Appl. Math. and Comp. Sci., 1995, vol.5, No.3, 481–490

PARTIALLY DEFINED CONTROL SYSTEMS. CONTROLLABILITY AND STABILIZABILITY[†]

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The paper gives a setting for studying controllability and stabilizability of partially defined systems. Some results known for globally defined systems are transferred to this more general setting, some are shown to be not valid for partially defined systems. Global reachability and local stabilizability for partially defined linear systems are studied in more detail.

1. Introduction

In many cases the vector fields forming a non-linear system are not defined on the entire state space. This happens e.g. if the vector fields have rational components. Moreover, different vector fields may have different domains and this leads to the concept of partially defined control system. Bartosiewicz and Johnson (1995) introduced necessary formalism, based on Johnson's theory of universes, to study observability properties. In this paper, we follow that way but in a more intuitive and less formal fashion. One of the key points in (Bartosiewicz and Johson, 1995) was the calculus of partially defined functions, making use of the "phantom" which represented an undefined scalar. In order to study controllability properties we introduce here the phantom vector (which could be deduced from the general theory of (Johson, 1986). This allows for similar calculations with partially defined vector fields. In particular, we can amalgamate several vector fields that agree on common domains into one vector field defined on the union of their domains. Having this concept we can construct the Lie universe of a partially defined control system which plays the same role as the Lie algebra for globally defined systems.

This paper is aimed rather at introducing the proper setting for studying controllability problems than presenting final results. We feel that global reachability is an especially interesting problem for partially defined systems. Even if the system has poor local control properties, it can still be globally reachable. This may be caused by a reasonable number of directions available globally, while locally only few of them may be used. The starting point for this problem should be global reachability of partially defined linear (or affine) systems. We make a first step in this direction studying a linear hybrid system in which we have two controls: the standard continuous one and a discrete control which allows switching between different dynamics (systems). We also give a short overview of some basic results for the globally defined

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non-linear systems see e.g. (Isidori, 1985) and show which of them remain valid in our setting. We present counter-examples when the results are no longer true for a partially defined system. This is a very selective and by no means exhaustive account. We believe that future papers will go deeper into that area.

Feedback transformation of a partially defined system produces a control-free global system. Hence, a natural stabilization problem consists of two parts: standard local stabilization and globalization. Another problem, maybe more interesting, is local stabilization at an equilibrium point that lies on the boundary of the domain. In this case, we are interested in system's behaviour in the neighbourhood of the equilibrium point, so we do not require the resulting system to be global. We solve this problem in a particular case when the system is linear. A similar problem, but in a more global form, was studied by Bacciotti and Boieri (1990).

2. Partially Defined Functions

For a complete description of the theory of partially defined functions see e.g. (Bartosiewicz and Johson, 1995; Johson 1986). Here we give only basic definitions and properties.

Let A_0 be the real line extended by an element representing the "undefined scalar" which is called the phantom and denoted by $\emptyset_0$. If X is any set and f is a real function defined on a subset U of X (called *domain*), then we treat f as a function $X \rightarrow A_0$ assigning $f(x) = \emptyset_0$ whenever $x \notin U$. Since the result of any operation involving $\emptyset_0$ (like addition or multiplication) is again $\emptyset_0$, we can extend all those operations to partially defined functions on X. In particular, we shall need substitutions for partially defined functions on $\mathbb{R}^k$.

If F is a partially defined function on $\mathbb{R}^k$ and $\varphi_1, \dots, \varphi_k$ are partially defined functions on X, then the partially defined function $F(\varphi_1, \dots, \varphi_k)$ is defined on X by

$$F(\varphi_1, \dots, \varphi_k)(x) = F(\varphi_1(x), \dots, \varphi_k(x)) \tag{1}$$

where we put $\emptyset_0$ whenever the composition is not defined.

Another operation we shall need is amalgamation. Two partially defined functions are *matching* if they take on the same values at all the points in the intersection of their domains. A family H of functions is matching, if any two functions in this family are matching. One can show that for any matching family H of functions one can construct a function denoted by $\underline{H}$ and defined on the union of domains of all functions in H such that $\underline{H}$ is matching with every function in H. This process is called *amalgamation*.

The set of all the partially defined functions on X will be denoted by $\mathbb{R}_X$. If M is an analytic manifold, then by A_M we shall mean the set of all the partially defined analytic functions on M. If $a \in A_0$ and $V \subset X$, then a_V denotes the constant function on X equal to a in the domain V.

Let us introduce several classes of functions defined on open domains in $\mathbb{R}^k$ for all $k > 0$. By L we denote the class of linear functions, by P the class of polynomial functions, and by A the class of analytic functions. Let K denote any class of functions. By *K-universe* on X we mean any family C of partially defined functions on X, which contains 0_X (global zero function on X), is closed under substitutions (1) for $F \in K$, and is closed under amalgamations, i.e. for every matching family $M \subset C$, $\underline{M} \in C$.

In particular, *a linear universe* is any L-universe and *an analytic universe* is any A-universe. Observe that if M is an analytic manifold, then A_M is an analytic universe.

3. Partially Defined Vector Fields and Distributions

Most of the material in this section can be developed more formally using the language of (Bartosiewicz and Johson, 1995). We prefer here a more staightforward and intuitive approach.

Similarly as the undefined scalar, we introduce the undefined vector in $\mathbb{R}^n$ denoted by $\emptyset_n$ and called the *phantom vector* (one may interpret it as the point or the vector at infinity). Then $\mathbb{R}_n := \mathbb{R}^n \cup \{\emptyset_n\}$ is an extension of the standard real vector space. By a *subspace* of $\mathbb{R}_n$ we shall mean a linear subspace of $\mathbb{R}^n$ together with $\emptyset_n$. A *linear subspace* of $\mathbb{R}_n$ will be just a linear subspace of $\mathbb{R}^n$.

For a C^ω n-dimensional manifold M, T_xM denotes as usual the tangent space to M at x, and P_xM will mean the extended tangent space at x (one can say "isomorphic" to $\mathbb{R}_n$), containing one more element $\emptyset_n$. For any vector $v \in T_xM$ and element $\alpha \in A_0$ we define $v + \emptyset_n = \emptyset_n + v = \emptyset_n$, $\emptyset_0 v = \emptyset_n$ and $\alpha\emptyset_n = \emptyset_n$. Any element of T_xM may be interpreted as a derivation mapping from the set A_M of partially defined analytic functions on M, into A_0. Similarly, $\emptyset_n$ is a derivation which assigns $\emptyset_0$ to each function of A_M.

A *partially defined vector field* on M is a vector field f defined on an open subset of M denoted by $\mathrm{dom}\, f$. Such a vector field may be identified with an extended vector field on M where we put $f(x) = \emptyset_n$ if $x \notin \mathrm{dom} f$. It may also be viewed as a derivation from A_M into A_M (see Bartosiewicz and Johson, 1995). Then, for $\varphi \in A_M$, we have $\mathrm{dom}\, f\varphi = \mathrm{dom} f \cap \mathrm{dom}\varphi$. In particular, $\mathrm{dom} f$ may be decoded as $\mathrm{dom}(f\varphi)$ where φ is any globally defined analytic function (e.g. 0_M). Similarly, $\mathrm{dom}(f_2 f_1) = \mathrm{dom} f_2 \cap \mathrm{dom} f_1$, where $f_2 f_1$ denotes the composition of derivations. Finally, $\mathrm{dom}[f_1, f_2] = \mathrm{dom} f_1 \cap \mathrm{dom} f_2$. The last equality may also be obtained within the framework of the standard calculus on $\mathbb{R}^n$, where a partially defined vector field would be treated as an analytic mapping $f : U \to \mathbb{R}^n$, $U \subset \mathbb{R}^n$ is open.

Let us extend the meaning of 0_U to a zero vector field with the domain equal to U (as we denote by 0 a real number and a vector). A convenient way to restrict a vector field f to a subset U is to write $f + 0_U$.

A *trajectory* of a partially defined vector field f is a mapping $\gamma : I \to \mathrm{dom} f$ defined on any interval I, satisfying the following differential equation

$$\dot{\gamma}(t) = f(\gamma(t)) \tag{2}$$

for all $t \in I$.

Let $V_\omega(M)$ be the the set of all the partially defined analytic vector fields on M. Although it is no longer a linear space, $V_\omega(M)$ has a structure of a linear universe, i.e. it is closed under substitutions into linear partially defined functions and closed under amalgamation. The latter operation is defined in the same way as for functions. Hence, we can add two partially defined vector fields, multiply such a field by an element of A_0, and glue together a matching family of partially defined vector fields.

Let $\mathcal{F}$ be any family of partially defined vector fields on a real analytic manifold M. By $LC(\mathcal{F})$ we denote the smallest linear universe containing $\mathcal{F}$, and by $\mathcal{L}(\mathcal{F})$ — the *Lie universe* generated by $\mathcal{F}$, i.e. the smallest set of partially defined vector fields on M containing $\mathcal{F}$ and closed with respect to linear substitutions, Lie brackets and amalgamation.

By a *(partially defined) distribution* D on M we mean a map which assigns a subspace $D(x)$ of P_xM to each $x \in M$. A distribution is *analytic* if there exists a family $\mathcal{F}$ of partially defined vector fields on M such that for every $x \in M$, $D(x) = LC(\mathcal{F})(x)$. We admit the case where $D(x)$ consists only of $\emptyset_n$ for some or all $x \in M$. We say that a partially defined vector field f *belongs to* a distribution D, if $f(x) \in D(x)$ for all $x \in M$.

A distribution D is *integrable* if for any $x \in M$ there exits an immersed submanifold N of M (maybe empty) such that for each $y \in N$ we have $P_yN = D(x)$. For the empty submanifold N the extended tangent space consists only of $\emptyset_n$ (or, more precisely, $\emptyset_0$). A distribution D is *involutive* if for every pair of vector fields f and g belonging to D, their Lie bracket belongs to D, too.

Partially defined distributions have much richer structures and behave more freely than globally defined ones. It is clear however that some facts of local nature will remain true also for partially defined distributions.

Proposition 1. *If D is integrable, then it is also involutive.*

Proof. It is analogous as for globally defined distributions. ∎

Unfortunately, the Nagano Theorem cannot be extended to partially defined (analytic) distributions as the following example shows.

Example 1. Let D be given by two vector fields f and g on $M = \mathbb{R}^2$, where $f = \partial_1$ and $g = 1_U\partial_2$, U being the right-hand side open half-plane, and $\partial_i = \frac{\partial}{\partial x_i}$. The distribution is obviously involutive, but there is no integral manifold passing through any point x with $x_1 = 0$. What happens here is quite similar to pathologies observed in the C^∞case.

One can easily change this picture to have still partially defined, but now integrable distribution.

Example 2. Let $M = \mathbb{R}^2$ and let D be generated by vector fields f and g, where $f(x) = x_1\partial_1$ and g is defined as in Example 1. Then for all x with $x_1 > 0$ the integral manifold of x is the right-hand side half-plane. Any point x with $x_1 = 0$ is itself a zero-dimensional integral manifold. And finally, the integral manifold passing through x with $x_1 < 0$ is a half-line parallel to the x_1-axis. Observe that one cannot find globally defined vector fields which define D.

Fortunately, the Frobenius Theorem is still valid for partially defined analytic distributions.

Theorem 1. *If* $\dim D(x)$ *is constant in the state space, then* D *is integrable.*

Proof. Take a point $x \in M$ and choose vector fields $f_1, \ldots, f_k$ belonging to D, linearly independent at x and spanning $D(x)$. Then they are linearly independent in some neighbourhood U of x, and since the dimension of $D(y)$ is constant, they span $D(y)$ for all $y \in U$. Hence, on U, D is spanned by global vector fields and has a constant dimension, so it is integrable by the standard Frobenius theorem. ∎

Example 3. Let $M = \mathbb{R}^2$ and let D be generated by the vector fields f, g_1 and g_2, where $\mathrm{dom} f = M$, $f = \partial_1$, $\mathrm{dom} g_1 = U_1 = \{x : x_1 < 1\}$, $g_1 = -\partial_1 + \partial_2$ and $\mathrm{dom} g_2 = U_2 = \{x : x_1 > 0\}$, $g_2 = \partial_1 + \partial_2$. Then at each $x \in M$, $\dim D(x) = 2$, so D is integrable. For each x there exists a neighbourhood on which D is generated by two vector fields globally defined on U, but this cannot be done globally on M (using only vector fields from $\mathcal{L}(f, g_1, g_2)$).

4. Controllability

By a *partially defined control system* Σ we shall mean a family of partially defined vector fields $\{f_\omega, \omega \in \Omega\}$ on an analytic manifold M, parametrized by some set Ω (the set of control values or commands). This is an abstract version of the standard description of a control system given by a differential equation of the form

$$\dot{x}(t) = f(x(t), u(t)) \tag{3}$$

with the common identification $f_\omega = f(\cdot, \omega)$. Partial definiteness of the system means that for any fixed command ω (i.e. the control value $u(t)$ for some t), the domain of the vector field $f(\cdot, \omega)$ may not be the entire manifold M. The *set of controls* $\mathcal{U}$ will consist of all piecewise constant functions

$$u : [0, T_u] \to \Omega \tag{4}$$

where $T_u \geq 0$ depends on u.

If U is an open subset of M, then $\Sigma_{|U}$ denotes the *restriction of* Σ *to* U which is the family of the vector fields $\{f_\omega + 0_U\}$. Observe that M is still the state space for the restricted system.

The *trajectory* of Σ starting at $x_0 \in M$ and corresponding to a control $u \in \mathcal{U}$ is a function γ on $[0, T_u]$ with the values in M such that $\gamma(t) \in \mathrm{dom} f_{u(t)}$ and $\gamma_{|[t_i, t_{i+1}]}$ is a trajectory of f_ω if $u(t) = \omega$ for $t \in [t_i, t_{i+1}]$. We shall also write $\gamma(t, x_0, u)$ instead of $\gamma(t)$ to stress the dependence on x_0 and u. For a given control u and an initial condition x_0, the trajectory may not exist. One reason for this might be incompleteness of one of the vector fields involved, another one — partial definiteness of those vector fields. We do not define "partial" trajectories, existing only on some subinterval of $[0, T_u]$. Instead, we can always consider shorter controls, i.e. restrictions of u to $[0, T]$, where $T < T_u$. Let $\mathcal{U}(x_0)$ denote the set of the controls $u \in \mathcal{U}$ such that the trajectory for x_0 and u exists.

The following definitions are standard, once we have the precise notion of trajectory. The *reachable set of* Σ *from* x_0 *at time* $t \geq 0$ is defined by

$$\mathcal{R}_t(\Sigma, x_0) = \{\gamma(t, x_0, u) : u \in \mathcal{U}(x_0) \text{ and } T_u = t\} \tag{5}$$

Then the *reachable set from* x_0 is

$$\mathcal{R}(\Sigma, x_0) = \bigcup_{t \geq 0} \mathcal{R}_t(\Sigma, x_0) \tag{6}$$

When the system is fixed, we often skip Σ in (5) and (6).

We say that Σ is *accessible from* x_0 if $\mathcal{R}(\Sigma, x_0)$ has a non-empty interior, is *strongly accessible from* x_0 if $\mathcal{R}_t(\Sigma, x_0)$ has a non-empty interior for all $t > 0$, and is *locally accessible from* x_0 if for every open neighbourhood U of x_0, the restricted system $\Sigma_{|U}$ is accessible from x_0.

We say that Σ is *reachable from* x_0 if $\mathcal{R}(\Sigma, x_0) = M$, and is *locally reachable from* x_0 if for every open neighbourhood U of x_0, $\mathcal{R}(\Sigma_{|U}, x_0)$ contains some open neighbourhood V of x_0 (of course $V \subset U$).

It is known that reachability properties for non-linear systems are often replaced by accessibility ones.

Let the Lie universe of a partially defined system Σ be defined by $\mathcal{L}(\Sigma) = \mathcal{L}\{f_\omega, \omega \in \Omega\}$. The proof of the following result is the same as for globally defined systems (Sussmann and Jurdjevic, 1972).

Theorem 2. *If* $\mathcal{L}(\Sigma)_x = P_x M$, *then* Σ *is locally accessible (and accessible) from* x. ∎

However, for partially defined systems Theorem 2 cannot be reciprocated.

Example 4. Let $M = \mathbb{R}^2$ and $\Sigma = \{f_1, f_2\}$ where $f_1 = \partial_1$ and $f_2 = \partial_2 + 0_U$, U being the right-hand-side open half-plane. Then Σ is strongly accessible (and then accessible), while $\mathcal{L}(\Sigma)$ evaluated at $x = 0$ is spanned only by ∂_1 (and contains $\emptyset_2$ by definition).

We say that Σ is *symmetric* if for every $\omega \in \Omega$ there is $\omega' \in \Omega$ such that $-f_\omega = f_{\omega'}$. The following theorem is a simple extension of the results known for globally defined systems.

Theorem 3.

a) If Σ *is symmetric and* $\mathcal{L}(\Sigma)_x = P_x M$, *then* Σ *is locally reachable from* x.

b) If Σ *is symmetric and for every* $x \in M$, $\mathcal{L}(\Sigma)_x = P_x M$, *then* Σ *is reachable from any* $x \in M$. ∎

Because the structure of a partially defined system may heavily depend on the current state of the sytem, we may have global reachability property even though locally the system is poorly controllable. We think that such a phenomenon is an essential feature of a partially defined system and studying global reachability should be one of the most interesting problems in this theory. Though a general solution seems to be out of reach at the moment, one can try to characterize the reachability for systems with a simpler structure.

A natural starting point for such a journey is a linear partially defined system. Such systems appear e.g. as linear approximations to non-linear systems. Let us consider a (globally defined) non-linear system affine in control

$$\dot{x} = f(x) + g_1(x)u_1 + \ldots + g_m(x)u_m \tag{7}$$

where $x(t) \in M = \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}$ for $i = 1, \ldots, m$. Choose $x_0 \in M$. Then locally around x_0, (7) can be approximated by a linear (affine) system

$$\dot{x} = Ax + Bu + d \tag{8}$$

where $d = f(x_0)$, $A = \frac{\partial f}{\partial x}(x_0)$ and $B = (g_1(x_0), \ldots, g_m(x_0))$. We neglect here bilinear terms involving x and u and all the higher order terms. Instead of approximation, one can obtain (8) via suitable coordinate change around x_0 or using a feedback transformation of static or dynamic nature. Whatever the method that leads to (8), the common feature is that we usually get only the linearized system locally around x_0. If we are interested in many operating points, the procedure results in a family of affine systems

$$(\Sigma_i) \quad \dot{x} = A_i x + B_i u + d_i \tag{9}$$

where i belongs to some set of indices J (possibly infinite) and each system Σ_i is defined on some open neighbourhood U_i in $\mathbb{R}^n$ of a point x_i. Let Σ be the collection of all Σ_i, $i \in J$. Then Σ is a partially defined system with the set of commands $\Omega = J \times \mathbb{R}^m$ and the affine vector fields $f_{i\omega}(x) = A_i x + B_i \omega + d_i$ (written as maps) with $\text{dom} f_{i\omega} = U_i$.

Global controllability properties of such a system Σ must depend not only on matrices and vectors A_i, B_i and d_i, but also on domains U_i. It is natural to assume that the union of all the domains covers the entire state space $\mathbb{R}^n$. A common sense strategy of controlling system Σ from one point to another would be to use overlapping regions where two (or more) systems Σ_i and Σ_k are valid to switch from system Σ_i to system Σ_k (or vice versa).

In order to solve the problem described above it seems necessary to have a good theory for the systems we call *hybrid* after Brockett (1993). A hybrid system on $\mathbb{R}^n$ is a family of linear or affine globally defined systems Σ_i of the form (9) where $i \in J$.

To control a hybrid system we need to specify the index i and the standard control u — both are (piecewise constant) functions of time. One can easily see that the standard control theory of linear systems cannot accomodate this problem.

Let $K(\Sigma)$ be a linear subspace of $\mathbb{R}^n$ spanned by the columns of the matrices of the form $A_{i_k}^{j_k} \ldots A_{i_1}^{j_1} B_s$, where $s \in \{1, \ldots, m\}, k \geq 0, i_1, \ldots, i_k \in J$ and $j_1, \ldots, j_k$ are greater than or equal to 0.

Proposition 2. *Assume that $d_i = 0$ for all i. Then Σ is reachable from every $x \in \mathbb{R}^n$ iff $K(\Sigma) = \mathbb{R}^n$.*

Proof. Observe that $\mathcal{L}(\Sigma)_0 = K(\Sigma)$ and that Σ is globally defined. Hence Σ is locally accessible from 0 iff $K(\Sigma) = \mathbb{R}^n$. Moreover, one can show that the reachable set from 0 is a linear subspace of $\mathbb{R}^n$. This means that reachability from zero is equivalent to accessibility from 0. Extending the result to all points of the state space is a standard procedure. ∎

One can show that if we drop the requirement that $d_i = 0$, the condition given in Proposition 2 will still be sufficient for reachability. Now we can use this result in partially defined systems.

Corollary 1. *Suppose that Σ consists of Σ_1 and Σ_2, where Σ_1 is globally defined and Σ_2 is defined on an open half-space U_2 given by the inequality $c^T x < a$ (c is a vector and $a \in \mathbb{R}$). Assume that $K(\Sigma) = \mathbb{R}^n$ and $c \notin K(\Sigma_2)$. Then the system Σ is reachable from any $x \in \mathbb{R}^n$.*

Proof. It is clear from Proposition 2 that we can travel from any x to any y if $x, y \in U_2$. The condition $c \notin K(\Sigma_2)$ allows us to leave the set U_2 when we start from a point x in U_2, and to reach any point y in U_2 from outside of U_2. This gives reachability. ∎

5. Stabilizability

In order to define feedback transformation of the system Σ we need a regular dependence of the vector fields f_ω on the control parameter ω. In this section, we assume that the state space is $M = \mathbb{R}^n$ and the set of control values Ω is an open subset of $\mathbb{R}^m$. Define a partially defined mapping $f : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ by

$$f(x,u) = f_u(x)$$

Let us assume that dom$f$ is open and that f is analytic. Thus, as before, for every $\omega \in \Omega$, domf_ω is open, but also for every $x \in M$, the set $\{\omega : f(x,\omega) \neq \emptyset_n\}$ is open. It is also clear that dom$f$ projected onto M (Ω, respectively) is the entire M (Ω, respectively).

By a *global feedback* of Σ we mean an analytic mapping $v : M \to \Omega$ such that the resulted closed-loop system is global, i.e. $g(x) := f(x, v(x))$ is defined for every $x \in M$. This is in agreement with the requirement that every $x \in M$ is in the domain of some vector field of the system.

By a *local feedback* at $x_0 \in M$ we mean a mapping $v : U \to \Omega$, where U is a neighbourhood of x_0. We do not assume that the resulting vector field $x \mapsto f(x, v(x))$ is defined on U. But in applications, this vector field will be defined on an open subset of U with x_0 in its closure.

Suppose that g is a partially defined vector field on M, x_0 is in closure of dom$g$ and $\lim_{x \to x_0} g(x) = 0$. We say then that x_0 is an *asymptotic equilibrium point* for g. Of course, a standard equilibrium point is also an asymptotic one. The vector field g is *asymptotically stable* at such a point if for every open neighbourhood U of x_0 there is an open neighbourhood V of x_0 such that for every $x \in V \cap \mathrm{dom}g$ the trajectory of g starting from x is in $U \cap \mathrm{dom}g$ after some time and tends to x_0 when time goes to infinity.

We say that the system Σ is *locally feedback stabilizable* at a point x_0 if there is a local feedback v such that the point x_0 is in the closure of the domain of the resulting vector field g, x_0 is an asymptotic equilibrium point for g, and g is asymptotically stable at x_0. The system Σ is *globally asymptotically stabilizable* at x_0 if there is a global feedback v such that x_0 is a globally asymptotically stable equilibrium of the closed-loop system.

Remark 1. Assume that $f(x_0, u_0) = 0$ and consider only feedbacks v such that $v(x_0) = u_0$. Then the point x_0 belongs to the (open) domain of the closed-loop vector field g and is an equilibrium point for g. We can now study standard asymptotic stability of g at x_0 and standard stabilizability of f. It is clear that standard stabilizability implies the local one introduced above.

Now we are going to study local stabilizability in the linear case. Assume that Σ is defined by the linear mapping f

$$f(x,u) = Ax + bu \tag{10}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and

$$\mathrm{dom} f = \{(x,u) : c^T x + du < e\} \tag{11}$$

We are interested in a linear feedback

$$v(x) = k^T x \tag{12}$$

Observe that k, c and b are vectors in $\mathbb{R}^n$, whereas d and e are scalars. The point $x = 0$ is an asymptotic equilibrium if $e \geq 0$.

Global stabilizability is easy to deduce.

Proposition 3. *The linear system Σ is globally stabilizable iff $e > 0$, and the matrix $A - bc^T/d$ is stable.*

Proof. Σ is globally stabilizable iff there is k such that the matrix $A + bk^T$ is asymptotically stable and the inequality $c^T x + dk^T x < e$ holds for every $x \in \mathbb{R}^n$. This is possible if and only if $e > 0$, $c^T + dk^T = 0$ and $A + bk^T$ is asymptotically stable. ■

In a regular case, local stabilizability is equivalent to the standard stabilizability.

Proposition 4. *Assume that $e > 0$ in (11). Then Σ is locally stabilizable at* 0 *with feedback (12) iff the pair (A, b) is stabilizable.*

Proof. Observe that the closed-loop system is defined in an open half-space containing 0. Local asymptotic stability of this system is then equivalent to global asymptotic stability of the closed-loop system without restrictions. ■

The singular case, when $e = 0$ in (11), is much more interesting. So far, we have a complete answer for $n = 2$.

Theorem 4. *Assume that $n = 2$ and the pair (A, b) is controllable. The system $\dot{x} = Ax + bu$ with the constraints $c^T x + u < 0$ is locally stabilizable at* 0 *iff the characteristic polynomial of the matrix $A - bc^T$ has only real zeros.*

Proof. If the system had no constraints, it would be stabilizable because of the controllability. Hence, the only goal to achieve is to make sure that the trajectories stay in the half-space $(c^T + k^T)x < 0$. This holds iff the subspace $(c^T + k^T)x = 0$ is invariant under the linear map defined by the matrix $A + bk^T$. This, in turn, is equivalent to the fact that

$$(c+k)^T(A+bk^T) = \lambda(c+k)^T \tag{13}$$

for some real λ. Let us first assume that A and b have the form

$$A = \begin{pmatrix} 0 & 1 \\ \alpha_1 & \alpha_2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We can achieve it via a coordinate change since the system is controllable. From (13), eliminating λ, we get

$$(c_2 + k_2)^2(\alpha_1 + k_1) = (c_1 + k_1)^2 + (c_1 + k_1)(c_2 + k_2)(\alpha_2 + k_2)$$

This is equivalent to a quadratic equation in k_2

$$k_2^2(\alpha_1 - c_1) + k_2(2c_2\alpha_1 + c_2k_1 - c_1c_2 - c_1\alpha_2 - k_1\alpha_2)$$
$$+(c_2^2\alpha_1 + c_2^2k_1 - c_1^2 - 2c_1k_1 - k_1^2 - c_1c_2\alpha_2 - k_1c_2\alpha_2) = 0$$

The discriminant of the equation takes the form

$$\Delta = (k_1 + c_1)^2[(\alpha_2 - c_2)^2 + 4(\alpha_1 - c_1)]$$

The quadratic equation has a real solution iff $\Delta \geq 0$, i.e. iff the polynomial $\chi_{A-bc^T}(\lambda) = \lambda^2 - \lambda(\alpha_2 - c_2) - (\alpha_1 - c_1)$ has real zeros.

Hence, the last condition is necessary for existence of the required k. However, it is also sufficient, since choosing sufficiently negative k_1 we get also sufficiently negative k_2 in order to achieve stability of the matrix $A + bk^T$. Finally, observe that χ_{A-bc^T} does not depend on a particular coordinate system in $\mathbb{R}^n$. This means that our condition is necessary and sufficient also for the general form of A and b. ∎

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Received: November 6, 1994