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BOUNDED INPUT BOUNDED OUTPUT STABILIZATION OF NONLINEAR SYSTEMS USING STATE DETECTORS*

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This paper is devoted to the study of relations between Lyapunov stabilization and bounded input bounded output (BIBO) stabilization for those nonlinear systems whose state is not available for measurement. It is found that when a state detector is used to construct asymptotic state, the local Lyapunov stabilization implies the local BIBO stabilization, and in the global case, this conclusions remain if the initial error between the output of state detector and state of compensated system is small enough.

1. Introduction

In the past several decades, the problem of feedback stabilization for nonlinear systems was an active research area, and it is now taken more seriously (Aeyels, 1985; Brockett, 1983; Kalouptsidis and Tsinias, 1984; Marino, 1988; Pan *et al.*, 1990; Sussmann, 1979). Nowadays, many researchers are engaged in studying the problem of stabilization for nonlinear systems around their equilibrium points (Aeyels, 1985; Brockett, 1983; Marino, 1988; Pan *et al.*, 1990; Sussmann, 1979), i.e., working in the sense of Lyapunov stability, which is an important subject in this area. However, for many practical applications, such as robot manipulator control and output regulation of chemical processes, it is far insufficient to merely study stability of the closed systems without external control. In such cases, a more useful concept for stability should be bounded input bounded state (or bounded output) stability. This subject was investigated early by Varaiya and Liu, (1966). Recently, Sontag (1989) furthered this research by illustrating with an example that the system which is stable in Lyapunov sense may not be BIBO stable and proving that systems which are Lyapunov stabilizable are then BIBO stabilizable by a suitable feedback (Sontag, 1989).

For those systems whose states are not directly measurable, state feedback control is always realized through an asymptotical state observer. It is well known that system's stabilization will not be changed for linear systems when the asymptotical state estimated from an observer substitutes for the system's state in feedback.

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But this property does not hold any more when nonlinear systems are concerned. We shall show this fact by the following example.

Consider nonlinear system

$$\dot{x} = -H(x) + 2x + v \tag{1}$$

where

$$H(x) = \begin{cases} 1/x & |x| \geq 1/2 \\ a \sin bx & |x| < 1/2 \end{cases} \tag{2}$$

In the Appendix of this paper, we show that there exist positive numbers a and b with $b < 3\pi/2$ such that $H(x)$ belongs to C^1–class.

Taking feedback control law in the form

$$u = -2x + v,$$

then the closed system becomes

$$\dot{x} = -H(x) + v \tag{3}$$

Using Lyapunov function $V(x) = x^2/2$, it is easily verified that the origin is a globally stable equilibrium point for system (3) i.e., for any initial condition $x(0) = x_0$, the trajectory $x(t, x_0)$ possesses the property $\lim_{t\to\infty} x(t, x_0) = 0$. If we apply feedback control law

$$u = -2(x + e) + v$$

where $x + e$ is the asymptotic state got from a state observer and the error e satisfies differential equation $\dot{e} = -8/9e^3$. Then the closed system is

$$\begin{aligned} \dot{x} &= -H(x) - 2e + v \\ \dot{e} &= -8/9e^3 \end{aligned} \tag{4}$$

origin is still an equilibrium point of system (4), but, this system is not globally stable. For example, let $x(0) = 1$, $e(0) = -3/4$, then,

$$x(t) = \sqrt{t+1}$$

$$e(t) = -3/\left(4\sqrt{t+1}\right),$$

and $x(t)$ diverges when t grows to infinite.

The above example shows that the stabilization for nonlinear systems may vary when asymptotical state observers are applied. Therefore, it is necessary to find the conditions under which the systems's stabilization will be invariant no matter whether an asymptotical state is adopted or not. We shall discuss the problem of BIBO stability by extending the results obtained by Sontag, (1989) to the cases where the only available signal for feedback is asymptotical state estimated from a state detector (we use the word *detector* proposed by Vidyasagar, (1980) instead of *observer*).

This paper is organized as follows: In section 2, we give some preliminary materials which include definitions and lemmas. In section 3, we present the main theorems concerning BIBO stability when different kinds of detector are applied. Section 4 contains conclusions.

2. Preliminaries

Consider the following affine nonlinear system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i = f(x) + G(x)u \tag{5a}$$

$$y = h(x) \tag{5b}$$

where $u \in R^m$, $x \in R^n$ and $y \in R^r$ are input, state and output of system (5), respectively. $G(x) = [g_1(x), ..., g_m(x)]$. Let M denote the set $\{1, 2, ..., m\}$. We assume f and g_i, $i \in M$ and h are smooth functions. Without loss of generality, we suppose $f(0) = 0$. Therefore, origin is an equilibrium point of (5a) if $u = 0$. Let $x(t) = x(t, x_0, u)$ be the solution of (5a) with a specified input u and initial condition $x(0) = x_0$.

Following (Hahn, 1967), we denote R^+ for the set of nonnegative real numbers and present the definitions of class $\mathcal{K}$, $\mathcal{K}^\infty$ and $\mathcal{L}$. A function $\alpha : R^+ \to R^+$ is said to belong to class $\mathcal{K}$ if it is continuous, strictly increasing and satisfies $\alpha(0) = 0$. It is of class $\mathcal{K}^\infty$, in addition, $\lim_{s\to\infty} \alpha(s) = \infty$. A function $\nu : R^+ \to R^+$ is said to be of class $\mathcal{L}$ if it is continuous, strictly decreasing and $\lim_{s\to\infty} \nu(s) = 0$. A function $\beta(s,t) : R^+ \times R^+ \to R^+$ is said to be of class $\mathcal{KL}$ if for every fixed t, the function is of class $\mathcal{K}$ and for every fixed s it is of class $\mathcal{L}$. In a similar way, we can define the class $\mathcal{KKL}$ for the functions $\gamma(s_1, s_2, t)$. The details are omitted.

For every integer p, R^p is considered as a normed linear space with Euclid norm, i.e., for any $x \in R^p$, $x^T = (x_1 ... x_p)$, $|x| = \sqrt{\sum_{i=1}^{p} x_i^2}$, where $'T'$ denotes the transposition. For any real number $\delta > 0$, let $B_\delta = \{x, x \in R^p, |x| \leq \delta\}$.

In this paper, we adopt the following definition for stability proposed by Sontag, (1989).

Definition 1. If there exist $\delta > 0$ $(\delta = \infty)$ and $\mathcal{KL}$ function $\beta(s,t)$ such that, for any $x_0 \in B_\delta$ $(x \in R^n)$,

$$|x(t)| = |x(t, x_0, 0)| \leq \beta(|x_0|, t), \tag{6}$$

then system (5) is said to be Locally (Globally) Asymtotically Stable (LAS(GAS)).

If there exist $\delta > 0$ $(\delta = \infty)$, and two functions $\beta(s,t) \in \mathcal{KL}$ and $\nu(s) \in \mathcal{K}$, such that for any $x_0 \in B_\delta$ $(x \in R^n)$,

$$|x(t)| = |x(t, x_0, u)| \le \beta(|x_0|, t) + \nu(|u|) \tag{7}$$

then system (5) is said to be Locally (Globally) Input–State Stable (LISS(GISS)).

Definition 1 is a stronger concept for stability than the usual one. It is clear from the above definition that ISS implies AS. The inverse conclusion we give below was verified by Sontag, (1989). In order to describe it, we give the definition of stabilization.

Definition 2. If there exists a smooth mapping $k : R^n \to R^m$, satisfying $k(0) = 0$, such that for the closed system

$$\dot{x} = f(x) + G(x)k(x) = \overline{f}(x) \tag{8}$$

equilibrium point $x = 0$ is LAS (GAS), then the system (5) is called LAS–able (GAS–able).

Furthermore, if

$$\dot{x} = \overline{f}(x) + G(x)u \tag{9}$$

is LISS (GISS), then the system (5) is called LISS–able (GISS–able).

Lemma 1. (Sontag, 1989). *The necessary and sufficient condition for existing a state feedback to make (5) GISS–able is that (5) is GAS–able.*

In this paper, we are going to discuss the conditions under which lemma 1 holds when a state detector is used. For the sake of convenience, we hereby present the definition of state detectors, (more details about detectors can be found in (Vidyasagar, 1980)).

Consider the following smooth system

$$\dot{z} = g(z, y, u) = g(z, h(x), u) \tag{10}$$

where $z \in R^n$, g vanishes when all of its arguments do. Combining (10) with (5), we get a composite system

$$\dot{x} = f(x) + G(x)k(x)u \tag{11a}$$

$$\dot{z} = g(z, y, u) \tag{11b}$$

$$y = h(x) \tag{11c}$$

Definition 3. If there exists a function $V_2 : R^n \times R^n \to R^+$, with $V_2(x,x) = 0$, $\psi_i \in \mathcal{K}$, $(i \in \mathbf{3})$, $\delta > 0$ and $\delta_u > 0$, such that for any $x, z \in B_\delta$ and $|u| \le \delta_u$,

$$\psi_1(|x-z|) \le V_2(x,z) \le \psi_2(|x-z|) \tag{12a}$$

$$\nabla_x V_2(x,z)[f(x)+G(x)u] + \nabla_z V_2(x,z)g(z,h(x),u) \le -\psi_3(|x-z|) \tag{12b}$$

where ∇_s means the gradient with respect to the variable s, then system (5) is called to be weakly detectable and (11b) is its weak detector.

It is easily seen from (10) that if $|u| \le \delta_u$, and both $x(t)$ and $z(t)$ do not leave B_δ, then $|z(t)-x(t)| \to 0$ $(t \to \infty)$. If $\psi_i \in \mathcal{K}^\infty$ and $\delta = \infty$ in Definition 2, then system (5) is said to be detectable and (11b) is a detector.

3. Main Results

Throughout this section, we suppose system (5) is LAS–able or GAS–able, i.e., there exists a feedback control law

$$u = k_1(x) + v \tag{13}$$

where $k_1 : R^n \to R^m$ is a smooth mapping with $k_1(0) = 0$, such that origin is a locally (globally) asymptotically stable equilibrium point of the following closed system

$$\dot{x} = \overline{f}(x) + G(x)v \tag{14a}$$

$$y = h(x) \tag{14b}$$

where $\overline{f}(x)$ is defined as (8). By inverse Lyapunov theorems (Hahn, 1967), there exist $\delta > 0$, Lyapunov function $V_1 : R^n \to R^+$ and $\mathcal{K}$–class ($\mathcal{K}^\infty$–class) functions φ_i, $i \in \mathbf{3}$, such that for any $x \in B_\delta$

$$\varphi_1(|x|) \le V_1(x) \le \varphi_2(|x|) \tag{15a}$$

$$\nabla_x V_1(x) f(x) \le -\varphi_3(|x|) \tag{15b}$$

If (14) is GAS, then $\delta = \infty$.

Although system (14) is LAS (GAS), it may not be LISS (GISS) (Sontag 1989), Therefore, it is necessary to give another feedback for achieving ISS. We propose the following feedback control law

$$u = k_1(x) + k_2(x) + v \tag{16}$$

where $k_1(x)$ is that in (13) which makes (14) stable and

$$k_2(x) = -\frac{1}{2m} a(x) b(x)$$

where $a(x)$ and $b(x)$ are defined as follows:

$$a(x) = -\nabla_x V_1(x) f(x), \quad b^T(x) = [b_1(x) \ldots b_m(x)] \text{ and } b_i(x) = \nabla_x V_1(x) g_i(x).$$

By (15b), $a(x) \geq \varphi_3(|x|) \geq 0$. With a detector, the closed system becomes (cf. Figure 1)

$$\dot{x} = f(x) + G(x)k_1(z) + G(x)k_2(z) + G(x)v \tag{17a}$$

$$\dot{z} = g(z, y)k_1(z) + k_2(z) + v) \tag{17b}$$

$$y = h(x) \tag{17c}$$

To prevent confusion, we denote $v^T = [u_1, ..., u_m]$.

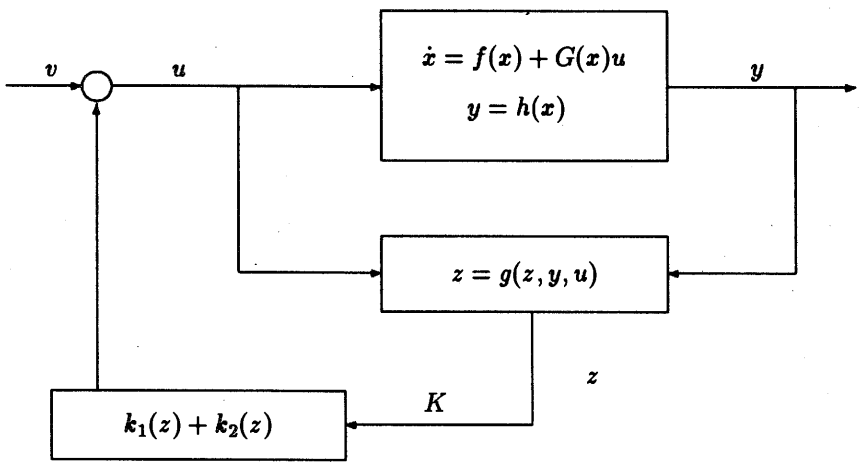

Fig. 1. The feedback system using a state detector.

We still use V_1 as a Lyapunov function for partial state x of system (17). The derivative of $V_1(x)$ along trajectory $x(t)$ may be calculated as follows

$$\frac{d}{dt}V_1(x(t)) = \nabla_x V_1(x)\,[f(x) + G(x)k_1(z) + G(x)k_2(z) + G(x)v] =$$

$$-a(x) + \sum_{i=1}^{m}\left[b_i(x)u_i - \frac{a(x)}{2m}b_i(x)b_i(z)\right] + b^T(x)\,[k_1(z) - k_1(x)] =$$

$$-\frac{a(x)}{2} - \frac{a(x)}{2m}\sum_{i=1}^{m}\left[\left(b_i(x) - \frac{mu_i}{a(x)}\right)^2 + \left(1 - \left(\frac{mu_i}{a(x)}\right)^2\right)\right] +$$

$$+\frac{1}{2m}b^T(x)\,[a(x)b(x) - a(z)b(z)] + b^T(x)\,[k_1(z) - k_1(x)] \tag{18}$$

Let $[x(t)\; z(t)]^T$ denote the trajectory of the closed system (11) with initial conditions $x(0) = x_0$, $z(0) = z_0$ and input v.

Lemma 2. *Suppose there exist two functions V_1 and V_2 satisfying*

$$V_1 : R^n \to R^+ \quad \text{with} \quad V_1(0) = 0,$$

$$V_2 : R^n \times R^+ \to R^+ \quad \text{with} \quad V_2(x, x) = 0,$$

such that in some neighbourhood B of $\begin{bmatrix} x_0 \\ z_0 \end{bmatrix}$, the following inequalities are valid

$$\varphi_1(|x|) \leq V_1(x) \leq \varphi_2(|x|) \tag{19a}$$

$$\psi_1(|x - z|) \leq V_2(x, z) \leq \psi_2(|x - z|) \tag{19b}$$

$$\frac{d}{dt} V_1(x) \leq -\varphi_3(|x|) + L|x - z| \tag{19c}$$

$$\frac{d}{dt} V_2(x, z) \leq -\psi_3(|x - z|) \tag{19d}$$

for any $(x\ z)^T \in B$, where φ_i and ψ_i, $i \in \mathbf{3}$, are $\mathcal{K}$−functions and L is a positive constant, then, there exist three $\mathcal{KL}$−functions β_1, β_2 and β_3 such that

$$|x(t) - z(t)| \leq \beta_1(|x_0 - z_0|, t) \tag{20a}$$

$$|x(t)| \leq \beta_2(|(x_0\ z_0)^T|, t) \tag{20b}$$

$$|z(t)| \leq \beta_3(|(x_0\ z_0)^T|, t) \tag{20c}$$

so long as $[x(t)\ z(t)]^T \in B$.

Proof: Because (20a) and (20b) imply (20c), it is sufficient to verify the above two inequalities. From (19b), we get

$$|x - z| \geq \psi_2^{-1}(V_2(x, z)). \tag{21}$$

Substituting (21) for the right side of (19d), we have

$$\frac{d}{dt} V_2(x, z) \leq -\psi_3 \psi_2^{-1}(V_2(x, z)) \tag{22}$$

Using Lemma 6.1 of (Sontag, 1989), (22) together with the initial condition $V(x_0, z_0) \geq 0$, we find that there exists a $\mathcal{KL}$−function ν such that for every $t \geq 0$, $[x(t)\ z(t)]^T \in B$,

$$\begin{aligned} V_2(x(t), z(t)) &\leq \nu(V_2(x_0, z_0), t) \\ &\leq \nu(\psi_2(|x_0 - z_0|), t) \end{aligned}$$

then, (19b) leads to the following inequality

$$|x(t) - z(t)| \leq \psi_1^{-1} \nu\left(\psi_2(|x_0 - z_0|, t)\right).$$

Let $\beta_1(s,t) = \psi_1^{-1}\nu(\psi_2(s),t)$, it is easily verified that $\beta_1 \in \mathcal{KL}$. Thus, (20a) is proved.

Furthermore, (19a) and (19b) together with (20a) result in

$$\frac{d}{dt}V_1(x) \le -\psi_3\psi_1^{-1}(V_1(x)) + L\beta_1(|x_0 - z_0|,t). \tag{23}$$

Consider now the following first–order differential equation

$$\frac{d}{dt}V = \psi_3\psi_1^{-1}(V) + L\beta_1(|x_0 - z_0|,t) \text{ and } V(0) = V_1(x_0) \tag{24}$$

where z_0 is considered a constant. Using Lasalle–Levin invariant theorem (Michel *et al.,* 1978), we easily get that for any $t > 0$ the solution of (24), denoted by $V(x_0, x_0 - z_0, t)$, exists and $V(x_0, x_0 - z_0, t) \to 0$ $(t \to \infty)$.

By the comparison principle (Hahn, 1967; Michel *et al.*, 1978) the above discussion leads to the following inequality

$$V(x_0, x_0 - z_0, t) \ge V_1(x(t)) \ge \varphi_1(x(t)),$$

$$|x(t)| \le \varphi_1^{-1}V(x_0, x_0 - z_0, t).$$

Because $\varphi_1^{-1}V(x_0, x_0 - z_0, t) \to 0$ $(t \to \infty)$, the method presented by Sontag, (1989) yields that $\varphi_1^{-1}V$ is $\mathcal{KKL}$–bounded, i.e., there exists a $\mathcal{KKL}$–function ν_1 satisfying.

$$\varphi_1^{-1}V(s_1, s_2, t) \le \nu_1(s_1, s_2, t) \tag{25}$$

hence,

$$\begin{aligned} |x(t)| &\le \nu_1(|x_0|, |x_0 - z_0|, t) \\ &\le \nu_1(|x_0| + |z_0|, |x_0| + |z_0|, t) \\ &\le \nu_1(\sqrt{2}|(x_0, z_0)^T|, \sqrt{2}|(x_0, z_0)^T|, t) \end{aligned}$$

The last inequality comes from the fact that $|x_0| + |z_0| \le \sqrt{2}|(x_0, z_0)^T|$. Let $\beta_2(s,t) = \nu_1(\sqrt{2}s, \sqrt{2}s, t)$, it is clear that $\beta_2(s,t) \in \mathcal{KL}$, (20b) is then verified.

Denote $\mu(s) = \varphi_3^{-1}(ms)$, then $\mu(s) \in \mathcal{K}$ or $\mu(s) \in \mathcal{K}^\infty$ according to the classification of φ_3. For the sake of convenience, we suppose that (12) and (15) hold in the same closed sphere B_c.

We are now ready to deal with the problem of BIBO stability. At first, we give a theorem to describe the boundedness of trajectories.

Theorem 1. *If $\mu(|v|) \le C$, then for any ε, with $\mu(|v|) \le \varepsilon \le C$, there exist $\delta_0 > 0$ and $\delta_v > 0$ such that for all $x_0, z_0 \in B\delta_0$ and $|v| \le \delta_v$, the following estimations are valid.*

$$|x(t)| = |x(t, x_0, z_0, v)| \le \varepsilon$$

$$|z(t)| = |z(t, x_0, z_0, v)| \le \varepsilon$$

where $x(t, x_0, z_0, v)$ and $z(t, x_0, z_0, v)$ are the trajectories of (17) with initial conditions $x(0) = x_0$, $z(0) = z_0$ and input v.

Proof: It is sufficient to consider only the case of $|x(t)| \ge \mu(|v|)$, it then implies that for every $i \in \boldsymbol{m}$,

$$\varphi_3(|x(t)|) \ge m|v| \ge mu_i. \tag{26}$$

Within the closed sphere B_c, $b(x)$ is bounded, $a(x)b(x)$ and $k_1(x)$ are Lipschitz bounded, i.e., there exist M_b, L_1 and L_2 such that

$$|b(x)| \le M_b,$$

$$|a(x)b(x) - a(z)b(z)| \le L_1|x - z|,$$

$$|k_1(x) - k_1(z)| \le L_2|x - z|.$$

Let $L = M_b(L_1/2m+L_2)$. Provided with (26), (18) leads to the following inequality

$$\frac{d}{dt}V_1(x(t)) \le -\frac{1}{2}a(x(t)) + L|x(t) - z(t)|.$$

Since $k_1(0) = 0$ and $a(0) = 0$, the continuity of $k_1(z)$ and $a(z)$ implies that there exists a positive δ_1 with $\delta_1 < \varepsilon$ such that for any $z \in B\delta_1$ the following is valid

$$|k_1(z) + k_2(z)| = |k_1(z) - \frac{1}{2m}a(z)b(z)| \le \frac{2}{3}C.$$

Furthermore, one can select ordinally positive numbers δ_2, δ_3 and δ_4 to satisfy the following inequalities, respectively. The existence of δ_i, $(i = 2,3,4)$ is a guaranty of the property of $\mathcal{K}-$class functions.

$$\varphi_2(\delta_2) < \varphi_1(\delta_1/2), \quad \delta_3 < \min\left(\frac{1}{2L}\varphi_3(\delta_2), \delta_2\right), \quad \psi_2(\delta_4) < \psi_1(\delta_3).$$

At last, let $\delta_0 = \delta_4/2$ and $\delta_v = \min(\mu^{-1}(C), C/3)$. Now it only remains to verify these δ_0 and δ_v. Because x_0, $z_0 \in B\delta_0$, there exists a positive T such that for any $t \in [0, T]$,

$$|k_1(z(t)) + k_2(z(t)) + v(t)| \le C.$$

From (12b), for any $t \in [0, T]$

$$\frac{d}{dt}V_2(x(t))\ z(t)) \le 0 \tag{27}$$

Due to the continuity of $\frac{d}{dt}V_1(x(t))$, it is possible that for this T the following inequality is also valid (if necessary, one may alter T)

$$\frac{d}{dt}V_1(x(t)) \leq 0 \tag{28}$$

Thus, (27) leads to

$$\begin{aligned}\psi_1(|x(t) - z(t)|) &\leq V_2(x(t)z(t)) \leq V_2(x_0, z_0) \\ &\leq \psi_2(|x_0 - z_0|) \leq \psi_2(\delta_4) < \psi_1(\delta_3),\end{aligned} \tag{29}$$

hence

$$|x(t) - z(t)| < \delta_3 \quad \text{for every} \quad t \in [0, T]$$

By a similar consideration, it can be verified from (28) that

$$|x(t)| < \delta_1/2 \qquad \text{for every} \quad t \in [0, T]$$

We shall fulfill the proof of Theorem 1 by showing $T = \infty$. This fact will be verified by reduction to a contradiction. If $T < \infty$, then there exists some t_1 with $\infty > t_1 > T$, such that one of the following two cases has to take place:

i) $|x(t_1)| = \delta_1/2$, but $|x(t) - z(t)| < \delta_3$ for every $t \in [0, t_1]$; or

ii) $|x(t)| \leq \delta_1/2$, for every $t \in [0, t_1]$, but $|x(t_1) - z(t_1)| = \delta_3$

First, we prove that it is impossible to happen to case ii). As a matter of fact, if $|x(t_1) - z(t_1)| = \delta_3$, then there exists a positive $t_0 \in [0, t_1]$ such that $|x(t_0) - z(t_0)| = \delta_4$ and $\delta_3 \geq |x(t) - z(t)| \geq \delta_4$ for all $t \in [t_0, t_1]$. Because for $t \in [t_0, t_1]$, $|x(t)| \leq \delta_1/2 \leq C$, $|z(t)| \leq \delta_3 + \delta_1/2 \leq C$. Using (12b) and repeating the calculation of (29), we get

$$|x(t) - z(t)| < \delta_3 \qquad \text{for every} \quad t \in [t_0, t_1]$$

Especially, $|x(t_1) - z(t_1)| < \delta_3$. A contradiction now comes.

If case i) takes place, i.e., $|x(t_1)| = \delta_1/2$, then there exists $t_0 \in [0, t_1)$ such that $|x(t_0)| = \delta_2$ and $t \in (t_0, t_1]$ $\delta_1/2 \geq (x(t)) > \delta_2$. Using (20), we get

$$\frac{d}{dt}V_1(t) \leq -\frac{a(x)}{2} + L\delta_3 \leq -\frac{1}{2}\varphi_3(x(t)) + L\delta_3 \leq -\frac{1}{2}\varphi_3(\delta_2) + L\delta_3 < 0,$$

hence, for every $x \in [t_0, t_1]$

$$\begin{aligned}\varphi_1(|x(t)|) &\leq V_1(x(t)) \leq V_1(x(t_0)) \\ &\leq \varphi_2(|x(t_0)|) = \varphi_2(\delta_2) < \varphi_1(\delta_1/2)\end{aligned}$$

$$|x(t)| < \delta_1/2$$

Especially, $|x(t_1)| < \delta_1/2$, it also contradicts the hypothesis. Thus, $T = \infty$, the conclusion is implied.

A more specified conclusion is the following theorem which reveals the property of LISS for closed system (17).

Theorem 2. *If system (5) is LAS–able and weakly detectable, then, there exists a positive number* δ_v *such that the closed system (17) is LISS with regarding to* x *whenever* $|v| \leq \delta_v$.

Proof: Let $\varepsilon = C$, from Theorem 1, δ_0 and δ_v can be found such that when x_0, $z_0 \in B\delta_0$ and $|v| \leq \delta_v$,

$$|x(t)| = |x(t, x_0, z_0, v)| \leq C$$

$$|z(t)| = |z(t, x_0, z_0, v)| \leq C$$

where $x(t, x_0, z_0, v)$ and $z(t, x_0, z_0, v)$ are the solutions of closed system (17). Recall the proof of Lemma 2, when $\mu(|v|) \leq |x(t)| \leq C$, $\mu(|v|) \leq |z(t)| \leq C$ and $|v| \leq \delta_v$, there exists a function $\nu_1(s_1, s_2, t)$ belonging to $\mathcal{KKL}$–class such that

$$|x(t)| \leq \nu_1(|x_0|, |x_0 - z_0|, t) \tag{30}$$

Choose $\overline{\delta}_v$ to satisfy the following inequality

$$\nu_1(\mu(\overline{\delta}_v), 2C, 0) \leq C \tag{31}$$

Denote $\nu^*(s) = \nu_1(\mu(s), 2C, 0)$. Now (30) implies that

$$\mu(s) \leq \nu^*(s) \qquad \text{for all} \quad s \tag{32}$$

Let $\delta_v^* = \min(\overline{\delta}_v, \delta_v)$, we now discuss the following two cases provided with $|v| \leq \delta_v^*$ and $|x_0| \in B\delta_0$.

First, let $|x_0| \leq \mu(|v|)$, we show that $|x(t)| \leq \nu^*(|v|)$ for all t. If $|x(t)| \leq \mu(|v|)$ for all t, then $|x(t)| \leq \nu^*(|v|)$ by (32). If for some t_1, $|x(t_1)| > \mu(|v|)$, then there exists $t_0 < t_1$ such that $|x(t_0)| = \mu(|v|)$. From (30) and (31), we have

$$|x(t)| \leq \nu_1(|x_0|, |x_0 - z_0|, t - t_0) \leq \nu_1(\mu(|v|), 2C, 0) = \nu^*(|v|)$$

Second, if $\delta_0 \geq |x_0| \geq \delta(|v|)$, Theorem 1 assures that $|x(t)| \leq C$, then using Lemma 2, there exists a function $\beta_1(s, t) \in \mathcal{KL}$ such that

$$|x(t)| \leq \beta_1(|(x_0\ z_0)^T|, t)$$

Because $\beta_1((x_0\ z_0)^T, t) \to 0$ $(t \to \infty)$, there exists some $t_1 > 0$ $|x(t_1)| \leq \mu(|v|)$. Furthermore, the discussion for the first case shows that for all $t \geq t_1$,

$$|x(t)| \leq \nu^*(|v|).$$

Concluding the above two cases, we get

$$|x(t)| \leq \beta_1(|(x_0\ z_0)|^T, t) + \nu^*(|v|)$$

for all $x_0, z_0 \in B\delta_0$ and $|v| \leq \delta_v^*$.

In Theorem 2, there is a constraint on the magnitude of system input, this is the consequence of the fact that the weak detector can only track the original system state in a small area around equilibrium point. This constraint possess some limit on the use and causes inconvenience to some practical applications. In the following theorem, this constraint, together with constraint on x_0 are removed.

Theorem 3. *If system (5) is GAS–able and detectable, then for any* $x_0 \in R^n$, *there exists* $\delta = \delta(x_0\ v) > 0$, *for all* $z_0 \in B(x_0, \delta)$, *where* $B(x_0, \delta) = \{x;\ x \in R^n,\ |z_0 - x_0| \leq \delta\}$, *the closed system (17) is ISS–able.*

Proof: We still adopt feedback shown in (16) and the $\mathcal{K}$–function $\mu(s) = \varphi_3^{-1}(ms)$. If $|x(t)| \leq \mu(|v|)$, then the theorem is verified. Hence, we only need to discuss the case that there exist some t, such that $|x(t)| > \mu(|v|)$.

For any $x_0 \in R^n$, denote $l_1 = \max\{\mu(|v|), |x_0|\}$, and let l_2 satisfy inequality $\varphi_2(l_1) < \varphi_1(l_2/2)$. Within closed sphere B_{l_2}, $b(x)$ is bounded, $a(x)b(x)$ and $k_1(x)$ satisfy Lipschitz condition, let M, L_1, L_2 and L be the same as those defined in the proof of Theorem 2. When $\mu(|v|) \leq |x(t)| \leq l_2$, $\mu(|v|) \leq |z(t)| \leq l_2$, we can get the following inequality from (18)

$$\frac{d}{dt}V_1(x(t)) \leq -a(x)/2 + L|x(t) - z(t)|. \tag{33}$$

If there exists $\delta > 0$, such that, for all $z_0 \in B(x_0, \delta)$, the inequalities $|x(t)| = |x(t, x_0, z_0)| \leq l_2$ and $|z(t)| = |z(t, x_0, z_0)| \leq l_2$ hold for all $t \geq 0$, then we can complete the proof by only repeating the proof of Theorem 2.

We now prove the statement about boundedness of $x(t)$ and $z(t)$. Let $N = \min\{a(x) : \mu(|v|) \leq |x| \leq l_2\}$ then $N > 0$ as $a(x) \geq \varphi_3(|x|)$. Denote $p = \min(N/2L, l_2/2)$. Select δ_1, such that

$$\psi_2(\delta_1) < \psi_1(p). \tag{34}$$

At last, let $\delta = \min(\delta_1, l_1)$. From (11b), for all $t > 0$, $dV_2(x(t), z(t))/dt < 0$, therefore

$$\begin{aligned} \psi_1(|x(t) - z(t)|) &\leq V_2(x(t)z(t)) \leq V_2(x_0, z_0) \\ &\leq \psi_2(|x_0 - z_0|) \leq \psi_2(\delta_1) < \psi_1(p) \end{aligned}$$

$$|x(t) - z(t)| < p \tag{35}$$

Consider now the set $S = \{t; dV_1(x(t))/dt \leq 0,\ \mu(|v|) \leq |x(t)| \leq l_2/2\}$. Without loss of generality, we may assume that $\mu(|v|) \leq |x_0| \leq l_1$, then at $t = 0$, (33) holds. By the definition of p and (35), we know $dV_1(x(t))/dt|_{t=0} < 0$, hence S is a nonempty set. Let $[0,T] \subset S$, if $T \neq \infty$, we show that there exists $\varepsilon > 0$, such that $[T, T+\varepsilon) \subset S$. Because, when $t \in [0,T]$, $dv_1(x(t))/dt \leq 0$

$$\varphi_1(|x(t)|) \leq V_1(x(t)) \leq V_1(x_0) \leq \varphi_2(l_i) < \varphi_1(l_2/2)$$

$$|x(t)| < l_2/2$$

Particularly, $x(T)| < l_2/2$. By the continuity of the function $x(t)$, there exists $\varepsilon > 0$, such that for every $t \in [T, T+\varepsilon)$, $|x(t)| < l_2/2$. Thus (33) is still true over $[T, T+\varepsilon)$. By (35), for every $t \in [T, T+\varepsilon)$, $dv_1(x(t)/dt < 0$, thus $[T, T+\varepsilon) \subset S$. It implies that $T = \infty$, i.e., $t \geq 0$, $|x(t)| \leq l_2/2$. Moreover, we have

$$|z(t)| \leq |x(t)| + |z(t) - x(t)| \leq l_2/2 + p \leq l_2$$

Using Lemma 2, Theorem 3 allows a corollary which extends partially the result derived by Vidyasagar, (1980).

Corollary 1. *If system (5) is GAS-able and detectable, then composite system (17) is also GISS-able provided that* $z_0 \in B(x_0, \delta)$.

Theorem 3 and Corollary 1 illustrate that if z_0 approaches sufficiently to x_0, the system will remain ISS or GAS when asymptotical state estimation is used. For a large class of systems that initially work in a small area, the constraint on the initial point can be easily satisfied. For those systems whose state varies over a considerable area, we suggest to install a switch at the place indicated by "K" on Figure 1. At the start, the switch is put off. Since the norm of $x(t) - z(t)$ will degenerate as long as the system runs, after a few minutes, the trajectories $x(t)$ and $z(t)$ will enter an identical sphere B_c, when the switch is put on, the system may be ISS or GAS.

4. Conclusions

In this paper, we have shown that asymptotical stabilization of a nonlinear system implies its input to state stabilization when some constraints are imposed on the initial point of the detector. As for any continuous output map $y = h(x)$, there always exists $\phi \in \mathcal{K}$, such that $y = h(x) \leq \phi(|x|)$. Therefore, the conditions given in Theorem 2 and Theorem 3 also serve as the sufficient conditions for input–output stabilization.

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Appendix

In this appendix, we show that there exist positive numbers a and b such that the following function $H(x)$ belongs to C^1–class:

$$H(x) = \begin{cases} 1/x & |x| \geq 1/2 \\ a \sin bx & |x| < 1/2 \end{cases}$$

Let $f_1(x) = 1/x$ and $f_2(x) = a \sin bx$. It is sufficient to verify that there exist positive numbers a and b such that

$$f_1(1/2) = f_2(1/2) \quad \text{and} \quad \dot{f}_1(1/2) = \dot{f}_2(1/2) \tag{A.1}$$

Equations (A.1) are equivalent to the following

$$a \sin(b/2) = 2 \tag{A.2}$$

$$ab \cos(b/2) = -4 \tag{A.3}$$

Dividing (A.2) by (A.3), we get

$$\mathrm{tg}(b/2) = -b/2 \tag{A.4}$$

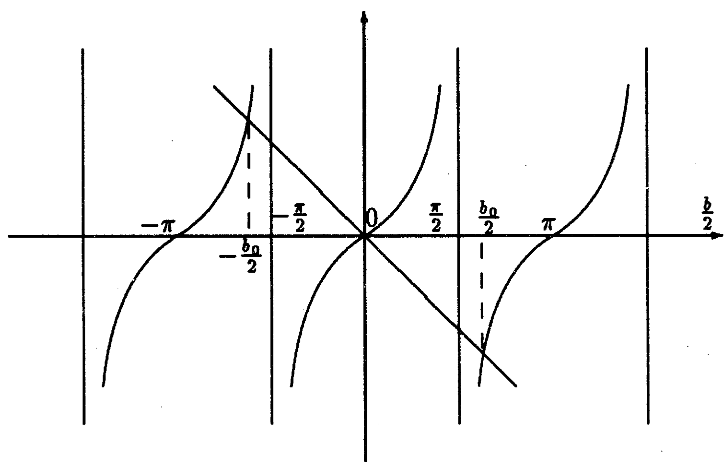

Fig. 2. Solutions of Equation (A.4).

It is clear, from Figure 2, that (A.4) is solvable, and there are infinite solutions, but on the travel $(\pi/2, 3\pi/4)$ the solution is unique, we denote this solution by $b_0/2$ (note: $\pi < b_0 < 3\pi/2$).

From (A.2), we get

$$a = \frac{2b_0}{\sqrt{4 + b_0^2}}$$

Thus

$$H(x) = \begin{cases} 1/x & |x| \geq 1/2 \\ ab_0/\sqrt{4 + b_0^2} \sin b_0 x & |x| < 1/2 \end{cases}$$

where b_0 is the unique solution of equation $x + \mathrm{tg}x = 0$ on the travel $(\pi, 3\pi/2)$.

The above discussion also suggests that one can construct smoother function $H(x)$ if he defines another function $f_2(x)$ with more free coefficients.