WHAT IS NOT CLEAR IN FUZZY CONTROL SYSTEMS?

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The paper presents a number of unclear, unsolved or partly solved problems of fuzzy logic, which hinder precise transformation of expert knowledge about proper control of a plant in a fuzzy controller. These vague problems comprise the realization of logical and arithmetic operations and another basic problem, i.e., the construction of membership functions. The paper also indicates how some of the above problems can be solved.

Keywords: fuzzy control, fuzzy systems, fuzzy arithmetic, fuzzy logic, necessity, possibility

1. Introduction

Control systems with fuzzy controllers are often successfully applied in practice. Their great advantage is the possibility to introduce the knowledge of human experts about proper and correct control of a plant in the controller (Piegat, 2001; Yager and Filev, 1994; von Altrock, 1995). Owing to their advantages, fuzzy control systems were universally accepted by engineers. Many examples of these systems were mentioned by Prof. L. Zadeh in his lectures at various international conferences, e.g., the 7-th International Conference on *Artificial Intelligence and Soft Computing, ICAISC 2004* in Zakopane, Poland. Fuzzy controllers were applied to industrial control, quality control, elevator control and scheduling, train control, traffic control, loading crane control, reactor control, automobile transmissions and climate control, automobile body panting control, automobile engine control, paper manufacturing, steel manufacturing, power distribution control, and other applications. Figure 1 presents the general scheme of the fuzzy control system.

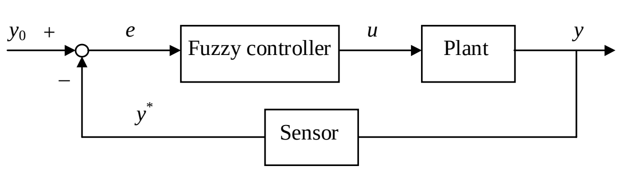


Fig. 1. General scheme of the fuzzy control system.

A fuzzy controller, often (but not always) a fuzzy PID one, consists of a dynamic and a fuzzy static part, see Fig. 2.

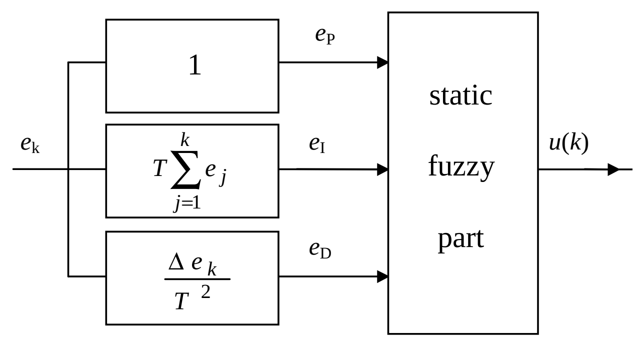


Fig. 2. Inner structure of the fuzzy PID controller.

The static fuzzy part of the controller contains linguistic knowledge about proper plant control. It also has its inner structure presented in Fig. 3.

Below we shall present uncertainty issues and vague problems connected with fuzzy control systems.

2. Uncertainty Connected with the Control Algorithm

Uncertainty connected with the control algorithm is mainly contained in the rule base and especially in linguistic notions such as very small, mean, large, etc. Below, an expert rule base for the control of a bridge trolley, which transports containers from a store place to a loading place, Fig. 4, is presented (Piegat, 2001).

The rule base contains the following rules:

$R1$: *IF* (d = large) *THEN* (P = positive large),

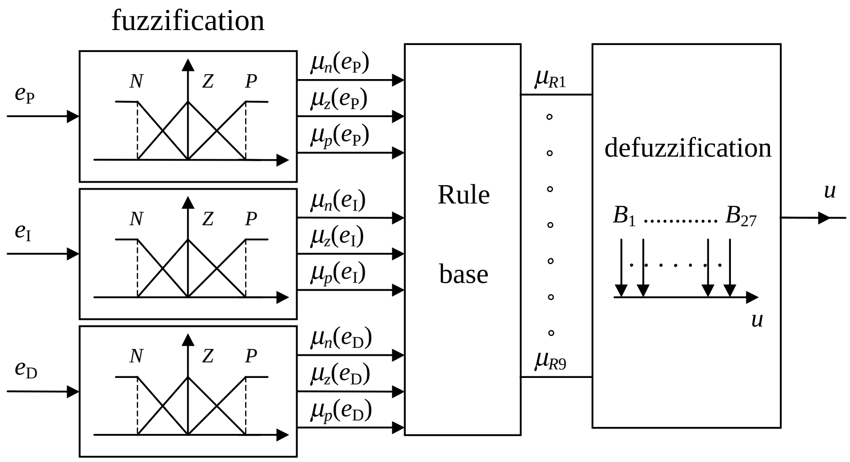


Fig. 3. Static part of the fuzzy PID controller: Ri – control rules, B_j – singletons representing output fuzzy sets. The block “Rule base” contains the inference engine of the controller.

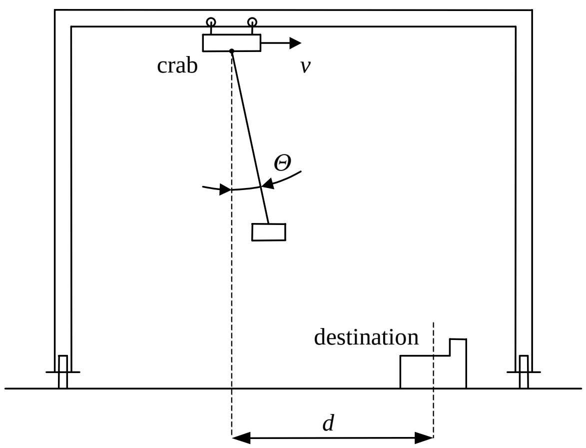


Fig. 4. Transport of containers with a bridge crane.

$R2$: *IF* (d = small) *AND* (Θ = negative large) *THEN* (P = negative medium),

$R3$: *IF* (d = small) *AND* (Θ = negative small *OR* zero *OR* positive small) *THEN* (P = positive medium),

$R4$: *IF* (d = small) *AND* (Θ = positive large) *THEN* (P = positive large),

$R5$: *IF* (d = zero) *AND* (Θ = positive large *OR* small) *THEN* (P = negative medium),

$R6$: *IF* (d = zero) *AND* (Θ = zero) *THEN* (P = zero),

$R7$: *IF* (d = zero) *AND* (Θ = negative small) *THEN* (P = positive medium),

$R8$: *IF* (d = zero) *AND* (Θ = negative large) *THEN* (P = positive large),

where:

- d – the distance between the trolley and the destination place expressed with the linguistic evaluations large (L), small (S), zero (Z),
- Θ – the angular displacement of the container line expressed with the linguistic evaluations positive large (PL), positive small (PS), zero (Z), negative small (NS), negative large (NL),
- P – the electrical power supplying the motor (operator controls the power shifting the lever) expressed with the evaluations negative large (NL), negative medium (NM), zero (Z), positive medium (PM), positive large (PL).

Linguistic evaluations used by the crane control expert are of uncertain character. They are described by the membership functions presented in Fig. 5.

The parameters of these functions are identified by the expert interview. The expert is usually unable to precisely give the parameters of fuzzy notions because they are partly settled in his or her subconsciousness. Therefore, he or she can give these parameters only roughly. Also, control rules given by the plant expert are not always certain because two different control experts of the same plant can sometimes give different control rules.

3. Uncertainty in the Realization of Logical Operations in Fuzzy Controllers

An inherent part of fuzzy control rules are the logical connections *AND* and *OR*. An example can be the rule $R3$ of the crane control:

$R3$: *IF* (d = small) *AND* (Θ = negative small *OR* zero *OR* positive small) *THEN* (P = positive medium).

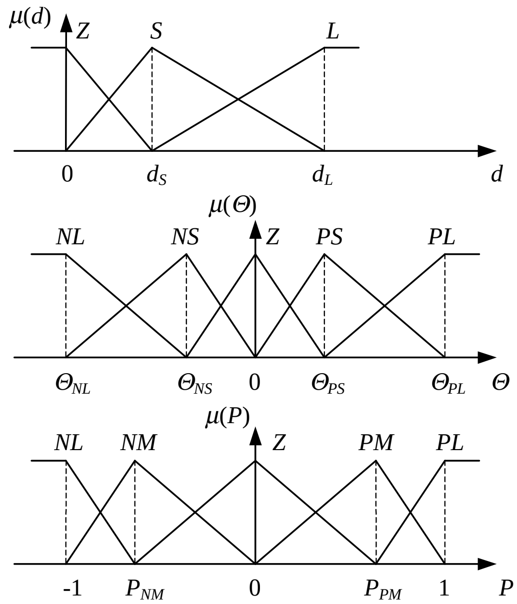


Fig. 5. The membership functions of the linguistic variables: distance (d), angular displacement (Θ) and power supply (P).

Scientists do not know precisely how people mathematically realize logical operations in their mind. This problem was recognized very early and described, e.g., in (Zimmermann, 1980). Therefore, scientists elaborated many proposals for implementing these operations (Driankov *et al.*, 1993; Yager and Filev, 1994; Piegat, 2001). Some of these proposals satisfying certain conditions are specified as t-norms (*AND* – operators) and s-norms (*OR* – operators). The operators proposed by scientists are, as a matter of fact, a hypothesis of how logical operations are accomplished in the human brain. However, it is possible that each person accomplishes these operations in a different way, which additionally varies with time. It hinders the transformation of expert knowledge in the fuzzy controller, because we must always insert in the controller some specific operator *AND* and *OR*. We do not know how the logical operators inserted differ from the operators used by a plant expert. Table 1 contains some existing logical *AND*-operators.

The number of possible *AND*-operators is by far higher than the number of operators shown in Table 1. The application of different operators gives different calculation results in fuzzy controllers. These differences are sometimes considerable. Figure 6 presents membership functions of linguistic evaluations of low, medium and high fever, and Fig. 7 provides the results of the operation "medium *AND* high" fever accomplished with five different t-norm operators.

As can be seen from Fig. 7, the results of various *AND*-operators differ considerably. Therefore, a question

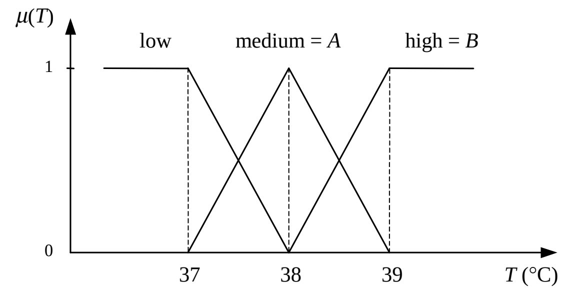


Fig. 6. Membership functions of linguistic values of fever.

arises: Which *AND*-operator should be used in the controller to be constructed? Which of the operators correctly models the logical operator used by the brain of the expert who provides us with knowledge about the plant control? An uncertainty identical with that relating to the operation *AND* pertains to the operation *OR*, where we can also use a large number of operators, e.g., maximum, algebraic sum, Hamacher sum, Einstein sum, drastic sum, bounded sum, etc., and to the implication operation, where we can use various implication operators, e.g., the Mamdani operator, Łukasiewicz operator, Kleene-Dienes operator, Kleene-Dienes-Łukasiewicz operator, Godel operator, Yager operator, Zadeh operator, etc. The application of each particular implication operator changes often considerably the results of implication. Which of the operators is the most suitable one for a given fuzzy control system?

4. Uncertainty Connected with Arithmetic Operations Realized in Control Algorithms and in the Design of Fuzzy Control Systems

In fuzzy control systems not only logical operations such as *AND*, *OR*, negation and implication are realized, but also various arithmetic operations, e.g., the calculation of the control error $e = d_0 - d$ and other operations contained in the control algorithm. Let us now analyze the example of an automatic control system for the distance between two cars illustrated in Fig. 8.

The control system must not keep here a certain, strictly determined and constant distance between the cars

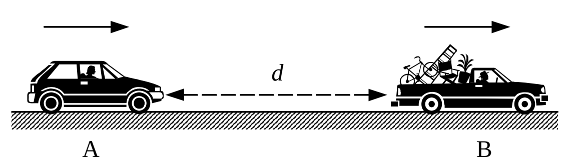


Fig. 8. Problem of the automatic control for a safe distance between two cars.

Table 1. Examples of t-norm operators for the realization of the logical *AND* operation.

Operator name	Formula
minimum (*MIN*)	$\mu_{A\cap B}(x) = MIN(\mu_A(x), \mu_B(x))$
product (*PROD*)	$\mu_{A\cap B}(x) = \mu_A(x) \cdot \mu_B(x)$
Hamacher product	$\mu_{A\cap B}(x) = \dfrac{\mu_A(x) \cdot \mu_B(x)}{\mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)}$
Einstein product	$\mu_{A\cap B}(x) = \dfrac{\mu_A(x) \cdot \mu_B(x)}{2 - (\mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x))}$
drastic product	$\mu_{A\cap B}(x) = \begin{cases} MIN(\mu_A(x), \mu_B(x)) & \text{for } MAX(\mu_A, \mu_B) = 1 \\ 0 & \text{otherwise} \end{cases}$
bounded difference	$\mu_{A\cap B}(x) = MAX(0,\ \mu_A(x) + \mu_B(x) - 1)$

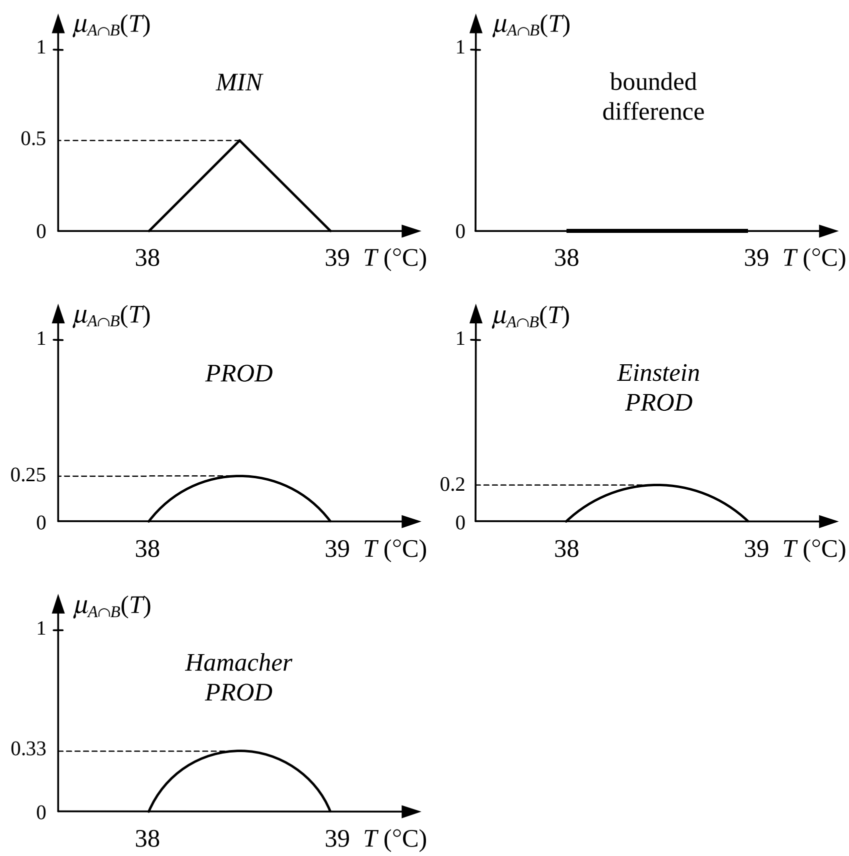


Fig. 7. Membership functions of the fuzzy set "medium *AND* high" fever calculated with five different t-norm operators.

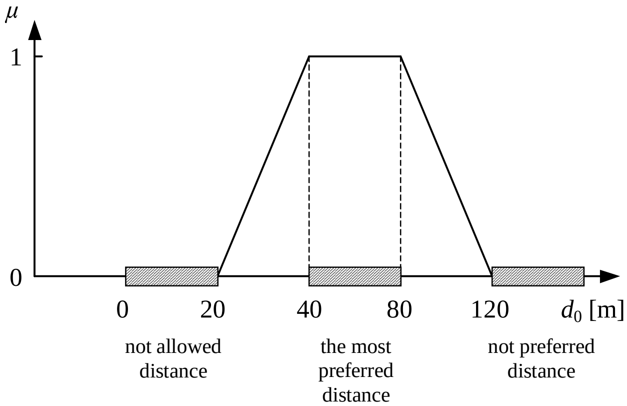


Fig. 9. Exemplary membership function of the recommended, safe distance d_0 [m] between two moving cars.

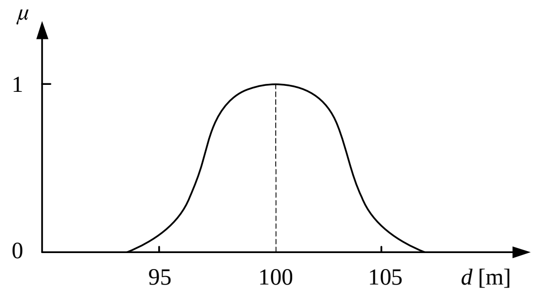


Fig. 10. Exemplary membership function of the actual distance d between two moving cars.

A and B, e.g., the distance of 60 [m]. It is sufficient to keep the distance lying in some safe interval, e.g., 40–80 [m]. However, this does not mean that the distance of, e.g., 39 [m] is a dangerous one. This distance is only less safe and preferred than 40 or 60 [m]. The preferred value of the distance between the two cars can be determined by a membership function. An example of such a function is presented in Fig. 9.

Apart from the distance, in this automatic control system we also have another fuzzy quantity. It is the actual distance d between the cars. This distance is measured with some error described by the Gaussian function. Therefore, the actual distance d between the cars can be described with the membership function of Fig. 10. The scheme of the automatic control system for the distance between the two cars is shown in Fig. 11.

In the comparison element of the fuzzy control system, the subtraction of two fuzzy numbers must be accomplished. Next, arithmetic operations may be contained in the control algorithm. Arithmetic operations on fuzzy numbers are realized not only during the operation of the control system, but also in the design process of the system. A fuzzy controller must be designed so that stable operation of the control system is secured. In the design process of fuzzy controllers also arithmetic operations on fuzzy numbers are accomplished. It can be exemplified by the design of the fuzzy controller based on the Lyapunov theory presented in (Zhou, 2002).

The author presents his method using the example of a fuzzy controller stabilizing the angle position x_1 of the inverted pendulum, cf. Fig. 12.

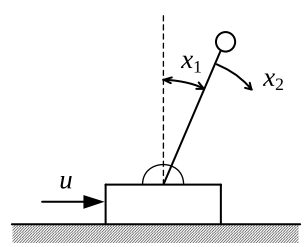


Fig. 12. Inverted pendulum and quantities important for the stabilization of its angle position.

In the problem of the inverted pendulum the following quantities are of importance:

x_1 – the deviation angle of the pendulum (controlled quantity),

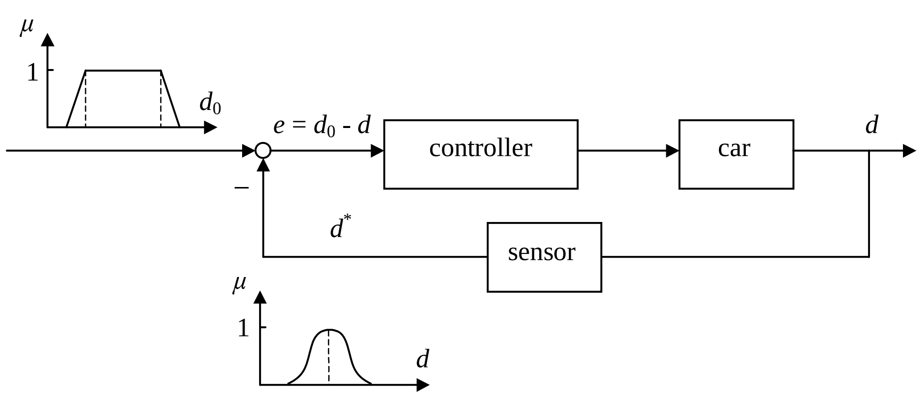


Fig. 11. Uncertain fuzzy signals in the control system for the distance between two cars.

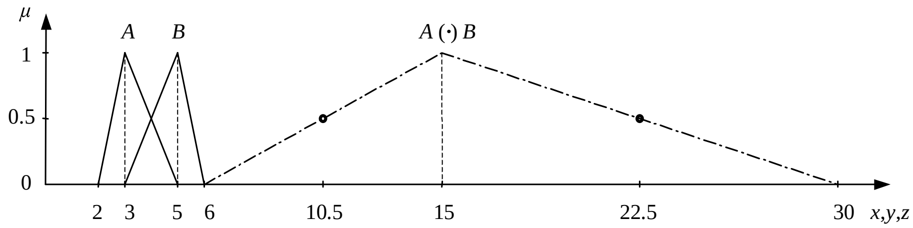


Fig. 13. Triangular result of multiplying two fuzzy numbers *about* 3 and *about* 5 according to the Zadeh extension principle.

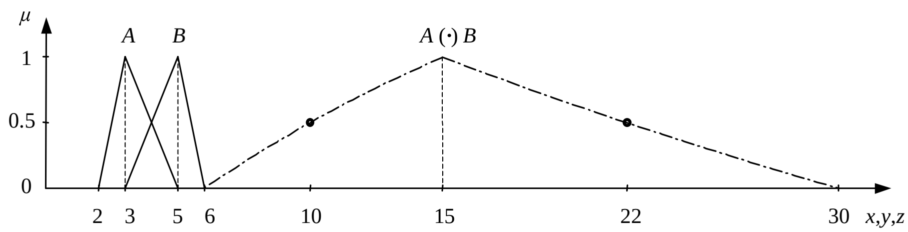


Fig. 14. Non-triangular result of multiplying fuzzy numbers *about* 3 and *about* 5 using the α-cut method (cf. parameters of points in Figs. 13 and 14).

x_2 – the angle velocity of the pendulum,

u – the force moving the pendulum base (control quantity).

In the synthesis process of the fuzzy controller the Lyapunov function candidate

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) \tag{1}$$

is used. Its derivative is given by

$$\dot{V} = x_1\dot{x_1} + x_2\dot{x_2} \cong x_2(x_1 + u). \tag{2}$$

In the synthesis process, the derivative of the Lyapunov function candidate is transformed into the linguistic form,

$$LV(\dot{V}) = LVx_2(LVx_1 + LVu), \tag{3}$$

where LV denotes the linguistic value of the candidate.

There following values are used:

- for the angle x_1: $LVx_1 \in$ {positive, negative},
- for the angular velocity x_2: $LVx_2 \in$ {negative, positive},
- for the force u: $LVu \in$ {negative big, about zero, positive big}.

To calculate the linguistic value $LV(\dot{V}(x))$ according to (3), arithmetic operations on fuzzy sets are necessary. An exemplary computation is

$$LV(\dot{V}(x)) = \text{positive } x_2(\text{negative } x_1 + \text{positive big } u). \tag{4}$$

The above example of fuzzy controller synthesis shows that for stability checking of a fuzzy control system, performing arithmetic operations on fuzzy (numbers) sets may be necessary. But, how can these operations be realized? It appears that there exist various methods of implementing fuzzy arithmetic operations, which give different results for one and the same operation. For instance, consider the multiplication of two fuzzy numbers A and B that takes place in (4). This operation can be accomplished with various methods from which two basic ones are the Zadeh extension principle:

$$\mu_{A(\cdot)B}(z) = \bigvee_{z=x\cdot y} \big(\mu_A(x) \wedge \mu_B(y)\big), \tag{5}$$

and the α-cut method (Kaufmann and Gupta, 1991):

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= \big[a_1^{(\alpha)}, a_2^{(\alpha)}\big](\cdot)\big[b_1^{(\alpha)}, b_2^{(\alpha)}\big] \\ &= \big[a_1^{(\alpha)} \cdot b_1^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}\big]. \end{aligned} \tag{6}$$

In the above formulas, μ denotes the membership grade and A_α denotes the α-cut of the fuzzy number at the α level. The borders of this cut are determined by $[a_1^{(\alpha)}, a_2^{(\alpha)}]$. A similar notation refers to the number B. Let us consider now, for instance, the multiplication of two numbers, $A = \text{about } 3$ and $B = \text{about } 5$, presented in Fig. 13. If the multiplication is realized with the Zadeh extension principle (5), the result shown in Fig. 13 is achieved. If it is realized with the α-cut method (6), we get a different result, presented in Fig. 14.

A different method of performing arithmetic operations on fuzzy numbers was proposed in (Kosiński *et al.*, 2003). The method introduces a special, new feature of fuzzy numbers called the *orientation*. It can be positive (counterclockwise) and negative (clockwise), and it strongly influences the results of arithmetic operations. If we add two fuzzy numbers A and B of the same, negative orientation, we get the result C with the support being the sum of the supports of A and B, Fig. 15.

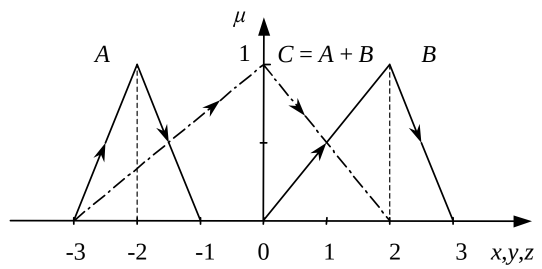


Fig. 15. Addition of ordered, negatively oriented fuzzy numbers according to the method of (Kosiński *et al.*, 2003).

We achieve a result that is the same as the one achieved with the Zadeh extension principle (5). However, if the orientation of the added numbers is not the same but opposite, we get the result C with a support considerably smaller than the sum of the supports of A and B, Fig. 16. The achieved result is different from the one of the Zadeh extension principle (5) or of the α-cut method (6).

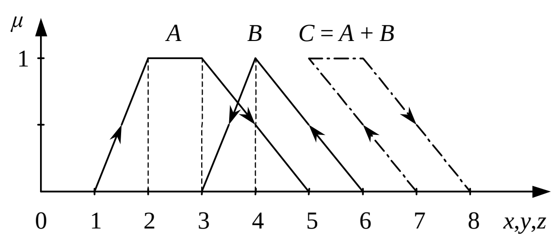


Fig. 16. Addition of ordered fuzzy numbers with opposite orientation according to the method of (Kosiński *et al.*, 2003).

Another method of implementing arithmetic operations on fuzzy numbers is proposed in (Rakus-Anderson, 2003). The addition of two fuzzy numbers, A and B, according to this method gives a result C with a support which, in the general case, is not equal to the sum of the supports of A and B. Accordingly, the computation results are different from those achieved with the Zadeh extension principle (5), the α-cut method (6) or the method proposed by Kosiński *et al.*. An example is shown in Fig. 17.

Summing up the above examples, the following conclusion can be drawn: there exist various methods of performing arithmetic operations that, in the general case, give different results for one and the same operation.

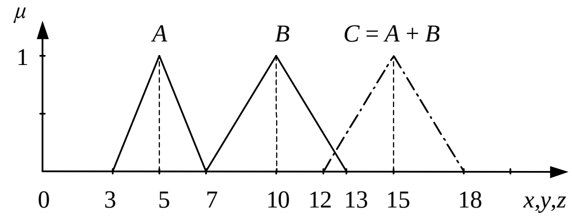


Fig. 17. Addition of two fuzzy numbers A and B according to the method of Rakus-Anderson.

Which of them should be used in the fuzzy control system? Which of them represents in the best way arithmetic operations realized in the human expert's brain?

5. Uncertainty Connected with the Essence of the Membership Function

All logical and arithmetic operations accomplished in fuzzy control systems are operations on membership functions of fuzzy sets. But, what is the membership function and what is its substance? An answer to this question was given by the creator of fuzziness, Professor L. Zadeh (1978): "…the possibility distribution function associated with X …is denoted by π_X and is defined to be numerically equal to the membership function of F, i.e.,

$$\pi_X \triangleq \mu_F. \tag{7}$$

Thus, $\pi_X(u)$, the possibility that $X = u$ is postulated to be equal to $\mu_F(u)$."

From the above definition it follows that determining the membership function of a fuzzy set amounts to the determination of the possibility distribution π_X of the set. For example, to determine a membership function of the income of a firm to a fuzzy set (linguistic evaluation) *high* income, possibility distribution of qualifying by the firm director particular, possible numerical values of the firm income as a *high* income should be identified. But what is the possibility distribution π_X and how can it be identified? What, in general, is the possibility of the occurrence of an event (in the case of the membership function the event is the qualification of the value u of the quantity X in the fuzzy set F)?

The notions of the possibility and necessity of event occurrence were introduced by Dubois and Prade (1983). Since then the authors have used these notions in numerous publications. One of the recent ones is (Dubois *et al.*, 2004). Apart from their definitions, there also exist other interpretations of possibility and necessity, e.g., the one given in (Borgelt and Kruse, 2003). However, the interpretation of Dubois and Prade is most well-known, widespread and used. According to the author of this paper, there exist serious doubts as to the definition of the possibility of Dubois and Prade and, consequently, to the substance of the membership function and to the method of its identification resulting from the definition. Further on, the author will explain the doubts. But first of all, the notion of possibility according to (Dubois and Prade, 1983) will be explained.

5.1. Notion of Possibility According to Dubois and Prade

Let us assume that a quantity x can take a finite number n of values x_i contained in the domain $X = \{x_i \mid i = 1, \dots, n\}$. As an elementary event A we shall further understand taking by x one of possible values x_i from the domain X, e.g.,

$$A: \ x = 7, \quad 7 \in X. \tag{8}$$

As a set event A we shall further understand taking by x one of many values x_i contained in some subset of the domain X, e.g.,

$$A: \ x \in \{5, 6, 7\}, \quad \{5, 6, 7\} \in X. \tag{9}$$

The possibility of the occurrence of an event A is generally defined by Dubois and Prade as follows:

$$\text{possibility}(A) = 1 - \text{impossibility}(A). \tag{10}$$

The possibility of event occurrence is full (equal to 1) only when the impossibility of its occurrence equals zero. If the impossibility of event occurrence is greater than zero, then its possibility is not full. But what is the impossibility of event occurrence and how can it be determined?

The impossibility of the event A is numerically equal to the necessity N of the occurrence of the opposite event $\bar{A}$ (of the event complementing the event A to the domain X), (11).

$$\text{impossibility}(A) = \text{necessity}(\bar{A}), \quad A \cup \bar{A} = X. \tag{11}$$

Thus, the possibility of the event A is defined by

$$\begin{aligned} \text{possibility}(A) &= 1 - \text{necessity}(\bar{A}), \\ \Pi(A) &= 1 - N(\bar{A}). \end{aligned} \tag{12}$$

However, at this point the next question arises: What is the necessity $N(A)$ of event occurrence and how can it be determined? Let us denote by p_i the value of the probability P of an elementary event occurrence $x = x_i$:

$$p_i = P\big(\{x_i\}\big), \quad \sum_{i=1}^{n} p_i = 1. \tag{13}$$

Dubois and Prade (1983) gave the following definition of the degree of the necessity of an event:

Definition 1. The *degree of the necessity of the event* $A \subseteq X$ is the extra amount of the probability of elementary events in A over the amount of the probability assigned to the most frequent elementary event outside A. In other words,

$$N(A) = \sum_{x_i \in A} \max\big(p_i - \max_{x_k \notin A} p_k,\ 0\big). \tag{14}$$

This definition is illustrated by Dubois and Prade with an example of coin tossing with a biased coin, which gives different probabilities for the head and the tail. However, further on, the notion of the possibility and necessity of an event will be explained not with coin tossing, but with an example of a roulette wheel, which enables us to explain the problem better.

Example 1. The roulette wheel has been divided into 2 parts: one part with an area of 60% of the full wheel area, to which the number 1 is assigned, and the other part with an area of 40% of the full wheel area, to which the number 2 is assigned, Fig. 18.

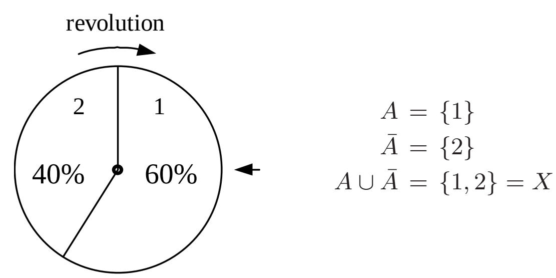


Fig. 18. Roulette wheel with unequal partition of the area.

Let us define the event A as producing the number 1. Thus, the opposite, complementing event $\bar{A}$ will be producing the number 2. The probabilities of the particular events are as follows:

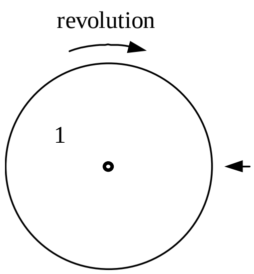
$$\begin{aligned} P\big(A = \{1\}\big) &= p_1 = 0.6, \\ P\big(\bar{A} = \{2\}\big) &= p_2 = 0.4. \end{aligned} \tag{15}$$

The necessities of the occurrence of particular events are calculated as

$$\begin{aligned} N(A = \{1\}) = n_1 &= \max(p_1 - p_2, 0) \\ &= \max(0.6 - 0.4, 0) = 0.2, \\ N(\bar{A} = \{2\}) = n_2 &= \max(p_2 - p_1, 0) \\ &= \max(0.4 - 0.6, 0) = 0. \end{aligned} \tag{16}$$

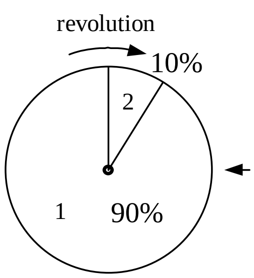
The probabilistic superiority of the number 1 over 2 $(0.6 - 0.4 = 0.2)$ means its domination or privilege. It can be said that there exists some fractional necessity of event $A = \{1\}$ occurrence. Thus, the number 2 is not probabilistically privileged in relation to 1 and therefore its occurrence necessity equals zero.

When is the event A fully necessary $(N(A) = 1)$? Such a situation is presented in Fig. 19. If we asign number 1 to the full roulette wheel, then producing 1 is fully necessary because no other number can be produced – the opposite event does not exist. Figure 20 presents an example of a highly necessary event A.



$N(A = \{1\}) = 1$
$\bar{A} = \emptyset,\ X = A \cup \bar{A} = \{1\}$

Fig. 19. Example of the fully necessary event: $n_1 = 1$.



$N(A = \{1\}) = \max(0.9 - 0.1, 0) = 0.8$
$\bar{A} = \{2\}, \quad X = A \cup \bar{A} = \{1, 2\}$

Fig. 20. Example of a highly necessary event $A = \{1\}$ with the necessity of occurrence $n_1 = 0.8$.

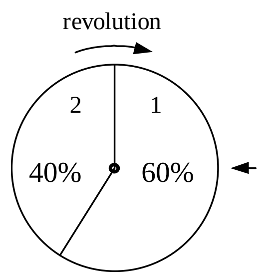
The probabilistic domination of the number 1 over 2 in Fig. 20 is very high and equals 0.8. Therefore, we can say that producing number 1 is highly necessary. Number 2 has no probabilistic domination over 1 and therefore its necessity equals zero.

Now let us come back to the example of Fig. 18 where 60% of the roulette wheel was assigned to 1 and 40% of the wheel to the number 2, and let us continue the calculations. The computed values of the necessity of 1 and 2 enable us to compute the possibilities of producing these numbers with the use of (12):

$$\begin{aligned}\text{possibility}\big(A = \{1\}\big) &= 1 - \text{impossibility}\big(A = \{1\}\big)\\ &= 1 - \text{necessity}\big(\bar{A} = \{2\}\big),\\ \pi_1 &= 1 - n_2 = 1 - 0 = 1, \end{aligned} \tag{17}$$

$$\begin{aligned}\text{possibility}\big(\bar{A} = \{2\}\big) &= 1 - \text{impossibility}\big(\bar{A} = \{2\}\big)\\ &= 1 - \text{necessity}\big(A = \{1\}\big),\\ \pi_2 &= 1 - n_1 = 1 - 0.2 = 0.8. \end{aligned} \tag{18}$$

The computational results of the necessity and possibility of particular events analyzed in Example 1 are collected in Fig. 21.



probability $p_2 = 0.4$	probability $p_1 = 0.6$
possibility $\pi_2 = 0.8$	possibility $\pi_1 = 1$
necessity $n_2 = 0$	necessity $n_1 = 0.2$
$\pi_2 = 1 - n_1$	$\pi_1 = 1 - n_2$

Fig. 21. Probabilities, possibilities and necessities of the events $A = \{1\}$ and $\bar{A} = \{2\}$ in Example 1.

The possibility of producing the number 1 is full (equals 1) because the necessity of producing the opposite the number 2 equals zero. The possibility of producing number 2 is not full (is less than 1) because the necessity of producing the opposite number 1 is greater than zero. ♦

5.2. Doubts as to the Correctness of the Definition of Necessity

The necessity $N(A)$ of the occurrence of the event A was defined by (14). Attention should be drawn to the fact that in the calculation of the occurrence necessity of the event A (in the general case, of the set event), the above definition does not take into account the complete opposite (complementing) event $\bar{A}$ and its total probability $P(\bar{A})$, but only one of the elementary events $x = x_k$ contained in that event whose probability p_k of occurrence is the greatest one (the term "$\max p_k$ for $x_k \in A$").

In (Dubois and Prade, 1983), some axioms are also given that are satisfied by the necessity and possibility function. Thus, we have

$$N(\emptyset) = 0, \quad N(X) = 1. \tag{19}$$

The necessity of the occurrence of the empty event is zero, and the necessity of the occurrence of one of all events contained in the universe X of events is full (equal to 1). The axiom

$$\forall A, B \subseteq X \quad N(A \cap B) = \min\big(N(A), N(B)\big) \tag{20}$$

allows us to calculate the necessity of logical intersection of two set events. The axiom

$$\forall A \subseteq X \quad \Pi(A) = 1 - N(\bar{A}) \tag{21}$$

informs us what the possibility of event A occurrence is and how it can be calculated. It is a very important axiom. The axiom

$$\Pi(\emptyset) = 0: \quad \Pi(X) = 1 \tag{22}$$

says that the occurrence possibility of the empty event is zero and the possibility of the occurrence of one of all events contained in the event universe X is full and equals 1. The axiom

$$\forall\ A, B \in X, \quad \Pi(A \cup B) = \max\big(\Pi(A), \Pi(B)\big) \tag{23}$$

tells us how to calculate the occurrence possibility of the event being the logical sum of two events A and B. It results from this axiom that the occurrence possibility of the event $C = A \cup B$ is equal to that of the possibilities $\Pi(A)$ or $\Pi(B)$, which is the greatest one. Let us check now whether (23) is correct.

Example 2. In Example 1 the roulette wheel was split into two parts. The number 1 was assigned to the greater part and the number 2 to the smaller part of the wheel, which is shown in Fig. 22.

Let us now divide the 60% part that was assigned the number 1 in Example 1 into two parts: the 35% part that will be assigned the number 3, and the 25% part that will be assigned number 4. The new partition of the roulette wheel is shown in Fig. 23.

Now let us consider the event $\{3, 4\}$ of producing the numbers 3 or 4 for the roulette wheel from Fig. 23. This event amounts to producing the number 1 in the case of the roulette wheel of Fig. 22 because it refers to the same section of the wheel. The necessity of the event is

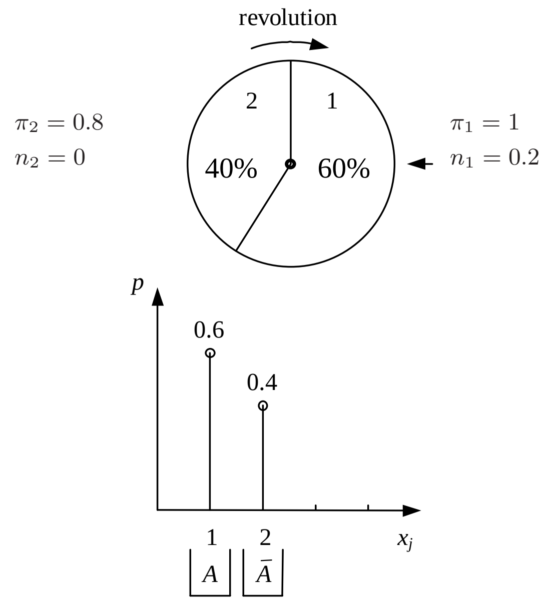


Fig. 22. Partition of the roulette wheel in Example 1 and the resulting values of the probability, necessity and possibility of particular events.

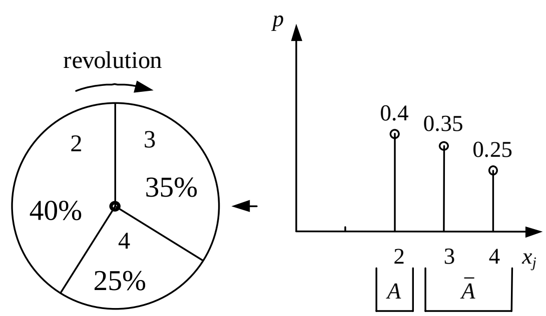


Fig. 23. Partition of the first part of the roulette wheel of Example 1 into two parts with the assigned numbers 3 and 4, and the probabilities of particular events resulting from the new partition.

calculated using (14):

$$\begin{aligned}
N(A) &= N\big(\{3,4\}\big) \\
&= \max(0.35 - 0.4, 0) + \max(0.25 - 0.4, 0) \\
&= 0 + 0 = 0, \\
N(\bar{A}) &= N\big(\{2\}\big) = \max(0.4 - 0.35, 0) = 0.05, \\
\Pi(A) &= \Pi\big(\{3,4\}\big) = 1 - N\big(\{2\}\big) = 1 - 0.05 = 0.95, \\
\Pi(\bar{A}) &= \Pi\big(\{2\}\big) = 1 - N\big(\{3,4\}\big) = 1 - 0 = 1.
\end{aligned} \tag{24}$$

The comparison of calculation results achieved before the partition of Section 1 of the roulette wheel and after its further partition into two sections is presented in Fig. 24.

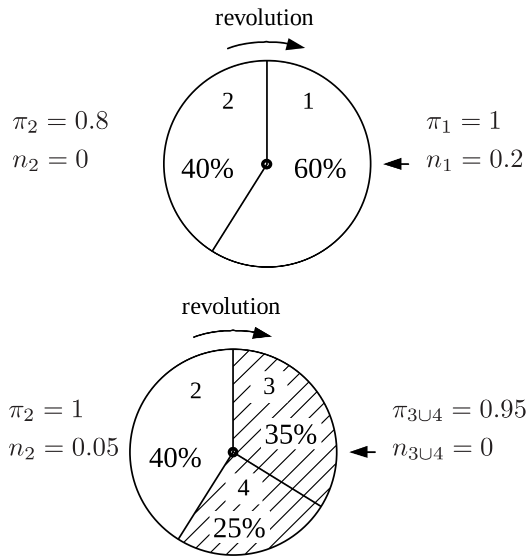


Fig. 24. Comparison of computational results of the necessity and possibility of event occurrence before and after partitioning Section 1 into Sections 3 and 4 of the roulette wheel according to the definition (14).

Example 2 and, especially, Fig. 24 show that the use of the definition (14) of the necessity of calculating the necessity and possibility of event occurrence results in paradoxes. The set event $\{3, 4\}$, which means producing numbers 3 or 4, has different necessity and possibility (0.2 and 1) than the event of producing the number 1, for which the necessity and possibility have values 0 and 0.95, respectively. This means that the event $\{3, 4\}$ has no probabilistic domination over producing the number 2, which is not true, because such a domination exists and equals 0.2. Before the partition, the number 2 had no domination over the number 1. After the partition of the first section, it suddenly achieved such superiority over it $(N(\{2\}) = 0.05)$. A reason behind this paradox is the feature of the definition (14) consisting in taking into account the probability of only one component event contained in the complement and not the probability of the complement.

Because the definition (14) results in calculation paradoxes and seems to be incorrect, the author of this publication proposes a new definition of event necessity.

5.3. New Definition of the Necessity of Event Occurrence

The degree of necessity $N(A)$ of the occurence of a set event $A \subseteq X$ is the surplus of the sum of the probabilities of all possible outcomes x_j contained in A over the same sum for possible outcomes x_k contained in the complement $\bar{A}$ if this surplus is positive. Otherwise, the necessity degree equals zero. The degree of necessity can be calculated as follows:

$$N(A) = \max\left(\sum_{x_j \in A} p_j - \sum_{x_k \in \bar{A}} p_k,\ 0\right). \tag{25}$$

Further on, the new definition of necessity (25) will be applied to solve the problem of Example 2, see Fig. 23 and the formulas

$$\begin{aligned}
N(A) &= N\big(\{3,4\}\big) = \max(0.6 - 0.4, 0) = 0.2,\\
N(\bar{A}) &= N\big(\{2\}\big) = \max(0.4 - 0.6, 0) = 0,\\
\Pi(A) &= \Pi\big(\{3,4\}\big) = 1 - N(\bar{A}) = 1 - 0 = 1,\\
\Pi(\bar{A}) &= \Pi\big(\{2\}\big) = 1 - N(A) = 1 - 0.2 = 0.8.
\end{aligned} \tag{26}$$

The comparison of results for the necessity and possibility of events before and after the partition of the roulette wheel according to the new definition is presented in Fig. 25. The formula (26) and Fig. 25 illustrate the fact that the new definition (25) of event necessity facilitates correct and sensible calculation of the necessity degree and does not cause paradoxes like the definition (14).

The new definition rests on the assumption that the necessity of event A occurrence is the surplus of the probability of this event over the total probability of the complement, i.e., over the probability sum of all component events contained in the complement and not of only one component event, as assumed by Dubois and Prade in their definition (14). The new approach (25) to necessity seems to be more sensible and convincing than the old one.

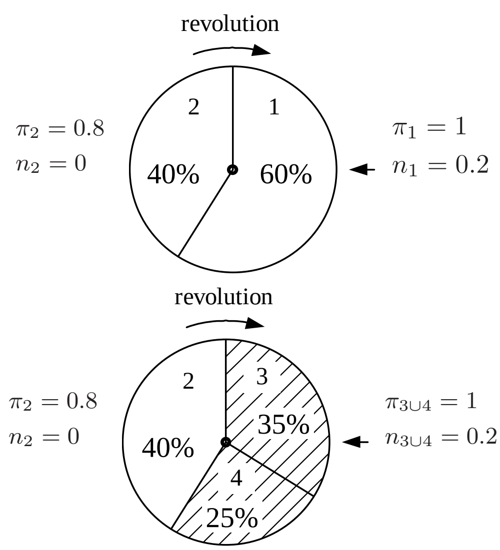


Fig. 25. Comparison of computation results of the necessity and possibility of events using the new definition (25) of necessity.

5.4. Meaning of the Error in the Actual Definition of Necessity for Fuzzy Control

The definition (14) is the oldest and the most propagated definition of necessity. It forms a basis for experimental construction of membership functions of fuzzy sets used everywhere, also in fuzzy control systems. Methods of constructing membership functions were elaborated by Dubois and Prade (1983; 1986), and by other scientists (Devi and Sarma, 1985) or (Civanlar and Trussel, 1986). These methods allow constructing a membership function on the basis of a probability density distribution or its simplified version, i.e., histograms. However, because the actual definition of necessity is not correct, all methods of constructing membership functions based on it provide incorrect membership functions. Investigations made by the author show that the new definition of necessity (25) gives quite different shapes of membership functions from those achieved with the definition of Dubois and Prade. The author's investigations also show that the same notion of membership to a fuzzy set being obligatory today, which can be found, e.g., in (Dubois and Prade, 1986; Yager and Filev, 1994; Zimmerman, 1991), is incorrect and should be revised. How? A proposal is given in (Piegat, 2005).

6. Concluding Remarks

In fuzzy control systems we have to do with many more or less obscure and unsolved problems. All the time, answers to the following questions are needed:

- How should logical operations such as *AND*, *OR* and implication in fuzzy control systems be realized if many operators for the accomplishment of these operations giving different calculation results can be applied?
- How should arithmetic operations in fuzzy control systems be realized if there exist many fuzzy arithmetic concepts, which give different calculation results?
- What is the membership function of a fuzzy set and how should it be constructed if the most commonly applied definition of the necessity of events, which is a basis for identifying the membership function, is not correct?

All the three unsolved problems hinder full transformation of expert knowledge about correct control of a plant. This kind of knowledge can today be transformed in the controller only roughly and approximately. Of course, this situation is disadvantageous for the quality of plant control.

From the above, the following question arises: If there exist so significant difficulties in the transformation of plant expert knowledge into a fuzzy controller, why do the contemporary fuzzy control systems operate satisfactorily and prove this in practice?

The answer to this question is as follows:

- Control rules provided by plant experts are usually correct and do not contain large errors.
- In fuzzy control systems, fuzzy controllers, rules and membership functions are frequently tuned with special self-learning methods (Piegat, 2001), which assure correct system operation and compensate for various errors and imprecise results of incorrect logical and arithmetic operations.

What should be done to improve the operation and constructing methods of fuzzy control systems? Suggestions are as follows:

- A new definition of a fuzzy set should be elaborated which would be better than today's definitions found, e.g., in (Dubois and Prade, 1986) and (Zimmermann, 1991) in the context of describing the substance of a fuzzy set. A proposal of the new definition is given in (Piegat, 2005).
- A new, experimentally verifiable fuzzy arithmetic should be elaborated, which would counteract doubts connected with today's existence of many different fuzzy arithmetic concept resulting in different calculation results. The same refers to logical operations.

The ultimate conclusion is that fuzzy control is not a fully developed area now and that much needs to be done to improve it.

References

Borgelt Ch. and Kruse R. (2003): *Learning possibilistic graphical models from data.* — IEEE Trans. Fuzzy Syst., Vol. 11, No. 2, pp. 159–172.

Civanlar M.R. and Trussel H.J. (1986): *Constructing membership functions using statistical data.* — Fuzzy Sets Syst., Vol. 18, No. 1, pp. 1–13.

Devi B.B. and Sarma V.V.S. (1985): *Estimation of fuzzy memberships from histograms.* — Inf. Sci., Vol. 35, No. 1, pp. 43–59.

Driankov D., Hellendoorn H. and Reinfrank M. (1993): *An introduction to fuzzy control*. —- Berlin, Heidelberg, Springer-Verlag.

Dubois D. and Prade H. (1983): *Unfair coins and necessity measures: Towards a possibilistic interpretation of histograms.* — Fuzzy Sets Syst., Vol. 10, No. 1, pp. 15–20.

Dubois D. and Prade H. (1986): *Possibility Theory.* — New York, London: Plenum Press.

Dubois D., Foulloy L. and Mauris G. (2004): *Probability – Possibility transformations, triangular fuzzy sets, and probabilistic inequalities.* — Reliable Computing, Vol. 10, No. 4, pp. 273–297.

Kaufmann A. and Gupta M.M. (1991): *Introduction to Fuzzy Arithmetic.* — New York: Van Nostrand Reinhold.

Kosiński W., Prokopowicz P. and Ślęzak D. (2003): *On algebraic operations on fuzzy reals*, In: Neural Networks and Soft Computing (Rutkowski L., Siekmann J., Tadeusiewicz R., Zadeh L.A., Eds.). — Heidelberg: Physica Verlag, pp. 54–61.

Piegat A. (2001): *Fuzzy Modeling and Control.* — Heidelberg: Physica Verlag.

Piegat A. (2005): *A new definition of the fuzzy set.* — Appl. Math. Comput. Sci., Vol. 15, No. 1, pp. 125–140.

Rakus-Anderson E. (2003): *The Newton interpolation method with fuzzy numbers as the entries*, In: Neural Networks and Soft Computing (Rutkowski L., Siekmann J., Tadeusiewicz R., Zadeh L.A., Eds.) — Heidelberg: Physica Verlag, pp. 310–315.

Yager R.R. and Filev D. (1994): *Essentials of Fuzzy Modeling and Control.* — New York: Wiley.

Von Altrock C. (1995): *Fuzzy Logic.* — Muenchen: R. Oldenburg Verlag.

Zadeh L.A. (1978): *Fuzzy sets as a basis for a theory of possibility.* — Fuzzy Sets Syst., Vol. 1, No. 1, pp. 3–28.

Zhou C. (2002): *Fuzzy-arithmetic-based Lyapunov synthesis to the design of stable fuzzy controllers: Computing with Words approach.* — Appl. Math. Comput. Sci., Vol. 12, No. 3, pp. 411–422.

Zimmermann H.J. and Zysno P. (1980): *Latent connectives in human decision making.* — Fuzzy Sets Syst., Vol. 4, No. 1, pp. 37–51.

Zimmermann H.J. (1991): *Fuzzy Set Theory and Its Applications.* — Boston: Kluwer.

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