$\begin{aligned}-\rho(\tilde{\Lambda}_m)&\le-(m+\frac{1}{2})\gamma_e<0,\\-\rho(\tilde{\Lambda}_0)&\le-\gamma_e/4<0,\end{aligned} \tag{42}$$

$$\begin{aligned}|u(x,t)|&\le 2||g||\left(e^{-\rho(\tilde{\Lambda}_0)t}+\sum_{m=1}^{\infty}e^{-\rho(\tilde{\Lambda}_m)t}\right)\\&\le 2||g||\left(e^{-\gamma_e\,t/4}+\sum_{m=1}^{\infty}e^{-(m+\frac{1}{2})\gamma_e\,t}\right)\\&=2||g||\left(e^{-\gamma_e\,t/4}+\frac{e^{-3\gamma_e\,t/2}}{1-e^{-\gamma_e t}}\right),\end{aligned} \tag{43}$$

where constant $\gamma_e$ is given by (A2). We observe that the series majorizing our series solution is convergent in any compact subset of $[-ML,ML]\times(0,\infty)$. From the Weierstrass test, we infer that the solution, described as the series (40), is absolutely and uniformly convergent. Therefore, its sum is a continuous function of two variables.

Now, we analyze properties of the series representing first-order partial derivatives

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t)=&-\frac{1}{ML}\sum_{m=0}^{\infty}e^{-\rho(\tilde{\Lambda}_m)t}\rho(\tilde{\Lambda}_m)\\&\times\langle g,Y_{\tilde{\Lambda}_m}\rangle Y_{\tilde{\Lambda}_m}(x),\end{aligned} \tag{44}$$

$$\begin{aligned}\frac{\partial}{\partial x}u(x,t)=&-\frac{\pi}{M^2L^2}\sum_{m=0}^{\infty}e^{-\rho(\tilde{\Lambda}_m)t}\langle g,Y_{\tilde{\Lambda}_m}\rangle\\&\times(m+\frac{1}{2})\sin\left(\frac{(m+\frac{1}{2})\pi x}{ML}\right).\end{aligned} \tag{45}$$

For the series determining the time derivative, we obtain the following estimate:

$$\begin{aligned}\left|\frac{\partial}{\partial t}u(x,t)\right|\le&\frac{1}{ML}\Bigg(e^{-\gamma_e t/4}\rho(\tilde{\Lambda}_0)|\langle g,Y_{\tilde{\Lambda}_0}\rangle|\\&+\sum_{m=1}^{\infty}e^{-(m+\frac{1}{2})\gamma_e t}\rho(\tilde{\Lambda}_m)|\langle g,Y_{\tilde{\Lambda}_m}\rangle|\Bigg)\\\le&\frac{4||g||\gamma_e M}{\left(\cos\left(\frac{\pi}{4M}\right)\right)^2}\Bigg(\frac{1}{4}e^{-\gamma_e t/4}\\&+\sum_{m=1}^{\infty}e^{-(m+\frac{1}{2})\gamma_e t}(m+\frac{1}{2})^2\Bigg),\end{aligned} \tag{46}$$

where we applied Propositions A1 and A2 with constant $\gamma_e$ defined in Eqn. (A2). The series on the right-hand side is convergent for points from any open subset of $[-ML,ML]\times(0,\infty)$. This fact can be tested using the ratio test, namely, we have the following limit for the ratio of two consecutive terms of this series:

$$\lim_{m\to\infty} \frac{e^{-(m+\frac{3}{2})\gamma_e t}\cdot (m+\frac{3}{2})^2}{e^{-(m+\frac{1}{2})\gamma_e t}\cdot (m+\frac{1}{2})^2} = \lim_{m\to\infty}\left(1+\frac{1}{m+\frac{1}{2}}\right)^2 e^{-\gamma_e t} = e^{-\gamma_e t} < 1, \quad t>0.$$

From the ratio test, we infer that the series on the right-hand side of the above inequalities is convergent in any open subset $[-ML, ML]\times(0,\infty)$; therefore, by the Weierstrass test, the series representing partial derivative $\frac{\partial u}{\partial t}$ is uniformly and absolutely convergent in this set. Thus, its sum, the derivative $\frac{\partial u}{\partial t}$, is a continuous function in any open subset $[-ML, ML]\times(0,\infty)$.

Next, we investigate the convergence of the series representing the spatial derivative and we get the estimate

$$\begin{aligned}\left|\frac{\partial}{\partial x}u(x,t)\right| &\le \frac{\pi}{M^2L^2}\sum_{m=0}^{\infty} e^{-\rho(\tilde{\Lambda}_m)t}|\langle g, Y_{\tilde{\Lambda}_m}\rangle|(m+\frac{1}{2}) \\ &\le \frac{2||g||\pi}{ML}\left(\frac{1}{2}e^{-\gamma_e/4} + \sum_{m=1}^{\infty} e^{-(m+\frac{1}{2})\gamma_e t}(m+\frac{1}{2})\right) \end{aligned} \tag{47}$$

with constant $\gamma_e$ given by Eqn. (A2). Again, we apply the ratio test calculating the limit for the ratio of two consecutive terms of the series on the right-hand side

$$\lim_{m\to\infty} \frac{e^{-(m+3/2)\gamma_e t}\cdot (m+\frac{3}{2})}{e^{-(m+\frac{1}{2})\gamma_e t}\cdot (m+\frac{1}{2})} = \lim_{m\to\infty}\left(1+\frac{1}{m+\frac{1}{2}}\right) e^{-\gamma_e t} = e^{-\gamma_e t} < 1, \quad t>0.$$

From ratio the test, we infer that the series on the right-hand side of the above inequalities is convergent in any open subset $[-ML, ML]\times(0,\infty)$; therefore, by the Weierstrass test, the series representing the partial derivative $\frac{\partial u}{\partial x}$ is uniformly and absolutely convergent in this set. Thus, its sum, the derivative $\frac{\partial u}{\partial x}$, is a continuous function in any open subset $G\subset[-ML, ML]\times(0,\infty)$.

We point out that the continuity of the function $u$ and its partial derivatives implies that the solution is of class $C^1(G)$, in any such open subset.

To prove the main result, we use the mathematical induction principle. Assuming that the function and all its partial derivatives of order up to $n-1$, $\quad n>1$, are continuous in any open subset of $[-ML, ML]\times(0,\infty)$ and, therefore, function $u$ is of class $C^{n-1}(G)$ we investigate the continuity properties of all the $n$-th order partial derivatives. In general, the $n$-th order partial derivatives are as follows ($n\in\mathbb{N},\ l=0,\dots,n$):

$$\frac{\partial^n u(x,t)}{\partial t^{n-l}\partial x^l} = C_{n,l}\sum_{m=0}^{\infty} e^{-\rho(\tilde{\Lambda}_m)\,t}\langle g, Y_{\tilde{\Lambda}_m}\rangle \left(\rho(\tilde{\Lambda}_m)\right)^{n-l} \times (m+\frac{1}{2})^l \cos\left(\frac{(m+\frac{1}{2})\pi x}{ML}+\frac{l\pi}{2}\right), \tag{48}$$

where $C_{n,l}$ is a constant dependent solely on the order of the derivative and the divide of derivatives between time and space derivatives. We observe that the series on the right-hand side can be estimated as follows by using Propositions A1 and A2:

$$\begin{aligned}\left|\frac{\partial^n u(x,t)}{\partial t^{n-l}\partial x^l}\right| &\le |C_{n,l}|\cdot||g||\cdot 2ML\left(\frac{\pi}{ML}\right)^l \\ &\times\left(\frac{2ML\cdot\gamma_e}{\cos^2\left(\frac{\pi}{4M}\right)}\right)^{n-l}\left(\frac{1}{2^{2n-l}}e^{-\gamma_e t/4} + \sum_{m=1}^{\infty} e^{-(m+\frac{1}{2})\gamma_e t}(m+\frac{1}{2})^{2n-l}\right). \end{aligned} \tag{49}$$

Applying the ratio convergence test, we get the following limit for the ratio of two consecutive terms:

$$\lim_{m\to\infty} \frac{e^{-(m+3/2)\gamma_e t}\cdot (m+3/2)^{2n-l}}{e^{-(m+\frac{1}{2})\gamma_e t}\cdot (m+\frac{1}{2})^{2n-l}} = \lim_{m\to\infty}\left(1+\frac{1}{m+\frac{1}{2}}\right)^{2n-l} e^{-\gamma_e t} = e^{-\gamma_e t} < 1, \quad t>0.$$

Therefore, we conclude that the above series majorizing the series determining the $n$-th order derivative is convergent. From the Weierstrass test, we infer that all the series representing the $n$-th order partial derivatives are absolutely and uniformly convergent. Hence these derivatives are continuous in any open subset $G\subset[-ML, ML]\times(0,\infty)$.

Finally, we note that since the function $u$ and all its partial derivatives of order up to $n$ are continuous in any open subset $G$, the function is of class $C^n(G)$. In turn, as the order $n$ is arbitrary and we can extend the above discussion and results for any order, we conclude that solution $u$ given by formula (41) is a function of class $C^\infty(G)$. ■

We point out that analogous procedures can be applied in the investigation of solutions generated by an odd initial function in Eqn. (30). We shall use them in estimation formulas (A4) and (A12) to determine the lower and upper bounds for eigenvalues $\rho(\Lambda_k),\ k \in \mathbb{N}$ of the operator $\mathcal{L}$ from Propositions A1 and A2, respectively. All the calculations and reasoning are similar to the ones presented above in detail. Therefore we omit the proof and formulate the result for the odd solutions below.

**Theorem 2.** *Let $g \in C[-ML, ML]$ be an odd function obeying homogeneous Dirichlet boundary conditions. Then the solution $u$ resulting from (40) has the following form:*

$$u(x,t) = \frac{1}{ML} \sum_{k=1}^{\infty} e^{-\rho(\Lambda_k)t} \langle g, Y_{\Lambda_k} \rangle Y_{\Lambda_k}(x). \tag{50}$$

*The above series is absolutely and uniformly convergent in any compact subset of $[-ML, ML] \times (0, \infty)$. Therefore its sum is a continuous function of two variables. It is also of class $C^\infty(G)$, where $G$ is any open subset of $[-ML, ML] \times (0, \infty)$.*

In addition, from Theorems 1 and 2, the following result is easily deduced, where the case with a general, continuous initial function $g$ is described.

**Corollary 1.** *Let function $g \in C[-ML, ML]$ obey the homogeneous Dirichlet boundary conditions. Then the solution $u$ represented by the series from formula (40) is absolutely and uniformly convergent in any compact subset of $(-ML, ML) \times [0, \infty)$. Therefore, its sum is a continuous function of two variables. It also is of class $C^\infty(G)$, where $G$ is any open subset of $[-ML, ML] \times (0, \infty)$*

**Remark 2.** We note that the estimate (43) for the solution generated by an initial even function $g$ and the estimate

$$|u(x,t)| \le 2||g|| \sum_{k=1}^{\infty} e^{-\rho(\Lambda_k)t} \le 2||g|| \sum_{k=1}^{\infty} e^{-k\gamma_o\ t} \tag{51}$$

for a solution corresponding to an odd initial function $g$ clearly indicate that the solutions described in the above theorems tend exponentially to zero as time $t \to \infty$.

We shall close this section with a simple calculation of the exact solution for the case when the initial function is of the class $C^2(-ML, ML)$. The results are given below for the cases of even and odd initial functions, respectively.

**Proposition 4.** *Let $g \in C^2(-ML, ML)$ be an even function obeying homogeneous Dirichlet boundary conditions. Then the solution $u$, given by (40), is of the form*

$$\begin{aligned} u(x,t) = -\frac{ML}{\pi^2} \sum_{m=0}^{\infty} & e^{-\rho(\tilde{\Lambda}_m)t} \\ & \times \frac{\langle g'', Y_{\tilde{\Lambda}_m} \rangle}{(m+\frac{1}{2})^2} Y_{\tilde{\Lambda}_m}(x), \end{aligned} \tag{52}$$

*Proof.* For the even version of the solution (40), the coefficients $A_k$ vanish

$$A_k = 0, \quad k \in \mathbb{N},$$

while for the coefficients $B_m,\ m \in \mathbb{N}_0$ the following relation holds:

$$\begin{aligned} B_m &= \frac{1}{ML} \int_{-ML}^{ML} g(x) \cos\left(\frac{(m+\frac{1}{2})\pi x}{ML}\right) \mathrm{d}x \\ &= -\frac{ML}{\left((m+\frac{1}{2})\pi\right)^2} \int_{-ML}^{ML} g''(x) \\ &\quad \times \cos\left(\frac{(m+\frac{1}{2})\pi x}{ML}\right) \mathrm{d}x \\ &= -\frac{ML}{\left((m+\frac{1}{2})\pi\right)^2} \langle g'', Y_{\tilde{\Lambda}_m} \rangle, \end{aligned} \tag{53}$$

which results from the integration by parts formula and the homogeneous Dirichlet boundary conditions (29). Substituting coefficients $B_m$ into (40), we get the even solution in the form

$$u(x,t) = -\frac{ML}{\pi^2} \sum_{m=0}^{\infty} e^{-\rho(\tilde{\Lambda}_m)t} \cdot \frac{\langle g'', Y_{\tilde{\Lambda}_m} \rangle}{(m+\frac{1}{2})^2} Y_{\tilde{\Lambda}_m}(x). \tag{54}$$

■

We shall now formulate the analogous proposition for the odd solution of problem (28)–(30) and omit its proof since it is analogous to the one presented above for the even solution.

**Proposition 5.** *Let function $g \in C^2(-ML, ML)$ be an odd function obeying homogeneous Dirichlet boundary conditions. Then the solution $u$ resulting from (40) has the form*

$$u(x,t) = -\frac{ML}{\pi^2} \sum_{k=1}^{\infty} e^{-\rho(\Lambda_k)t} \cdot \frac{\langle g'', Y_{\Lambda_k} \rangle}{k^2} Y_{\Lambda_k}(x). \tag{55}$$

**3.2. Approximation.** In the previous part of our paper, we constructed the exact solutions of the problem (28)–(30) in the form of series (40) or (52), respectively, for even solutions corresponding to even initial functions, and (50) or (55) for odd solutions generated by the odd initial function. We also presented the continuity and $C^\infty$ class results proved for solutions generated by the continuous initial function appearing in condition (30). The respective theorems were formulated separately for even and odd solutions in view of Remark 1. The result for an arbitrary continuous initial function is given in Corollary 1.

Now, we shall consider approximate solutions, where we simplify solutions to the finite sums and investigate the estimation of approximation error. Again, we start with even solutions and first study approximation of the solution to the problem with a continuous initial function.

**Proposition 6.** *Let $g \in C[-ML, ML]$ be an even function obeying homogeneous Dirichlet boundary conditions (29). Then, the approximation of the solution $u$ determined by formula (41) is given as*

$$u_{\text{app}}(x,t) = \frac{1}{ML} \sum_{m=0}^{M_{\text{app}}} e^{-\rho(\tilde{\Lambda}_m)t} \langle g, Y_{\tilde{\Lambda}_m} \rangle Y_{\tilde{\Lambda}_m}(x) \quad (56)$$

*and generates the approximation error with the upper bound given below*

$$|u(x,t) - u_{\text{app}}(x,t)| \le e^{-\,(M_{\text{app}}+3/2)\gamma_e\, t} \cdot \frac{2\,||g||}{1-e^{-\gamma_e t}}, \quad x \in [-ML, ML], \quad t > 0 \quad (57)$$

*with constant $\gamma_e$ given by (A2).*

*Proof.* First, from Proposition A1, we get the following set of inequalities for the eigenvalues of the $\mathcal{L}$ operator:

$$\begin{aligned} \rho(\tilde{\Lambda}_m) &\ge (m+\frac{1}{2})\gamma_e \\ &> (M_{\text{app}}+\frac{1}{2})\gamma_e, \quad m > M_{\text{app}}, \\ -\rho(\tilde{\Lambda}_m) &\le -(m+\frac{1}{2})\gamma_e \\ &< -(M_{\text{app}}+\frac{1}{2})\gamma_e, \quad m > M_{\text{app}}. \end{aligned} \quad (58)$$

Now, we calculate the difference between the full solution (41) and its approximation (56):

$$u(x,t) - u_{\text{app}}(x,t) = \frac{1}{ML} \sum_{m=M_{\text{app}}+1}^{\infty} e^{-\rho(\tilde{\Lambda}_m)t} \langle g, Y_{\tilde{\Lambda}_m} \rangle Y_{\tilde{\Lambda}_m}(x). \quad (59)$$

We estimate the approximation error at any point of set $[-ML, ML] \times (0, \infty)$ applying the above inequalities for the eigenvalues resulting from Proposition A1 and obtain Eqn. (57)

$$\begin{aligned} |u(x,t) - u_{\text{app}}(x,t)| &\le 2\,||g|| \sum_{m=M_{\text{app}}+1}^{\infty} e^{-\rho(\tilde{\Lambda}_m)t} \\ &\le 2\,||g|| \sum_{m=M_{\text{app}}+1}^{\infty} e^{-\,(m+\frac{1}{2})\gamma_e t} \\ &= e^{-\,(M_{\text{app}}+3/2)\gamma_e t} \cdot \frac{2\,||g||}{1-e^{-\gamma_e t}}. \end{aligned} \quad (60)$$

■

**Remark 3.** Let us note that the estimating function on the right-hand side of (57) is a rapidly decreasing function of time. It consists of the exponential part and function $(1-e^{-\gamma_e t})^{-1}$ which is also decreasing.

Thus, the error is an exponentially and decreasing function of both the parameter $M_{\text{app}}$ the time variable. The estimation of approximation error (57) can be improved when we move to the problem with the initial function $g$ being of class $C^2$ as described in Proposition 4. Such a case is discussed in the proposition below.

**Proposition 7.** *Let $g \in C^2(-ML, ML)$ be an even function obeying homogeneous Dirichlet boundary conditions. Then, the approximation of solution $u$ determined by (52) is given as*

$$u_{\text{app}}(x,t) = -\frac{ML}{\pi^2} \sum_{m=0}^{M_{\text{app}}} e^{-\rho(\tilde{\Lambda}_m)t} \frac{\langle g'', Y_{\tilde{\Lambda}_m} \rangle}{(m+\frac{1}{2})^2} Y_{\tilde{\Lambda}_m}(x) \quad (61)$$

*and generates the approximation error with the upper bound given below:*

$$\begin{aligned} |u(x,t) - u_{\text{app}}(x,t)| &\le \frac{e^{-\,(M_{\text{app}}+3/2)\gamma_e\, t}}{(M_{\text{app}}+3/2)^2} \cdot \frac{2\,M^2L^2\,||g''||}{\pi^2(1-e^{-\gamma_e t})}, \\ & x \in [-ML, ML], \quad t > 0 \end{aligned} \quad (62)$$

*with constant $\gamma_e$ given by (A2).*

*Proof.* Let us note that in the case when the initial function is of class $C^2$, we get the exact form of the solution to the problem (28)–(30) described in (52). Therefore, the difference between the full solution and its approximation (61) is given by the series

$$\begin{aligned} u(x,t) - u_{\text{app}}(x,t) &= -\frac{ML}{\pi^2} \sum_{m=M_{\text{app}}+1}^{\infty} e^{-\rho(\tilde{\Lambda}_m)t} \\ &\quad \times \frac{\langle g'', Y_{\tilde{\Lambda}_m} \rangle}{(m+\frac{1}{2})^2} Y_{\tilde{\Lambda}_m}(x). \end{aligned} \quad (63)$$

■

In the estimation of the approximation error, we again apply Proposition A1 and resulting inequalities for eigenvalues (58). In this manner, we get Eqn. (62) valid for any point in set $[-ML, ML] \times (0, \infty)$:

$$\begin{aligned} |u(x,t) - u_{\text{app}}(x,t)| &\le \frac{2\,M^2L^2\,||g''||}{\pi^2} \sum_{m=M_{\text{app}}+1}^{\infty} \frac{e^{-\rho(\tilde{\Lambda}_m)t}}{(m+\frac{1}{2})^2} \\ &\le \frac{2\,M^2L^2\,||g''||}{\pi^2(M_{\text{app}}+\frac{3}{2})^2} \sum_{m=M_{\text{app}}+1}^{\infty} e^{-\,(m+\frac{1}{2})\gamma_e t} \\ &= \frac{e^{-\,(M_{\text{app}}+\frac{3}{2})\gamma_e\, t}}{(M_{\text{app}}+\frac{3}{2})^2} \cdot \frac{2\,M^2L^2\,||g''||}{\pi^2(1-e^{-\gamma_e t})}. \end{aligned} \quad (64)$$

Now, we formulate the corresponding versions of Propositions 6 and 7 when the initial function $g$ is an odd function. First, we have the case of the continuous initial function for which we get the approximation error bound described in the proposition below. The proof is similar to the one of Proposition 6.

**Proposition 8.** *Let $g \in C[-ML, ML]$ be an odd function obeying homogeneous Dirichlet boundary conditions. Then, the approximation of solution $u$ determined by (50) is given as*

$$u_{\text{app}}(x,t) = \frac{1}{ML} \sum_{k=1}^{K_{\text{app}}} e^{-\rho(\Lambda_k)t} \langle g, Y_{\Lambda_k} \rangle Y_{\Lambda_k}(x) \quad (65)$$

*and generates the approximation error with the upper bound*

$$|u(x,t) - u_{\text{app}}(x,t)| \le e^{-\,(K_{\text{app}}+1)\gamma_o\, t} \cdot \frac{2\,||g||}{1-e^{-\gamma_o t}}, \quad x \in [-ML, ML], \quad t > 0 \quad (66)$$

*with constant $\gamma_o$ given in formula (A5).*

Similarly to the case of the even initial function of the $C^2$ class discussed in Proposition 7, we have an analogous proposition valid for an odd initial function, and we omit the proof as it is analogous to the one of Proposition 7.

**Proposition 9.** *Let $g \in C^2[-ML, ML]$ be an odd function obeying homogeneous Dirichlet boundary conditions. Then the approximation of solution $u$ determined by (52) is*

$$u_{\text{app}}(x,t) = -\frac{ML}{\pi^2} \sum_{k=1}^{K_{\text{app}}} e^{-\rho(\Lambda_k)\,t} \frac{\langle g'', Y_{\Lambda_k} \rangle}{k^2} Y_{\Lambda_k}(x) \quad (67)$$

*and generates the approximation error with the upper bound*

$$|u(x,t) - u_{\text{app}}(x,t)| \le \frac{e^{-\,(K_{\text{app}}+1)\gamma_o\, t}}{(K_{\text{app}}+1)^2} \cdot \frac{2\,M^2L^2\,||g''||}{\pi^2(1-e^{-\gamma_o t})}, \quad (x,t) \in [-ML, ML], \quad t > 0 \quad (68)$$

*with constant $\gamma_o$ given by (A5).*

We note that all the above results provide bounds on approximation errors dependent on both the length of the approximating trigonometric polynomial and the specific value of the time variable. First, we see that in the case of a continuous initial function $g$ for any chosen value of the time variable $t$ the approximation error exponentially decreases for an increasing number of terms in the approximate solution (56) and (65). When the function $g \in C^2(-ML, ML)$ and the approximate solution are given by (61) and (67), the error decreases even faster. Secondly, when we keep constant parameters $M_{\text{app}}$ or $K_{\text{app}}$, we observe that the error bound decreases faster than the corresponding exponential in (57), (62), (66) and (68).

**Remark 4.** Analyzing the above-mentioned results on approximation errors, we note that fixing the error bound at time $t_0 > 0$ and denoting it as $\Delta_{t_0}$, we obtain the following estimate of the approximation error at $t > t_0$. Results for even solutions (57) and (62) yield the relation:

$$\Delta_t \le \Delta_{t_0} e^{-(M_{\text{app}}+\frac{3}{2})\gamma_e(t-t_0)}$$

and Eqns. (66) and (68) describing the errors for odd solutions lead to the relation

$$\Delta_t \le \Delta_{t_0} e^{-(K_{\text{app}}+1)\gamma_o(t-t_0)}.$$

In conclusion, for the considered problems we can establish the required level of accuracy of the approximation at the fixed moment of time and the above inequalities show the increasing accuracy of the procedure for $t > t_0$.

**Remark 5.** In addition, let us observe that $\gamma_e = \gamma_e(\alpha, M, L)$, i.e., this constant is in fact a function of problem parameters (see (A2)). The function describing the bound of the approximation error includes the factor $(1 - e^{-\gamma_e\, t})^{-1}$ which leads to singularity whenever $\gamma_e\, t$ tends to zero. Therefore, the approximation accuracy must be analyzed carefully at each step of the investigation of the behavior of solutions. The same remark also holds for the case with an odd initial function as $\gamma_o = \gamma_o(\alpha, M, L)$ (cf. (A5)).

On the other hand, when we approximate the solution close to $t = 0$, then the estimates of the error bounds indicate that more terms in the respective approximation formula are necessary to keep the assumed accuracy of the procedure. It is possible to overcome this difficulty, at least in the case of initial function in class $C^2(-ML, ML)$. For even and odd versions of the solution, the estimates independent of time are valid.

In the next propositions we prove formulas which bound the approximation error by the inverse of parameters $M_{\text{app}}$ or $K_{\text{app}}$, respectively.

**Proposition 10.** *Let $g \in C^2[-ML, ML]$ be an even function obeying homogeneous Dirichlet boundary conditions. Then, the approximation of solution $u$ given in (61) generates the approximation error with the upper bound*

$$|u(x,t) - u_{\text{app}}(x,t)| \le \frac{2M^2L^2||g''||}{\pi^2(M_{\text{app}}+\frac{1}{2})}, \quad (x,t) \in [-ML, ML] \times (0, \infty). \quad (69)$$

*Proof.* We calculate the error based on the exact form of the solution (52) and its approximation (61) to obtain

$$\begin{aligned}
&|u(x,t) - u_{\text{app}}(x,t)| \\
&\quad \le \frac{2M^2L^2||g''||}{\pi^2} \sum_{m=M_{\text{app}}+1}^{\infty} \frac{1}{(m+\frac{1}{2})^2} \\
&\quad \le \frac{2M^2L^2||g''||}{\pi^2} \int_{M_{\text{app}}}^{\infty} \frac{1}{(u+\frac{1}{2})^2}\,\mathrm{d}u \\
&\quad = \frac{2M^2L^2||g''||}{\pi^2(M_{\text{app}}+\frac{1}{2})}.
\end{aligned}$$

■

Analogous calculations yield the the approximation error formula for the case of an odd initial function.

**Proposition 11.** *Let $g \in C^2[-ML, ML]$ be an odd function obeying homogeneous Dirichlet boundary conditions. Then the approximation of solution $u$ given in (67) generates the approximation error with the upper bound*

$$|u(x,t) - u_{\text{app}}(x,t)| \le \frac{2M^2L^2\,||g''||}{\pi^2\,K_{\text{app}}},$$
$$(x,t) \in [-ML, ML] \times (0,\infty). \quad (70)$$

The above results give bounds for approximation errors which will be particularly useful in the investigation of the quality of approximation close to $t = 0$. Let us point out that for specific examples, these bounds can be improved (see Example 3, formula (80)). In addition, we shall analyze the case with a continuous initial function $g \notin C^2(-ML, ML)$ in Example 2. It appears that the bound for approximation error, independent from the time variable, can be derived based on the exact form of the solution (cf. Eqn. (77)). Therefore, the approximation for the time variable close to $t = 0$ can also be developed with a reliable analysis of its accuracy.

In the examples below, we study various solutions with initial functions given in the form of the delta Dirac function, a continuous function not belonging to the $C^2$ class and function from the $C^2$ class. In all cases we calculate the explicit form of the solution, approximation errors and present graphs describing the time evolution as well as comparing the properties of the solution for various choices of problem parameters such as fractional order $\alpha$ and memory length $L$.

**Example 1.** (A Dirac delta function in the initial condition (30)). Assume the initial function in the form of the Dirac delta function

$$g(x) = \delta(x). \quad (71)$$

The solution $u$ is determined as a series of even terms in the following form:

$$u(x,t) = \sum_{m=0}^{\infty} \exp\left(-\rho(\tilde{\Lambda}_m)\,t\right) \cos\left(\frac{(m+\frac{1}{2})\pi x}{ML}\right). \quad (72)$$

We shall check the convergence as the initial function does not fulfill the assumptions of Theorems 1 and 2. In the estimation, we apply the property (58) resulting from Proposition A1 and for any point $(x,t) \in [-ML, ML] \times (0,\infty)$ we get

$$\begin{aligned}
|u(x,t)| &\le \sum_{m=0}^{\infty} \exp\left(-\rho(\tilde{\Lambda}_m)\,t\right) \\
&\le \sum_{m=0}^{\infty} \exp\left(-(m+\frac{1}{2})\gamma_e t\right) \\
&= \frac{e^{-\gamma_e t/2}}{1-e^{-\gamma_e t}}. \quad (73)
\end{aligned}$$

Analogously to the proof of Theorem 1, we apply the Weierstrass test of function series convergence and conclude that the solution $u$ as a sum of series convergent absolutely and uniformly in each compact subset of $[-ML, ML] \times (0,\infty)$ is a continuous function in any such subset.

Next, we assume the approximate solution in the form of the finite sum

$$u_{\text{app}}(x,t) = \sum_{m=0}^{M_{\text{app}}} e^{-\rho(\tilde{\Lambda}_m)t} Y_{\tilde{\Lambda}_m}(x) \quad (74)$$

and calculate the approximation error for any $(x,t) \in [-ML, ML] \times (0,\infty)$,

$$\begin{aligned}
|u(x,t) - u_{\text{app}}(x,t)| &\le \sum_{m=M_{\text{app}}+1}^{\infty} e^{-\rho(\tilde{\Lambda}_m)t} \\
&\le \sum_{m=M_{\text{app}}+1}^{\infty} e^{-(m+\frac{1}{2})\gamma_e t} \\
&= \frac{e^{-\,(M_{\text{app}}+3/2)\gamma_e t}}{1-e^{-\gamma_e t}} \quad (75)
\end{aligned}$$

with constant $\gamma_e$ given by (A2).

We analyze the behavior of solutions by applying their approximate versions. First, in the graphs included in Fig. 1, we compare time evolution of the solutions for orders $\alpha = 1$ and $\alpha = 0.7$ for fixed values of parameters $M = 10, L = 0.25$. Both solutions are decreasing functions of time, tend to zero as $t \to \infty$, which results from the estimate (73). However, comparing the movement of maximum of the solution, we see that the solution for a fractional order of $\alpha = 0.7$ decreases more slowly than in the classical case.

Next, Fig. 2 includes two graphs. In the first one we compare solutions for fixed values of parameters $M \cdot L = 12, \alpha = 0.8$, time moment $t = 25$ and a varying memory length. We observe that the evolution slows down when the memory length decreases for the constant length of the problem domain $2ML = const$. Next, we fix values $M = 10, L = 0,25$ and time moment $t = 3$. Then, we change the fractional order $\alpha \in \{0.7, 0.8, 0.9, 1.0\}$ and note that the evolution for the decreasing value of the fractional order becomes slower.

For the examples described above, all calculations, have been performed assuming that the approximation error is less than $10^{-6}$. This assumption required truncating the series to $M_{\text{app}} = 1100$. The $M_{\text{app}}$ value was determined based on the estimate (75). ♦

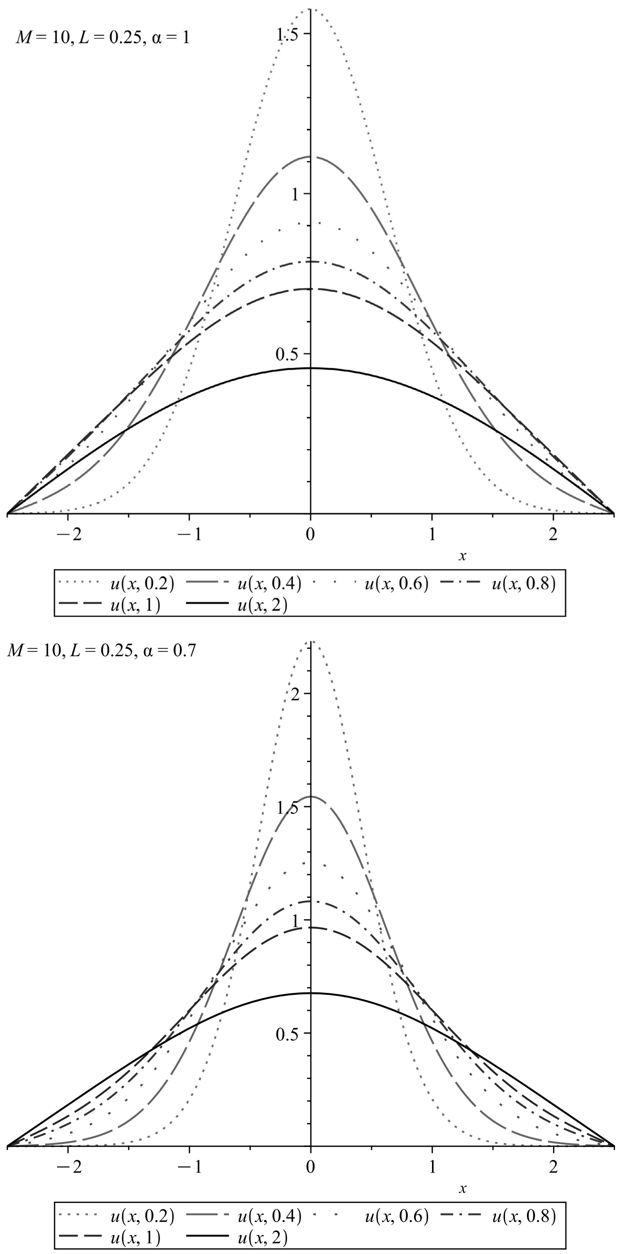


Fig. 1. The approximate solution of the problem (28)–(30) for the initial function $g(x) = \delta(x)$ and for different times.

**Example 2.** (A continuous and even initial function) Next, we consider the example with the even initial function $g(x) = ML - |x|$. Applying Theorem 1, we get coefficients

$$B_m = \frac{2ML}{\left((m+\frac{1}{2})\pi\right)^2}$$

and the even solution in the form of the series

$$u(x,t) = \frac{2ML}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(m+\frac{1}{2})^2} \times \exp\left(-\rho(\tilde{\Lambda}_m)t\right) \cos\left(\frac{(m+\frac{1}{2})\pi x}{ML}\right).$$

Applying Theorem 1 and the fact that the initial function is continuous here, we infer that the solution $u$ is a continuous function of two variables in any compact subset of $[-ML, ML] \times (0,\infty)$ and a function of the $C^\infty(G)$ in any open subset $G$.

In addition, using (57), we estimate the approximation error (here $||g|| = ML$)

$$|u(x,t) - u_{\text{app}}(x,t)| \leq \frac{e^{-\ (M_{\text{app}}+\frac{1}{2})\gamma_e\ t}}{1-e^{-\gamma_e t}} \cdot 2ML \tag{76}$$

with constant $\gamma_e$ given by (A2).

Let us note that the above estimate does not fully describe the approximation error when time $t$ is close to zero. The initial function in this example is not of $C^2(-ML, ML)$ class. Therefore, we cannot apply the result from Proposition 10. We calculate the error based on the exact form of the solution and obtain the following formula, which is useful for the time variable $t$ close to zero:

$$\begin{aligned} &|u(x,t) - u_{\text{app}}(x,t)| \\ &\quad\leq \frac{2ML}{\pi^2} \sum_{m=M_{\text{app}}+1}^{\infty} \frac{1}{(m+\frac{1}{2})^2} \\ &\quad\leq \frac{2ML}{\pi^2} \int_{M_{\text{app}}}^{\infty} \frac{1}{(u+\frac{1}{2})^2}\,\mathrm{d}u \\ &\quad= \frac{2ML}{\pi^2(M_{\text{app}}+\frac{1}{2})}. \end{aligned} \tag{77}$$

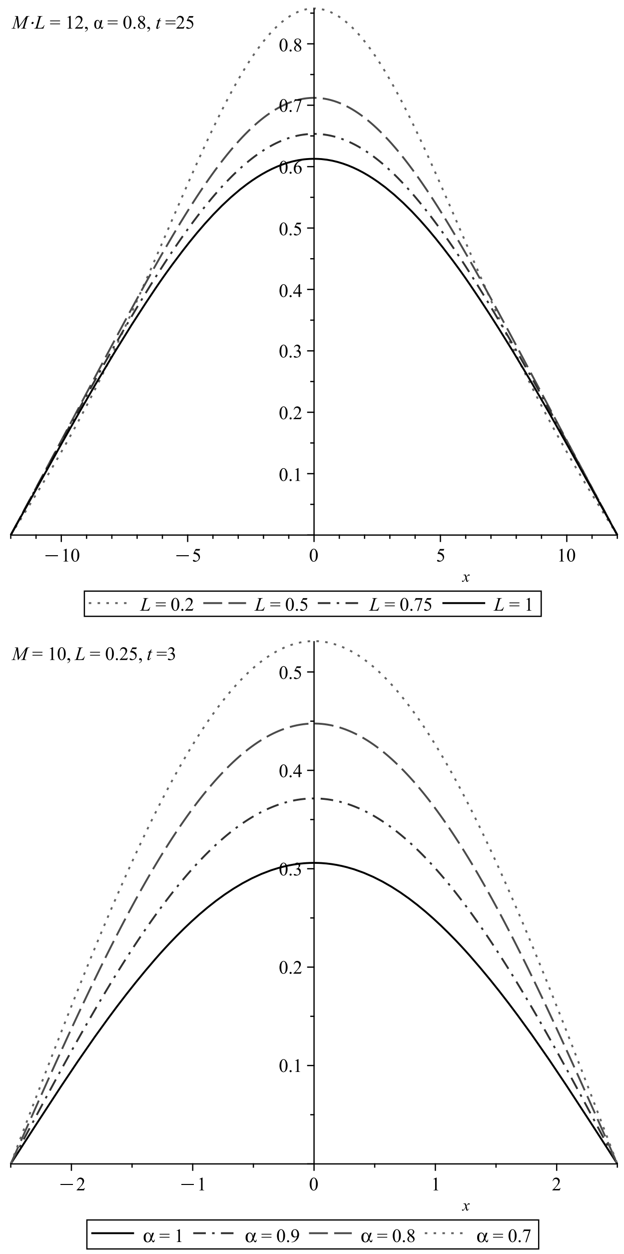


Fig. 2. The approximate solution of the problem (28)–(30) for the initial function $g(x) = \delta(x)$ and for different values of spatial memory length $L$ (left) and fractional order $\alpha$ (right).

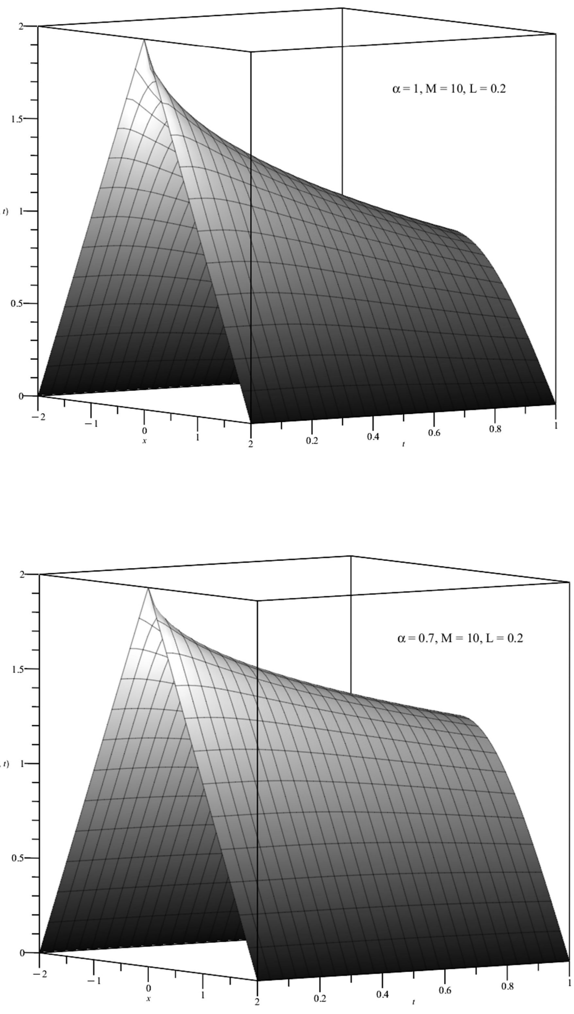


Fig. 3. The approximate solution of the problem (28)–(30) for the initial function $g(x) = ML - |x|$ and for different times.

In the graphs of Fig. 3, we again compare time evolution of the solutions for orders $\alpha = 1$ and $\alpha = 0.7$. By Remark 2, both solutions are rapidly decreasing functions of time and tend exponentially to zero when time $t \to \infty$. However, when we analyze the movement of the solution maximum, we observe that for fractional order $\alpha = 0.7$ it decreases more slowly than in the classical case.

Next, Fig. 4 includes two graphs. In the first one, we compare solutions for fixed values of parameters $M \cdot L = 12, \alpha = 0.8$, time moment $t = 20$ and a varying memory length. We note that the evolution slows down when the memory length decreases for a constant length of the problem domain $2ML = const$. In the next graph we fix values $M = 10, L = 0,25$ and time moment $t = 3$. Then, we change the fractional order $\alpha \in \{0.7, 0.8, 0.9, 1.0\}$. We observe that the evolution for a decreasing value of fractional order becomes more slow.

For the examples described above, all calculations, have been performed assuming that the approximation error is less than $10^{-6}$. This assumption required truncating the series to $M_{\text{app}} = 800$. The $M_{\text{app}}$ value was determined based on the estimation (76). ♦

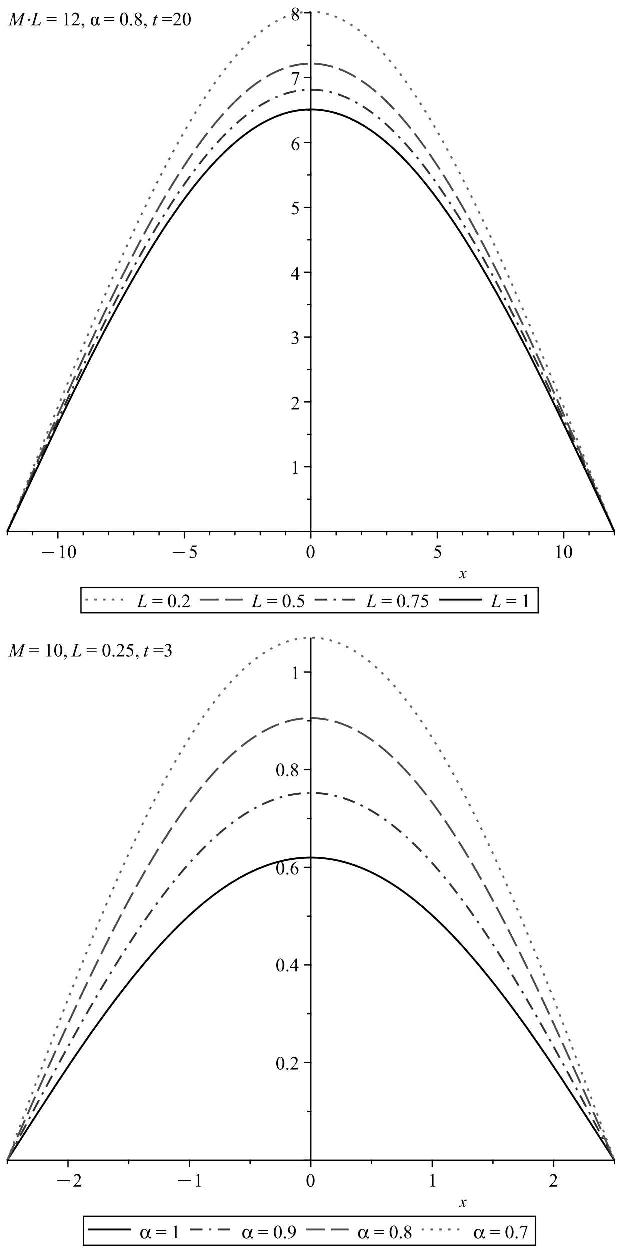


Fig. 4. The approximate solution of the problem (28)–(30) for the initial function $g(x) = ML - |x|$ and for different values of the spatial memory length $L$ (left) and the fractional order $\alpha$ (right).

**Example 3.** (The initial function in $C^2(-ML, ML)$) In the next case, we assume $g(x) = M^2L^2 - x^2$ which is clearly of the $C^2(-ML, ML)$ class. Then, applying Proposition 4, we get coefficients

$$B_m = \frac{4M^2L^2(-1)^m}{\big((m+\frac{1}{2})\pi\big)^3}$$

and the even solution in the form of the series

$$u(x,t) = \frac{4M^2L^2}{\pi^3}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+\frac{1}{2})^3} \times \exp\Big(-\rho(\tilde{\Lambda}_m)\, t\Big)\cos\left(\frac{(m+\frac{1}{2})\pi x}{ML}\right), \tag{78}$$

which is of $C^\infty(G)$ class for any open subset $G \subset [-ML, ML] \times (0, \infty)$ and a continuous function in any compact subset.

Applying Eqn. (62), we can give an explicit estimate of the approximation error as $||g''|| = 2$:

$$|u(x,t) - u_{\text{app}}(x,t)| \le \frac{e^{-\,(M_{\text{app}}+1/2)\gamma_e t}}{(M_{\text{app}}+\frac{3}{2})^2\,(1-e^{-\gamma_e t})}\cdot\frac{4\,M^2L^2}{\pi^3} \tag{79}$$

with constant $\gamma_e$ given by (A2).

Similarly to the previous example, the above estimate does not fully describe the approximation error when time $t$ is close to zero. The initial function in this example is in $C^2(-ML, ML)$ and we could apply a result from Proposition 10. Instead, we calculate the error based on the exact form of the solution and obtain the following formula providing a more accurate estimate

$$\begin{aligned}|u(x,t) - u_{\text{app}}(x,t)| &\le \frac{4M^2L^2}{\pi^3}\sum_{m=M_{\text{app}}+1}^{\infty}\frac{1}{(m+\frac{1}{2})^3} \\ &\le \frac{4M^2L^2}{\pi^3}\int_{M_{\text{app}}}^{\infty}\frac{1}{(u+\frac{1}{2})^3}\,\mathrm{d}u \\ &= \frac{2M^2L^2}{\pi^3(M_{\text{app}}+\frac{1}{2})^2}.\end{aligned} \tag{80}$$

We analyze the behavior of solutions by applying their approximate versions. First, in the graphs included in Fig. 5, we compare time evolution of the solutions for orders $\alpha = 1$ and $\alpha = 0.7$. By Remark 2, both solutions are decreasing functions of time, tend to zero as $t \to \infty$. However, analyzing the movement of the solution maximum, we see that the solution for a fractional order of $\alpha = 0.7$ decreases more slowly than in the classical case.

Next, Fig. 6 consists of two graphs. In the first one, we compare solutions for fixed values of parameters $M \cdot L = 12, \alpha = 0.8$, time moment $t = 20$ and a varying memory length. Again, we note that the evolution slows down when memory length decreases for the constant length of the problem domain $2ML = const$. Next, we fix values $M = 10, L = 0, 25$ and time moment $t = 3$. Then, we change fractional order $\alpha \in \{0.7, 0.8, 0.9, 1.0\}$. We observe that the evolution for the decreasing value of the fractional order becomes slower.

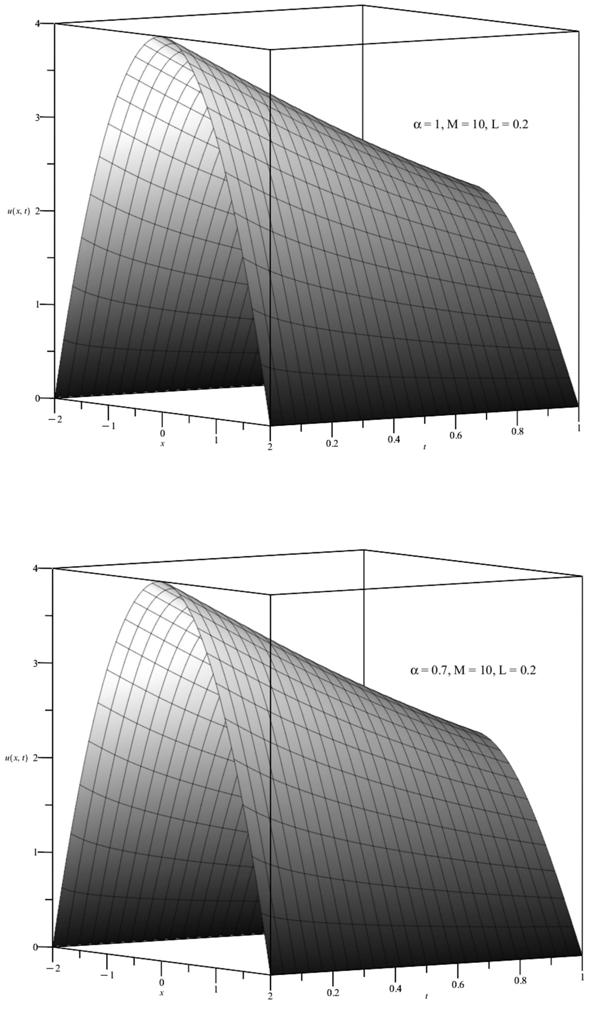


Fig. 5. The approximate solution of the problem (28)–(30) for the initial function $g(x) = M^2L^2 - x^2$ and for different times.

For the examples described above, all calculations, have been performed assuming that the approximation error is less than $10^{-6}$. This assumption required truncating the series to $M_{\text{app}} = 290$. The $M_{\text{app}}$ value was determined based on the estimate (79). ♦

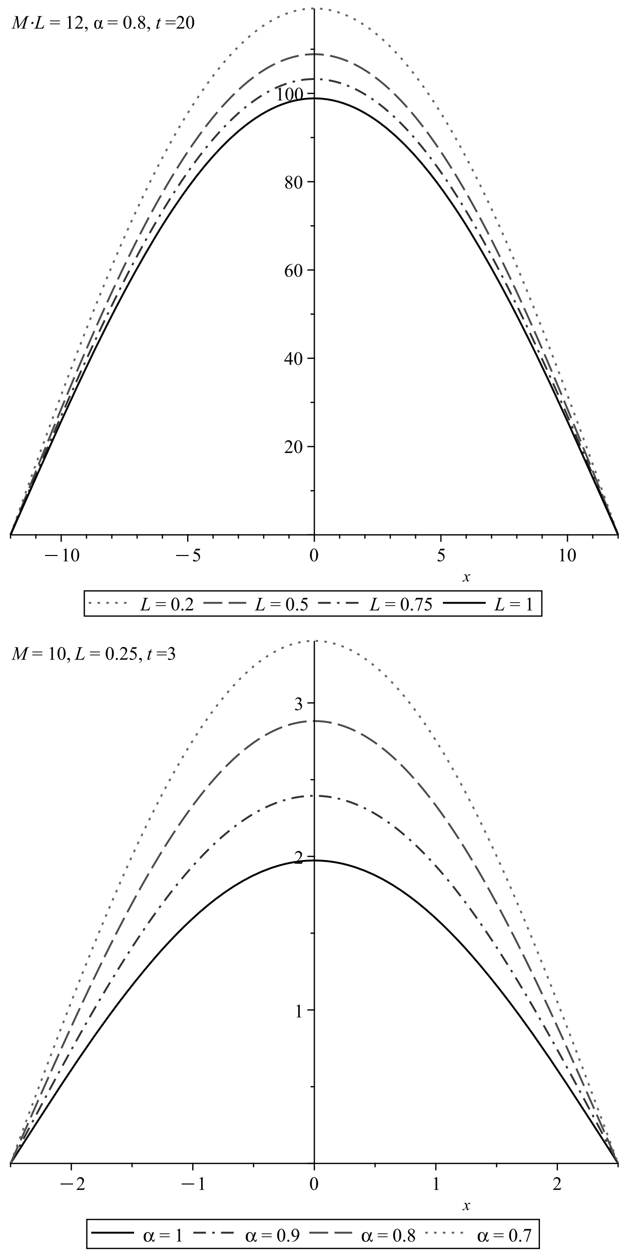


Fig. 6. The approximate solution of the problem (28)–(30) for the initial function $g(x) = M^2L^2 - x^2$ and for different values of spatial memory length $L$ (left) and fractional order $\alpha$ (right).

## 4. Conclusions

In the paper, we studied the partial fractional differential equation, where the fractional derivatives with fixed length of space memory replace the second-order derivative with respect to the spatial variable. The derivatives of this type seem particularly well suited for the finite domain problems where we assume the size of the domain to be expressed in terms of $L$, i.e., memory length. The problem considered here was complemented with the homogeneous Dirichlet boundary conditions. Applying our previous results on the eigenfunctions and eigenvalues of the oscillator operator with fixed memory length, we derived the exact solutions in the form of a trigonometric series with time-dependent coefficients.

As the exact solution of the fractional problem (28)–(30) is given in the form of a series, we studied approximate solutions in the form of partial sums. The length of the partial sum (parameter $M_{\text{app}}$ or $K_{\text{app}}$, respectively) and the time variable are fundamental parameters deciding on the accuracy of approximation. To control the accuracy of approximation, we derived formulas describing a bound of the approximation error.

First, the approximation error is controlled by parameters $M_{\text{app}}$ and $K_{\text{app}}$ as well as time $t$ in formulas (57), (62), (66) and (68). Rapid decreasing of the error for increasing values of time and length of the partial sum is to be noticed. Therefore, establishing the approximation accuracy at a given time moment and the value of parameters $M_{\text{app}}$ and $K_{\text{app}}$, we can be sure that this accuracy will be strongly improved in time according to Remark 4.

On the other hand, time-dependent estimates (57), (62), (66) and (68) predict that in order to approximate a solution near $t = 0$ with high accuracy, very long partial sums are needed. Our results allow us to keep control of the approximation error by dynamically changing parameters $M_{\text{app}}$ and/or $K_{\text{app}}$ according to time moment.

In addition, time independent formulas (69) and (70) are developed in the case when the initial function $g \in C^2(G)$. The obtained functions describing the error bound are decreasing functions of parameters $M_{\text{app}}$ and/or $K_{\text{app}}$, respectively. It seems that they will be particularly useful when approximating the solution near $t = 0$. An analogous formula is also valid for certain cases of continuous initial functions which was shown in Example 2.

The length of partial sums depends strongly on parameters such as the memory length $L$ and time $t$. For very large values of $M_{\text{app}}$ and $K_{\text{app}}$, the time necessary to calculate the partial sum increases significantly. In such a situation, the calculations become ineffective. Therefore, our future work will be focused on developing effective numerical algorithms for solving the studied diffusion problem.

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**Malgorzata Klimek** is a professor at the Department of Mathematics, Faculty of Computer Science and Artificial Intelligence, Czestochowa University of Technology. She is the author of one monograph and more than 50 research papers. Her main scientific interests focus on fractional eigenvalue problems, analytical and numerical methods for fractional differential equations and on their applications.

**Tomasz Blaszczyk** is an associate professor at the Department of Mathematics, Faculty of Computer Science and Artificial Intelligence, Czestochowa University of Technology. He is the author of more than 30 research papers. His scientific interests center around analytical and numerical methods for fractional differential equations.

## Appendix

## Estimation of lower and upper bounds for eigenvalues

In this section, we shall derive useful estimates of lower and upper bounds for the eigenvalues of the oscillator problem (19) and (20). To this aim, we apply the property of fractional derivatives with fixed memory length given in (13), which will be fundamental in our discussion. We start by estimating lower bounds for the eigenvalues connected to even and odd eigenfunctions (25) and (23), respectively.

**Proposition A1.** *Eigenvalues of the oscillator problem (19) and (20) obey the following inequalities:*

$$\begin{aligned}\rho(\tilde{\Lambda}_m) &\geq \left(\frac{(m+\frac{1}{2})\pi}{M}\right)^2 \left(\cos\left(\frac{\pi}{4M}\right)\right)^2 \\ &\quad \times \frac{L^{-2\alpha}}{(\Gamma(2-\alpha))^2} \times \frac{1}{M} \\ &\geq (m+\frac{1}{2})\gamma_e, \quad m \in \mathbb{N}, \end{aligned} \tag{A1}$$

*where $\rho(\tilde{\Lambda}_m)$, determined by (16), corresponds to eigenfunction $Y_{\tilde{\Lambda}_m}$ given in (25) and*

$$\gamma_e := \left(\cos\left(\frac{\pi}{4M}\right)\right)^2 \times \frac{\pi^2 L^{-2\alpha}}{M^3(\Gamma(2-\alpha))^2}. \tag{A2}$$

*For eigenvalue $\rho(\tilde{\Lambda}_0)$ we have*

$$\rho(\tilde{\Lambda}_0) \geq \gamma_e/4. \tag{A3}$$

*For eigenvalues $\rho(\Lambda_k)$, determined by (16), corresponding to eigenfunctions $Y_{\Lambda_k}$ given in (23) we have*

$$\begin{aligned}\rho(\Lambda_k) &\geq \left(\frac{k\pi}{M}\right)^2 \left(\sin\left(\frac{\pi}{4M}\right)\right)^2 \\ &\quad \times \frac{L^{-2\alpha}}{(\Gamma(2-\alpha))^2} \times \frac{1}{M} \geq k\gamma_o, \quad k \in \mathbb{N}, \end{aligned} \tag{A4}$$

*where*

$$\gamma_o := \left(\sin\left(\frac{\pi}{4M}\right)\right)^2 \times \frac{\pi^2 L^{-2\alpha}}{M^3(\Gamma(2-\alpha))^2}. \qquad (A5)$$

*Proof.* We begin with calculations of formula (A1) describing the lower bound for eigenvalues $\rho(\tilde{\Lambda}_m)$. Let us start with the following useful formula:

$$ {}_{x-L}I_x^{1-\alpha} 1 = \frac{L^{1-\alpha}}{\Gamma(2-\alpha)} = {}_xI_{x+L}^{1-\alpha} 1 \qquad (A6)$$

and write the oscillator equation with function $Y_{\tilde{\Lambda}_m}$ as a solution ($m \in \mathbb{N}_0$)

$$\rho(\tilde{\Lambda}_m) Y_{\tilde{\Lambda}_m}(x) = \mathcal{L} Y_{\tilde{\Lambda}_m}(x), \quad x \in [-ML, ML]. \quad (A7)$$

Now, we multiply both the sides of this equality with $Y_{\tilde{\Lambda}_m}$ and integrate over interval $[-ML, ML]$

$$\rho(\tilde{\Lambda}_m) \int_{-ML}^{ML} Y_{\tilde{\Lambda}_m}(x) Y_{\tilde{\Lambda}_m}(x)\,\mathrm{d}x = \int_{-ML}^{ML} Y_{\tilde{\Lambda}_m}(x) \mathcal{L} Y_{\tilde{\Lambda}_m}(x)\,\mathrm{d}x. \quad (A8)$$

Calculating the integral on the left-hand side and using formula (13) for the right operators, we obtain the relation

$$\rho(\tilde{\Lambda}_m) ML = \frac{1}{2}\Big( \int_{-ML}^{ML} \left({}_{x-L}D_x^{\alpha}\ Y_{\tilde{\Lambda}_m}(x)\right)^2 \mathrm{d}x + \int_{-ML}^{ML} \left({}_xD_{x+L}^{\alpha}\ Y_{\tilde{\Lambda}_m}(x)\right)^2 \mathrm{d}x \Big). \quad (A9)$$

Using the above relation, we estimate two terms on the right-hand side. For the first one, we get

$$\begin{aligned}
&\int_{-ML}^{ML} \left({}_{x-L}D_x^{\alpha}\ Y_{\tilde{\Lambda}_m}(x)\right)^2 \mathrm{d}x \\
&= \left(\frac{(m+\frac{1}{2})\pi}{ML}\right)^2 \times \int_{-ML}^{ML} \left({}_{x-L}I_x^{1-\alpha}\ \sin\left(\frac{(m+\frac{1}{2})\pi x}{ML}\right)\right)^2 \mathrm{d}x \\
&\geq 2(2m+1)\left(\frac{(m+\frac{1}{2})\pi}{ML}\right)^2 \times \int_{\frac{(M-\frac{1}{2})L}{2m+1}}^{\frac{ML}{2m+1}} \left({}_{x-L}I_x^{1-\alpha}\ \sin\left(\frac{(m+1/2)\pi x}{ML}\right)\right)^2 \mathrm{d}x \\
&\geq 2(2m+1)\left(\frac{(m+\frac{1}{2})\pi}{ML}\right)^2 \times \left(\sin\left(\frac{(m+\frac{1}{2})\pi}{ML} \times \frac{(M-\frac{1}{2})L}{2m+1}\right)\right)^2 \times \int_{\frac{(M-\frac{1}{2})L}{2m+1}}^{\frac{ML}{2m+1}} \left({}_{x-L}I_x^{1-\alpha}\ 1\right)^2 \mathrm{d}x \\
&= 2(2m+1)\left(\frac{(m+\frac{1}{2})\pi}{ML}\right)^2 \times \left(\sin\left(\frac{(m+\frac{1}{2})\pi}{ML} \times \frac{(M-\frac{1}{2})L}{2m+1}\right)\right)^2 \times \left({}_{x-L}I_x^{1-\alpha}\ 1\right)^2 \times \frac{L}{2(2m+1)} \\
&= \left(\frac{(m+\frac{1}{2})\pi}{ML}\right)^2 \left(\sin\left(\frac{\pi}{2} - \frac{\pi}{4M}\right)\right)^2 \times \frac{L^{2-2\alpha}}{(\Gamma(2-\alpha))^2} \cdot L.
\end{aligned}$$

Similarly, for the second term with the right-sided derivatives we obtain the same inequality:

$$\begin{aligned}
&\int_{-ML}^{ML} \left({}_xD_{x+L}^{\alpha}\ Y_{\tilde{\Lambda}_m}(x)\right)^2 \mathrm{d}x \\
&= \left(\frac{(m+\frac{1}{2})\pi}{ML}\right)^2 \times \int_{-ML}^{ML} \left({}_{x-L}I_x^{1-\alpha}\ \sin\left(\frac{(m+1/2)\pi x}{ML}\right)\right)^2 \mathrm{d}x \\
&\geq \left(\frac{(m+\frac{1}{2})\pi}{ML}\right)^2 \left(\sin\left(\frac{\pi}{2} - \frac{\pi}{4M}\right)\right)^2 \times \frac{L^{2-2\alpha}}{(\Gamma(2-\alpha))^2} \times L.
\end{aligned}$$

Taking into account the above estimates and relation (A9), we arrive at estimates (A1) and (A3),

$$\begin{aligned}
\rho(\tilde{\Lambda}_m) &\geq \left(\frac{(m+\frac{1}{2})\pi}{ML}\right)^2 \left(\cos\left(\frac{\pi}{4M}\right)\right)^2 \times \frac{L^{2-2\alpha}}{(\Gamma(2-\alpha))^2} \times \frac{1}{M} \\
&= \left(m+\frac{1}{2}\right)^2 \gamma_e \geq (m+\frac{1}{2})\gamma_e, \quad m \in \mathbb{N},
\end{aligned} \qquad (A10)$$

$$\rho(\tilde{\Lambda}_0) \geq \gamma_e/4,$$

where we define constant $\gamma_e$ by (A2). We omit the proof of the estimate (A4) as it is analogous to the proof of formulas (A1)–(A3) presented in detail. ∎

Next, we consider the upper bounds for eigenvalues (16) connected to even and odd eigenfunctions determined by Eqns. (25) and (23), respectively. The proof will also be based on the relation given in (13).

**Proposition A2.** *The eigenvalues of the oscillator problem (19) and (20) obey the following inequalities:*

$$\begin{aligned}\rho(\tilde{\Lambda}_m) &\le \left(\frac{(m+\frac{1}{2})\pi}{ML}\right)^2 \frac{2L^{2-2\alpha}}{(\Gamma(2-\alpha))^2}\\ &= \left(\frac{(m+\frac{1}{2})\pi}{M}\right)^2 \frac{2L^{-2\alpha}}{(\Gamma(2-\alpha))^2}, \quad m\in\mathbb{N}_0,\end{aligned} \tag{A11}$$

*where $\rho(\tilde{\Lambda}_m)$, the eigenvalue given in (16), corresponds to the eigenfunction $Y_{\tilde{\Lambda}_m}$ determined in (25) and*

$$\begin{aligned}\rho(\Lambda_k) &\le \left(\frac{k\pi}{ML}\right)^2 \frac{2L^{2-2\alpha}}{(\Gamma(2-\alpha))^2}\\ &= \left(\frac{k\pi}{M}\right)^2 \frac{2L^{-2\alpha}}{(\Gamma(2-\alpha))^2}, \quad k\in\mathbb{N},\end{aligned} \tag{A12}$$

*where eigenvalue $\rho(\Lambda_k)$, the given in (16), corresponds to the eigenfunction $Y_{\Lambda_k}$ determined by formula (23).*

*Proof.* In the calculations of the estimate (A11), we apply relation (A9) resulting from (13). We shall estimate both the terms on the right-hand side. To this aim, we apply the explicit form of the solution

$$Y_{\tilde{\Lambda}_m}(x) = \cos\left(\frac{(m+\frac{1}{2})\pi x}{ML}\right)$$

and the fact that the left derivative is given as

$${}_{x-L}D^{\alpha}_{x} = {}_{x-L}I^{1-\alpha}_{x}\frac{\mathrm{d}}{\mathrm{d}x}.$$

For the first of the two terms on the right-hand side, we obtain the following estimate

$$\begin{aligned}&\int_{-ML}^{ML} \left({}_{x-L}D^{\alpha}_{x}\; Y_{\tilde{\Lambda}_m}(x)\right)^2 \mathrm{d}x\\ &= \left(\frac{(m+\frac{1}{2})\pi}{ML}\right)^2\\ &\quad\times \int_{-ML}^{ML}\left({}_{x-L}I^{1-\alpha}_{x}\;\sin\left(\frac{(m+\frac{1}{2})\pi x}{ML}\right)\right)^2 \mathrm{d}x\\ &\le \left(\frac{(m+\frac{1}{2})\pi}{ML}\right)^2 \int_{-ML}^{ML}\left({}_{x-L}I^{1-\alpha}_{x}\; 1\right)^2 \mathrm{d}x\\ &= \left(\frac{(m+\frac{1}{2})\pi}{ML}\right)^2 \frac{L^{2-2\alpha}}{(\Gamma(2-\alpha))^2}\cdot 2ML,\end{aligned}$$

where we applied (A6).

The estimation for the second part on the right-hand side of (A9) can be derived analogously and is as follows:

$$\begin{aligned}&\int_{-ML}^{ML}\left({}_{x}D^{\alpha}_{x+L}\; Y_{\tilde{\Lambda}_m}(x)\right)^2 \mathrm{d}x\\ &\qquad\le \left(\frac{(m+\frac{1}{2})\pi}{ML}\right)^2 \frac{L^{2-2\alpha}}{(\Gamma(2-\alpha))^2}\cdot 2ML\end{aligned}$$

Taking into account the above inequalities, we arrive at the estimate of the upper bound of $\rho(\tilde{\Lambda}_m)$ given in (A11):

$$\rho(\tilde{\Lambda}_m) \le \left(\frac{(m+\frac{1}{2})\pi}{M}\right)^2 \frac{2L^{-2\alpha}}{(\Gamma(2-\alpha))^2}.$$

The second part of the estimation describing the eigenvalues $\rho(\Lambda_k)$ in formulas (16) and (23) can be proved analogously. ∎

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