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MINIMAL REALIZATION FOR POSITIVE MULTIVARIABLE LINEAR SYSTEMS WITH DELAY

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The realization problem for positive multivariable discrete-time systems with one time delay is formulated and solved. Conditions for the solvability of the realization problem are established. A procedure for the computation of a minimal positive realization of a proper rational matrix is presented and illustrated by an example.

Keywords: positive realization, discrete-time system, time delay, existence, computation

1. Introduction

In positive systems, inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models exhibiting positive linear systems behaviour can be found in engineering, management studies, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of the state-of-the-art in positive systems theory is given in the monographs (Farina and Rinaldi, 2000; Kaczorek, 2002). Recent developments in positive systems theory and some new results are given in (Kaczorek, 2003). Realization problems of positive linear systems without time delays have been considered in many papers and books (Benvenuti and Farina, 2003; Farina and Rinaldi, 2000; Kaczorek, 2002). An explicit solution of equations describing the discrete-time systems with time delay was given in (Busłowicz, 1982). Recently, the reachability, controllability and minimum energy control of positive linear discrete-time systems with time delays have been considered in (Busłowicz and Kaczorek, 2004; Xie and Wang, 2003).

In this paper the realization problem for positive multivariable discrete-time systems with time delay will be formulated and solved. Conditions for the solvability of the realization problem will be established and a procedure for the computation of a minimal positive realization of a proper rational matrix will be presented. To the best of the authors' knowledge, the realization problem for positive linear systems with time delays has not been considered yet.

2. Problem Formulation

Consider the multivariable discrete-time linear system with one time delay:

$$x_{i+1} = A_0 x_i + A_1 x_{i-1} + B u_i, \quad i \in \mathbb{Z}_+ = \{0, 1, \dots\}, \tag{1a}$$

$$y_i = C x_i + D u_i, \tag{1b}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are the state, input and output vectors, respectively, and $A_k \in \mathbb{R}^{n\times n}$, $k = 0, 1$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

The initial conditions for (1a) are given by

$$x_{-1}, x_0 \in \mathbb{R}^n. \tag{2}$$

Let $\mathbb{R}_+^{n\times m}$ be the set of $n \times m$ real matrices with non-negative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

Definition 1. (Busłowicz and Kaczorek, 2004). The system (1) is called (internally) positive if for every $x_{-1}, x_0 \in \mathbb{R}_+^n$ and all inputs $u_i \in \mathbb{R}_+^m$, $i \in \mathbb{Z}_+$ we have $x_i \in \mathbb{R}_+^n$ and $y_i \in \mathbb{R}_+^p$ for $i \in \mathbb{Z}_+$.

Theorem 1. (Busłowicz and Kaczorek, 2004). *The system (1) is positive if and only if*

$$A_0 \in \mathbb{R}_+^{n\times n}, \quad A_1 \in \mathbb{R}_+^{n\times n}, \quad B \in \mathbb{R}_+^{n\times m},$$
$$C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}. \tag{3}$$

The transfer matrix of (1) is given by

$$T(z) = C\left[I_n z - A_0 - A_1 z^{-1}\right]^{-1} B + D. \quad (4)$$

Definition 2. The matrices (3) are called a *positive realization* of a given proper rational matrix $T(z)$ if and only if they satisfy (4). A realization (3) is called minimal if and only if the dimension n of A_0 and A_1 is minimal among all realizations of $T(z)$.

A positive realization problem can be stated as follows: Given a proper rational matrix $T(z)$, find a positive realization (3) of the rational matrix $T(z)$. Conditions for the solvability of the problem will be established and a procedure for the computation of a positive realization will be presented.

3. Problem Solution

The transfer matrix (4) can be rewritten in the form

$$\begin{aligned} T(z) &= C\left[z^{-1}\left(I_n z^2 - A_0 z - A_1\right)\right]^{-1} B + D \\ &= \frac{Cz\mathrm{Adj}[I_n z^2 - A_0 z - A_1]B}{\det\left[I_n z^2 - A_0 z - A_1\right]} + D \\ &= \frac{zN(z)}{d(z)} + D, \end{aligned} \quad (5)$$

where

$$\begin{aligned} N(z) &= C\mathrm{Adj}\left[I_n z^2 - A_0 z - A_1\right] B \\ &= N_{2(n-1)} z^{2(n-1)} + N_{2n-3} z^{2n-3} \\ &\quad + \dots + N_1 z + N_0, \\ d(z) &= \det\left[I_n z^2 - A_0 z - A_1\right] \\ &= z^{2n} - a_{2n-1} z^{2n-1} - \dots - a_1 z - a_0, \end{aligned} \quad (6)$$

and $\mathrm{Adj}\,[I_n z^2 - A_0 z - A_1]$ denotes the adjoint matrix for $[I_n z^2 - A_0 z - A_1]$.

From (5) we have

$$D = \lim_{z\to\infty} T(z) \quad (7)$$

since $\lim_{z\to\infty}[z^{-1}(I_n z^2 - A_0 z - A_1)]^{-1} = 0$. The strictly proper part of $T(z)$ is given by

$$T_{sp}(z) = T(z) - D = \frac{zN(z)}{d(z)}. \quad (8)$$

Therefore, the positive realization problem has been reduced to finding matrices

$$A_0 \in \mathbb{R}_+^{n\times n}, \quad A_1 \in \mathbb{R}_+^{n\times n}, \quad B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{p\times n} \quad (9)$$

for a given strictly proper rational matrix (8).

Lemma 1. *The strictly proper transfer matrix (8) has the form*

$$T'_{sp}(z) = \frac{N(z)}{d'(z)} \quad (10)$$

if and only if $\det A_1 = 0$, *where*

$$d'(z) = z^{2n-1} - a_{2n-1} z^{2n-2} - \dots - a_2 z - a_1. \quad (11)$$

Proof. From the definition of $d(z)$ for $z = 0$ it follows that $a_0 = \det A_1$. Note that $d(z) = zd'(z)$ if and only if $a_0 = 0$ and (8) can be reduced to (10). ■

Lemma 2. *If the matrices A_0 and A_1 have one of the following forms:*

$$A_0 = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ a_1 & 0 & \dots & 0 & 0 \\ a_3 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2n-7} & 0 & \dots & 0 & 0 \\ a_{2n-5} & 0 & \dots & 0 & 0 \\ a_{2n-3} & 0 & \dots & 0 & a_{2n-1} \end{bmatrix} \in \mathbb{R}^{n\times n},$$

$$A_1 = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ a_0 & 0 & \dots & 0 & 0 & 0 \\ a_2 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{2(n-4)} & 0 & \dots & 0 & 0 & 0 \\ a_{2(n-3)} & 0 & \dots & 1 & 0 & 0 \\ a_{2(n-2)} & 0 & \dots & 0 & 1 & a_{2(n-1)} \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad (12a)$$

$$\bar{A}_0 = \begin{bmatrix} 0 & a_1 & a_3 & \dots & a_{2n-7} & a_{2n-5} & a_{2n-3} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & a_{2n-1} \end{bmatrix},$$

$$\bar{A}_1 = \begin{bmatrix} 0 & a_0 & a_2 & \dots & a_{2(n-4)} & a_{2(n-3)} & a_{2(n-2)} \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & a_{2(n-1)} \end{bmatrix}, \quad (12b)$$

$$\hat{A}_0 = \begin{bmatrix} a_{2n-1} & 0 & \dots & 0 & a_{2n-3} \\ 0 & 0 & \dots & 0 & a_{2n-5} \\ 0 & 0 & \dots & 0 & a_{2n-7} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_3 \\ 0 & 0 & \dots & 0 & a_1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

$$\hat{A}_1 = \begin{bmatrix} a_{2(n-1)} & 1 & 0 & \dots & 0 & a_{2(n-2)} \\ 0 & 0 & 1 & \dots & 0 & a_{2(n-3)} \\ 0 & 0 & 0 & \dots & 0 & a_{2(n-4)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a_2 \\ 0 & 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \qquad (12c)$$

$$\tilde{A}_0 = \begin{bmatrix} a_{2n-1} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ a_{2n-3} & a_{2n-5} & a_{2n-7} & \dots & a_3 & a_1 & 0 \end{bmatrix},$$

$$\tilde{A}_1 = \begin{bmatrix} a_{2(n-1)} & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ a_{2(n-2)} & a_{2(n-3)} & a_{2(n-4)} & \dots & a_2 & a_0 & 0 \end{bmatrix}, \qquad (12d)$$

then

$$\begin{aligned} &\det\left[I_n z^2 - A_0 z - A_1\right] \\ &\quad = \det\left[I_n z^2 - \bar{A}_0 z - \bar{A}_1\right] = \det\left[I_n z^2 - \hat{A}_0 z - \hat{A}_1\right] \\ &\quad = \det\left[I_n z^2 - \tilde{A}_0 z - \tilde{A}_1\right] \\ &\quad = z^{2n} - a_{2n-1} z^{2n-1} \\ &\qquad - a_{2(n-1)} z^{2(n-1)} - \dots - a_1 z - a_0. \end{aligned} \qquad (13)$$

Proof. The expansion of the determinant with respect to the first row yields

$$\begin{aligned} &\det\left[I_n z^2 - A_0 z - A_1\right] \\ &= \begin{vmatrix} z^2 & 0 & \dots & 0 & 0 & -1 \\ -a_1 z - a_0 & z^2 & \dots & 0 & 0 & 0 \\ -a_3 z - a_2 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -a_{2n-7} z - a_{2(n-4)} & 0 & \dots & z^2 & 0 & 0 \\ -a_{2n-5} z - a_{2(n-3)} & 0 & \dots & -1 & z^2 & 0 \\ -a_{2n-3} z - a_{2(n-2)} & 0 & \dots & 0 & -1 & z^2 - a_{2n-1} z - a_{2(n-1)} \end{vmatrix} \\ &= z^{2(n-1)}(z^2 - a_{2n-1} z - a_{2(n-1)}) \\ &\quad + (-1)^{n+2} \begin{vmatrix} -a_1 z - a_0 & z^2 & 0 & \dots & 0 & 0 \\ -a_3 z - a_2 & -1 & z^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{2n-7} z - a_{2(n-4)} & 0 & 0 & \dots & z^2 & 0 \\ -a_{2n-5} z - a_{2(n-3)} & 0 & 0 & \dots & -1 & z^2 \\ -a_{2n-3} z - a_{2(n-2)} & 0 & 0 & \dots & 0 & -1 \end{vmatrix} \\ &= \dots = z^{2n} - a_{2n-1} z^{2n-1} - a_{2(n-1)} z^{2(n-1)} - \dots - a_1 z - a_0. \end{aligned}$$

The forms (12b) follow from the fact that we have $\bar{A}_0 = A_0^T$, $\bar{A}_1 = A_1^T$ and $\det[I_n z^2 - \bar{A}_0 z - \bar{A}_1] = \det[I_n z^2 - A_0 z - A_1]^T$, where T stands for the transpose.

It is easy to verify that $\hat{A}_0 = P A_0 P$ and $\hat{A}_1 = P A_1 P$, where

$$P = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Taking into account that $P^{-1} = P^T = P$, we obtain

$$\det\left[I_n z^2 - \hat{A}_0 z - \hat{A}_1\right] = \det\left[I_n z^2 - A_0 z - A_1\right].$$

Finally, note that $\tilde{A}_0 = \hat{A}_0^T$ and $\tilde{A}_1 = \hat{A}_1^T$. ■

The matrices A_0 and A_1 having one of the forms (12) will be called the *matrices in canonical forms*.

Remark 1. The matrices (12) have non-negative entries if and only if the coefficients a_k, $k = 0, 1, \dots, 2n-1$ of the polynomial (13) are non-negative.

Remark 2. The dimension $n \times n$ of the matrices (12) is the smallest possible one for (8).

Definition 3. A pair (A_0, A_1) of square matrices $A_0, A_1 \in \mathbb{R}^{n\times n}$ is called *cyclic* if and only if its characteristic polynomial

$$\begin{aligned} d(z) &= \det\left[I_n z^2 - A_0 z - A_1\right] \\ &= z^{2n} - a_{2n-1}z^{2n-1} - \dots - a_1 z - a_0 \end{aligned} \tag{14}$$

is equal to the minimal polynomial $\Psi(z)$ of the pair, i.e. $d(z) = \Psi(z)$.

It is well known that the polynomials are related by

$$\Psi(z) = \frac{d(z)}{D_{n-1}(z)} \tag{15}$$

and that

$$\Psi(z) = d(z) \tag{16}$$

if and only if $D_{n-1}(z) = 1$ or, equivalently,

$$i_1(z) = i_2(z) = \dots = i_{n-1}(z) = 1, \tag{17}$$

where $D_{n-1}(z)$ is the greatest common divisor of all $n-1$ order minors of the matrix $[I_n z^2 - A_0 z - A_1]$ and $i_k(z)$, $k = 1, \dots, n-1$ are its monic invariant polynomials.

Lemma 3. *Every pair of the matrices (12) is cyclic for any values of its parameters* a_k, $k = 0, 1, \dots, 2n-1$.

Proof. The details of the proof will be given only for the pair (12a). In the remaining cases the proof is similar.

Note that the minor obtained by removing the second row and the first column of the matrix, i.e.

$$\left[I_n z^2 - A_0 z - A_1\right]$$
$$= \begin{bmatrix} z^2 & 0 & \dots & 0 & 0 & -1 \\ -a_1 z - a_0 & z^2 & \dots & 0 & 0 & 0 \\ -a_3 z - a_2 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -a_{2n-7}z - a_{2(n-4)} & 0 & \cdots & z^2 & 0 & 0 \\ -a_{2n-5}z - a_{2(n-3)} & 0 & \dots & -1 & z^2 & 0 \\ -a_{2n-3}z - a_{2(n-3)} & 0 & \dots & 0 & -1 & z^2 - a_{2n-1}z - a_{2(n-1)} \end{bmatrix}, \tag{18}$$

is equal to $(-1)^{n-1}$. Therefore $D_{n-1}(z) = 1$ and, by (15), $\Psi(z) = d(z)$. ∎

For any square matrices $A_0, A_1 \in \mathbb{R}^{n\times n}$ the inverse matrix $[I_n z^2 - A_0 z - A_1]^{-1}$ can be written in the form

$$\left[I_n z^2 - A_0 z - A_1\right]^{-1} = \frac{\bar{N}(z)}{d(z)}, \tag{19}$$

where $\bar{N}(z)$ is an $n \times n$ polynomial matrix and $d(z)$ is a polynomial. The matrix (19) is said to be in the standard form if the matrix $\bar{N}(z)/d(z)$ is irreducible and the leading coefficient of $d(z)$ is equal to 1.

Definition 4. The matrix (19) is called *normal* if and only if every non-zero second-order minor of the polynomial matrix $\bar{N}(z)$ is divisible (with zero remainder) by $d(z)$.

Lemma 4. *The standard matrix (19) for* $n \geq 2$ *is normal if and only if the pair* (A_0, A_1) *is cyclic.*

Proof. Let the pair (A_0, A_1) be cyclic. Then by Definition 3, (16) and (17) hold and the Smith canonical form of $[I_n z^2 - A_0 z - A_1]$ is equal to

$$\left[I_n z^2 - A_0 z - A_1\right]_S = \mathrm{diag}\left[1 \;\; 1 \;\; \dots \;\; 1 \;\; d(z)\right]. \tag{20}$$

The adjoint matrix to (20) is given by

$$\begin{aligned} &\mathrm{Adj}\left[I_n z^2 - A_0 z - A_1\right]_S \\ &\qquad = \mathrm{diag}\left[d(z) \;\; d(z) \;\; \dots \;\; d(z) \;\; 1\right] \end{aligned} \tag{21}$$

and every non-zero second-order minor of (21) is divisible by $d(z)$. By the Binet-Cauchy theorem, every non-zero second-order minor of the matrix $V(z)\mathrm{Adj}\,[I_n z^2 - A_0 z - A_1]_S U(z)$ is also divisible by $d(z)$ since it is the sum of the products of second-order minors of the unimodular matrices $V(z)$, $U(z)$ and of (21). The necessity will be shown by contradiction. By assumption, the matrix (19) is irreducible. If the characteristic polynomial (14) is not equal to the minimal one $\Psi(z)$, $\Psi(z) \neq d(z)$, then by (15) $D_{n-1}(z) \neq 1$ and every non-zero $(n-1)$-th order minor of $[I_n z^2 - A_0 z - A_1]$ is divisible by $D_{n-1}(z)$. In this case $\det\,[I_n z^2 - A_0 z - A_1] = D_{n-1}(z)\bar{d}(z)$ and the matrix (19) is reducible. So we get a contradiction.

Lemma 5. *If the pair* (A_0, A_1) *has the canonical form (12a), then the adjoint matrix* $\mathrm{Adj}\,[I_n z^2 - A_0 z - A_1]$ *can be decomposed as follows:*

$$\mathrm{Adj}\,[I_n z^2 - A_0 z - A_1] = \bar{P}(z)\bar{Q}(z) + d(z)\bar{G}(z), \tag{22a}$$

where

$$\bar{P}(z) = \begin{bmatrix} 1 \\ z^{2(n-1)} - a_{2n-1}z^{2n-3} - \dots - a_3 z - a_2 \\ z^{2(n-2)} - a_{2n-1}z^{2n-5} - \dots - a_5 z - a_4 \\ \vdots \\ z^4 - a_{2n-1}z^3 - a_{2(n-1)}z^2 - a_{2n-3}z - a_{2(n-2)} \\ z^2 \end{bmatrix},$$

$$\begin{aligned} \bar{Q}(z) = \big[&z^{2(n-1)} - a_{2n-1}z^{2n-3} - a_{2(n-1)}z^{2(n-2)} \\ &1 \;\; z^2 \;\; z^4 \;\; \dots \;\; z^{2(n-2)}\big], \end{aligned} \tag{22b}$$

$$\bar{G}(z) = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ * & 0 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & 0 & \dots & * & * \\ * & 0 & \dots & * & * \end{bmatrix},$$

and '$*$' denotes the entries that are not important in the considerations.

Similar decompositions hold for the pairs (12b), (12c) and (12d).

Proof. The adjoint matrix has the form

$$\mathrm{Adj}\left[I_n z^2 - A_0 z - A_1\right] = \begin{bmatrix} m_{11} & 1 & z^2 & \dots & z^{2(n-2)} \\ * & m_{22} & * & \dots & * \\ * & m_{32} & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & m_{n2} & * & \dots & * \\ * & z^2 & * & \dots & * \end{bmatrix}, \quad (23)$$

where

$$m_{11} = z^{2(n-1)} - a_{2n-1}z^{2n-3} - a_{2(n-1)}z^{2(n-2)},$$

$$m_{22} = z^{2(n-1)} - a_{2n-1}z^{2n-3} - \dots - a_3 z - a_2,$$

$$m_{32} = z^{2(n-2)} - a_{2n-1}z^{2n-5} - \dots - a_5 z - a_4,$$

$$m_{n2} = z^4 - a_{2n-1}z^3 - a_{2(n-1)}z^2 - a_{2n-3}z - a_{2(n-2)},$$

and it can be written in the form (22) since by Lemma 4 every non-zero second-order minor of (23) is divisible by $d(z)$. It is easy to verify that (22b) satisfies (22a). ■

The substitution of (22a) into (8) yields

$$T_{sp}(z) = \frac{Cz\mathrm{Adj}\left[I_n z^2 - A_0 z - A_1\right]B}{\det\left[I_n z^2 - A_0 z - A_1\right]} = \frac{zP_c(z)Q_b(z)}{d(z)} + Cz\bar{G}(z)B, \quad (24)$$

where

$$P_c(z) = C\bar{P}(z), \quad Q_b(z) = \bar{Q}(z)B. \quad (25)$$

Remark 3. From (24) it follows that the positive realization (9) of (8) is independent of the polynomial matrix $\bar{G}(z)(Cz\bar{G}(z)B)$.

Using (22b) and (25) we obtain

$$P_c(z) = C\bar{P}(z) = \begin{bmatrix} C_1 & C_2 & \dots & C_n \end{bmatrix} \times \begin{bmatrix} 1 \\ z^{2(n-1)} - a_{2n-1}z^{2n-3} - \dots - a_3 z - a_2 \\ z^{2(n-2)} - a_{2n-1}z^{2n-5} - \dots - a_5 z - a_4 \\ \vdots \\ z^4 - a_{2n-1}z^3 - a_{2(n-1)}z^2 - a_{2n-3}z - a_{2(n-2)} \\ z^2 \end{bmatrix}$$

$$\begin{aligned} &= C_2 z^{2(n-1)} - C_2 a_{2n-1}z^{2n-3} \\ &\quad + \left(C_3 - a_{2(n-1)}C_2\right) z^{2(n-2)} - \dots \\ &\quad + (C_n - a_{2(n-1)}C_{n-1} - \dots - a_6 C_3 - a_4 C_2)z^2 \\ &\quad + \dots - (a_{2n-3}C_{n-1} \dots + a_5 C_3 + a_3 C_2)\, z \\ &\quad + C_1 - a_2 C_2 - a_4 C_3 \dots - a_{2(n-2)}C_{n-1}, \end{aligned} \quad (26a)$$

$$\begin{aligned} Q_b(z) &= \bar{Q}(z)B \\ &= \left[z^{2(n-1)} - a_{2n-1}z^{2n-3} - a_{2(n-1)}z^{2(n-2)} \quad 1 \quad z^2 \quad z^4 \quad \dots \quad z^{2(n-2)}\right] \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} \\ &= B_1 z^{2(n-1)} - a_{2n-1}B_1 z^{2n-3} \\ &\quad + \left(B_n - a_{2(n-1)}B_1\right) z^{2(n-2)} \\ &\quad + \dots + B_3 z^2 + B_2. \end{aligned} \quad (26b)$$

From Lemma 5 it follows that the strictly proper matrix (8) can be decomposed as follows:

$$N(z) = P(z)\,Q(z) + d(z)G(z), \quad (27)$$

where

$$\begin{aligned} P(z) &= P_{2(n-1)}z^{2(n-1)} - P_{2n-3}z^{2n-3} \\ &\quad + P_{2(n-2)}z^{2(n-2)} - \dots + P_2 z^2 - P_1 z + P_0, \\ Q(z) &= Q_{2(n-1)}z^{2(n-1)} - Q_{2n-3}z^{2n-3} \\ &\quad + Q_{2(n-2)}z^{2(n-2)} - \dots + Q_2 z^2 - Q_1 z + Q_0, \\ G(z) &\in \mathbb{R}^{p\times n}[z]. \end{aligned} \quad (28)$$

The polynomial matrices $P(z)$, $Q(z)$ and $G(z)$ of (27) can be computed using the following procedure:

Procedure 1.

Step 1. Using elementary row and column operations perform the reduction

$$U(z)N(z)V(z) = p(z)\begin{bmatrix} 1 & r(z) \\ c(z) & M(z) \end{bmatrix},$$

where $U(z)$ and $V(z)$ are unimodular matrices of elementary operations, $p(z)$ is a polynomial, $r(z) \in \mathbb{R}^{1\times(m-1)}[z]$, $c(z) \in \mathbb{R}^{p-1}[z]$ and $M(z) \in \mathbb{R}^{(p-1)\times(m-1)}[z]$.

Step 2. Compute the matrices

$$P(z) = U^{-1}(z)p(z)\begin{bmatrix} 1 \\ c(z) \end{bmatrix},$$

$$Q(z) = \begin{bmatrix} 1 & r(z) \end{bmatrix} V^{-1}(z) \tag{29}$$

$$G(z) = U^{-1}(z)\begin{bmatrix} 0 & 0 \\ 0 & p(z)\big(M(z)-c(z)r(z)\big) \end{bmatrix} V^{-1}(z).$$

The comparison of (26) and (28) yields

$$\begin{aligned}
P_{2(n-1)} &= C_2, \quad P_{2n-3} = C_2 a_{2n-1}, \\
P_{2(n-2)} &= C_3 - a_{2(n-1)}C_2, \quad \dots, \\
P_0 &= C_1 - a_2C_2 - a_4C_3 - \dots - a_{2(n-2)}C_{n-1}, \\
Q_{2(n-1)} &= B_1, \quad Q_{2n-3} = a_{2n-1}B_1, \\
Q_{2(n-2)} &= B_n - a_{2(n-1)}B_1, \dots, Q_0 = B_2.
\end{aligned} \tag{30}$$

Given the matrices P_k and Q_k for $k = 0, 1, \dots, 2(n-1)$, from (30) we can find C_i and B_i, $i = 1, \dots, n$ corresponding to the matrices C and B.

From (30) it follows that $B_i \in \mathbb{R}_+^{1\times m}$ and $C_i \in \mathbb{R}_+^p$ for $i = 1, \dots, n$ if $P_k \in \mathbb{R}_+^p$ and $Q_k \in \mathbb{R}_+^{1\times m}$ for $k = 0, 1, \dots, 2(n-1)$ and $a_j \geq 0$ for $j = 0, 1, \dots, 2n-1$. Thus the following result was proven:

Theorem 2. *Let the transfer matrix (4) be normal. The positive realization problem has a solution if the following conditions are satisfied:*

(i) $T(\infty) = \lim_{z\to\infty} T(z) \in \mathbb{R}_+^{p\times m}$.

(ii) The coefficients a_k, $k = 0, 1, \dots, 2n-1$ *of the polynomial* $d(z)$ *are non-negative.*

(iii) The polynomial matrix $N(z)$ *of (8) can be decomposed so that the polynomials* $P(z)$ *and* $Q(z)$ *(defined by (28)) have non-negative coefficients matrices, i.e.* $P_k \in \mathbb{R}_+^p$ *and* $Q_k \in \mathbb{R}_+^{1\times m}$ *for* $k = 0, 1, \dots, 2(n-1)$ *and the relations (29) are satisfied.*

If the conditions of the theorem are satisfied, then a positive realization (3) of $T(z)$ can be found using the following procedure:

Procedure 2.

Step 1. Using (7) and (8), find D and the strictly proper rational matrix $T_{sp}(z)$.

Step 2. Given the coefficients a_k, $k = 0, 1, \dots, 2n-1$ of $d(z)$ find the matrices (12a) (or (12b), (12c), (12d)).

Step 3. Using Procedure 1 find the decomposition (27) of the polynomial matrix $N(z)$ of (8) and the coefficients matrices P_k and Q_k, $k = 0, 1, \dots, 2(n-1)$ of the polynomial matrices (28).

Step 4. Using (30), find B_i and C_i, $i = 1, \dots, n$ and the matrices B and C.

Example 1. Find a positive realization (3) of the transfer matrix

$$T(z) = \frac{1}{z^5 - z^4 - 2z^3 - 3z^2 - 2z - 1} \times \begin{bmatrix} 2z^5+z^4-2z^3-4z^2-3z-2 & z^5-2z^3-z^2-2z \\ z^5+z^4+2z^3-2z^2-z-1 & 2z^4-z^3-z^2-2z-2 \end{bmatrix}. \tag{31}$$

Using Procedure 1, we obtain successively the following results:

Step 1. From (7) and (8) we have

$$D = \lim_{z\to\infty} T(z) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \tag{32}$$

and

$$T_{sp}(z) = T(z) - D = \frac{N(z)}{d'(z)}, \tag{33}$$

where

$$N(z) = \begin{bmatrix} 3z^4+2z^3+2z^2+z & z^4+2z^2+1 \\ 2z^4+4z^3+z^2+z & 2z^4-z^3-z^2-2z-2 \end{bmatrix},$$

$$d'(z) = z^5 - z^4 - 2z^3 - 3z^2 - 2z - 1.$$

Step 2. Taking into account the fact that $a_0 = 0$, $a_1 = a_5 = 1$, $a_2 = a_4 = 2$, $a_3 = 3$ and using (12a), we obtain

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix}. \tag{34}$$

Step 3. Using Procedure 1, we decompose the matrix $N(z)$ in the form (27) with

$$\begin{aligned}
P(z) &= \begin{bmatrix} z^2+1 \\ z^4 - z^3 - z^2 - 3z - 2 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} z^4 - \begin{bmatrix} 0 \\ 1 \end{bmatrix} z^3 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} z^2 \\
&\quad - \begin{bmatrix} 0 \\ 3 \end{bmatrix} z + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= P_4z^4 - P_3z^3 + P_2z^2 - P_1z + P_0,
\end{aligned}$$

$$Q(z) = \begin{bmatrix} z^4 - z^3 & z^2 + 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \end{bmatrix} z^4 - \begin{bmatrix} 1 & 0 \end{bmatrix} z^3 + \begin{bmatrix} 0 & 1 \end{bmatrix} z^2 + \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$= Q_4 z^4 - Q_3 z^3 + Q_2 z^2 - Q_1 z + Q_0,$$
$$G(z) = \begin{bmatrix} -1 & 0 \\ -z^2 + z - 1 & -1 \end{bmatrix}. \tag{35}$$

Step 4. Using (30) and (35), we obtain

$$B_1 = Q_4 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad B_2 = Q_0 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$
$$B_3 = Q_2 + 2Q_4 = \begin{bmatrix} 2 & 1 \end{bmatrix},$$
$$C_1 = P_0 + 2P_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_2 = P_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$C_3 = P_2 + 2P_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and

$$B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \tag{36}$$

The desired positive minimal realization of (31) has the form (34), (36) and (32).

4. Concluding Remarks

The realization problem for positive multivariable discrete-time systems with one time delay has been formulated and solved. The canonical forms (12) of the system matrices A_0 and A_1 were introduced. It was shown that the pair (12) is cyclic. The conditions for the existence of a positive minimal realization (3) of a proper rational matrix $T(z)$ were established. A procedure for the computation of a minimal positive realization of a proper rational matrix was presented and illustrated by an example. The deliberations can be extended to multivariable discrete-time linear systems with many time delays. An extension to continuous-time linear systems with time delays is also possible.

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