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A MULTIDIMENSIONAL POLYNOMIAL FACTORIZATION METHOD BASED ON THE MULTIDIMENSIONAL LAGRANGE POLYNOMIALS

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This paper provides a factorization technique of multidimensional polynomials which is based on the concept of multidimensional Lagrange polynomials. These polynomials are used for finding the coefficients of the factors of the multidimensional polynomial. The result is that the given polynomial is factorized in general factors involving several independent variables. The technique is fully supported by a set of theorems and an illustrative example.

1. Introduction

Much research effort has been devoted in the recent years to problems involving multidimensional (m-D) signals and multidimensional (m-D) systems. An m-D signal is a function of more than one variable while an m-D system is an algorithm or a transformation that transforms an m-D input signal to an m-D output signal. If for a linear combination of two inputs, the same linear combination of their corresponding outputs is obtained in the output of the system, the system is said to be linear. If for each shifted m-D input signal a similarly shifted m-D output signal is obtained, the system is said to be shift invariant. Linear and shift invariant m-D systems, indicated by the symbol LSI, have recently attracted increasing attention. The reason is that many practical systems and applications lead to m-D models. Among these applications, one can mention m-D digital filtering and image processing, biomedical and geophysical data processing, remote sensing, computer vision, underwater acoustics, moving-objects recognition, x-ray enhancement, digital memory modelling and distributed-parameter system analysis (Kaczorek, 1985; Tzafestas, 1986).

An m-D, LSI, SISO (single input, single output) discrete system, with input $u(n_1, ..., n_m)$, output $y(n_1, ..., n_m)$ and corresponding m-D z-transform $U(z_1, ..., z_m)$, $Y(z_1, ..., z_m)$ may be defined by its m-D transfer function ($n_1, ..., n_m$ are positive integers).

$$G(z_1, ..., z_m) = \frac{Y(z_1, ..., z_m)}{U(z_1, ..., z_m)} = \frac{\sum\limits_{i_1=0}^{N_1} \cdots \sum\limits_{i_m=0}^{N_m} Q_{i_1,...,i_m} z_1^{i_1} \dots z_m^{i_m}}{\sum\limits_{i_1=0}^{N_1} \cdots \sum\limits_{i_m=0}^{N_m} F_{i_1,...,i_m} z_1^{i_1} \dots z_m^{i_m}}$$

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where $N_1, \ldots, N_m$ are positive integers, $Q_{i_1,\ldots,i_m}$ and $F_{i_1,\ldots,i_m} \in \mathbb{R}$. The polynomial $\sum_{i_1=0}^{N_1} \cdots \sum_{i_m=0}^{N_m} F_{i_1,\ldots,i_m} z_1^{i_1} \ldots z_m^{i_m}$ is the characteristic polynomial of the system and henceforth it is noted by $f_c(z_1, \ldots, z_m)$.

If an m-D polynomial is written as a product of other lower degree polynomials, then it is said to be factorizable. Factorization of m-D polynomials is one of the primary processes in the field of m-D systems, since among others it helps performing simpler realizations, simpler stability tests, and simpler controllers. More specifically, if the numerator and denominator of the transfer function $G(z_1, \ldots, z_m) = g(z_1, \ldots, z_m)/f_c(z_1, \ldots, z_m)$ are factorized as:

$$g(z_1, \ldots, z_m) = g_1(z_1, \ldots, z_m) \ldots g_N(z_1, \ldots, z_m)$$

$$f_c(z_1, \ldots, z_m) = f_1(z_1, \ldots, z_m) \ldots f_N(z_1, \ldots, z_m)$$

where the g_i's and f_i's are obviously simpler than g and f_c, respectively, one has to realize the simpler m-D transfer functions:

$$G_1(z_1, \ldots, z_m) = \frac{g_1(z_1, \ldots, z_m)}{f_1(z_1, \ldots, z_m)}, \ldots, G_N(z_1, \ldots, z_m) = \frac{g_N(z_1, \ldots, z_m)}{f_N(z_1, \ldots, z_m)}$$

As the stability tests are in the form: check if $f_c(z_1, \ldots, z_m) = 0$ (in appropriate regions of $z_1, \ldots, z_m$), it is important to factorize $f(z_1, \ldots, z_m)$ in $f_1(z_1, \ldots, z_m), \ldots, f_N(z_1, \ldots, z_m)$, because in this case the stability test is separated into simpler ones.

The factorization results of m-D polynomials are also useful in the theory of distributed-parameter systems (DPS), which are described by partial differential equations, since the characteristic polynomials of DPS are actually m-D polynomials.

Factorizing an m-D polynomial is not a simple problem and available 1-D factorization techniques cannot be extended to the m-D case in a straightforward way. Up to now, the general factorization problem, i.e. the factorization of each factorizable polynomial, has not been solved yet. For this reason, some more or less special types of m-D polynomial factorization has been studied. In Theodorou and Tzafestas (1985), the factorization in factors of one variable i.e. $f(z_1, \ldots, z_m) = f_1(z_1) \ldots f_m(z_m)$, or in factors with no common variables i.e. $f(z_1, \ldots, z_m) = f_1(\overline{z}_1) f_k(\overline{z}_k)$ ($\overline{z}_1, \ldots, \overline{z}_k$ are mutually disjoint groups of independent variables) was solved. In Mastorakis *et al.* (1991), the factorization succeeded by considering the given polynomial as (1-D) polynomial with respect to z_j and applying the well-known formulas from elementary algebra, when the given polynomial is 2nd, 3rd, 4th degree in z_j. In Mastorakis and Theodorou (1992), the factorization of the State-Space Model was presented. In Mastorakis *et al.* (1992), the factorization of an m-D polynomial in linear factors i.e. $f(z_1, \ldots, z_m) = \prod_{i=1}^{N_1} (z_1 + a_{i,2} z_2 + \ldots + a_{i,m} z_m + c_i)$ was studied. In Mastorakis and Theodorou (1990), the factorization of an m-D polynomial in factors where at least one factor contains no more than $m - 1$ variables was studied.

Here, a factorization method for m-D polynomials is presented that employs multidimensional Lagrange polynomials. The method is simple since it is based on the computation of the roots of 1-D polynomials. These 1-D polynomials are obtained from the given m-D polynomial (to be factorized) by considering all variables, except one, as parameters.

Section 2 introduces the concept of multidimensional Lagrange polynomials by extending the well known 1-D Lagrange polynomials. Section 3 presents the root factorization method and Section 4 provides an illustrative example. Two subsidiary theorems are shown in the Appendix.

2. Lagrange Polynomials

Let $p_N(z_1)$ be a polynomial of the complex variable z_1. If the values $p_N(z_1) = g(z_1)$ are known at the points $z_{1_0}, ..., z_{1_N}$, then the polynomial $p_N(z_1)$ can be found by the formula

$$p_N(z_1) = \sum_{i=1}^{N} \ell_i(z_i) g(z_{1_i}) \tag{1}$$

where:

$$\ell_i(z_1) = \frac{(z_1 - z_{1_0})...(z_1 - z_{1_{i-1}})(z_1 - z_{1_{i+1}})...(z_1 - z_{1_N})}{(z_{1_i} - z_{1_0})...(z_{1_i} - z_{1_{i-1}})(z_{1_i} - z_{1_{i+1}})...(z_{1_i} - z_{1_N})} \tag{2}$$

are the associated Lagrange polynomials.

This polynomial is uniquely defined, since if there exists another polynomial $p'_N(z_1)$, then $p_N(z_1) - p'_N(z_1)$ has $N+1$ roots (at the points $z_{1_0}, ..., z_{1_N}$). But $p_N(z_1) - p'_N(z_1)$ has degree N and so $p_N(z_1) - p'_N(z_1) \equiv 0$ i.e. $p_N(z_1) \equiv p'_N(z_1)$.

Now, consider a polynomial $p_{N_1,...,N_m}(z_1, ..., z_m)$ in m-dimensions. If the values $p_{N_1,...,N_m}(z_1, ..., z_m) = g(z_1, ..., z_m)$ are known at the points $(z_{1_{i_1}}, ..., z_{m_{i_m}})$, where $0 \le i_1 \le N_1, ..., 0 \le i_m \le N_m$, then one obtains:

$$p_{N_1,...,N_m}(z_1, ..., z_m) = \sum_{i_1=0}^{N_1} \cdots \sum_{i_m=0}^{N_m} \ell_{i_1,...,i_m}(z_1, ..., z_m) g(z_{1_{i_1}}, ..., z_{m_{i_m}}) \tag{3}$$

where:

$$\ell_{i_1,...,i_m}(z_1, ..., z_m) = \frac{\prod\limits_{\substack{k=0 \\ k \ne i_1}}^{N_1} (z_1 - z_{1_k}) \cdots \prod\limits_{\substack{k=0 \\ k \ne i_m}}^{N_m} (z_m - z_{m_k})}{\prod\limits_{\substack{k=0 \\ k \ne i_1}}^{N_1} (z_{1_{i_1}} - z_{1_k}) \cdots \prod\limits_{\substack{k=0 \\ k \ne i_m}}^{N_m} (z_{m_{i_m}} - z_{m_k})} \tag{4}$$

This polynomial is again uniquely defined, since if there exists another polynomial $p'_{N_1,...,N_m}$, then $p_{N_1,...,N_m} \equiv p'_{N_1,...,N_m}$, since $p_{N_1,...,N_m} = p'_{N_1,...,N_m}$ at the points $(z_{1_{i_1}}, ..., z_{m_{i_m}})$, where $0 \le i_1 \le N_1, ..., 0 \le i_m \le N_m$. The proof is given in the Appendix.

3. The Root Factorization Method

The factorization method that follows refers to the special class of m-D polynomials $f(z_1, ..., z_m)$, described by (5)

$$f(z_1, ..., z_m) = \sum_{i_1=0}^{N_1} \cdots \sum_{i_m=0}^{N_m} \alpha(i_1, ..., i_m) z_1^{i_1} ... z_m^{i_m} \tag{5}$$

that have the following property: There exists at least one independent variable z_j such that the only existing monomial including the maximum power of z_j is $z_j^{N_j}$, that is for at least one j $(j = 1, ..., m)$:

$$\alpha(i_1, ..., i_{j-1}, N_j, i_{j+1}, ..., i_m) = 0$$

when $i_1 + ... + i_{j-1} + i_{j+1} + ... + i_m > 0$ (and of course $\alpha(0, ..., 0, N_j, 0, ..., 0) \neq 0$). Then, without loss of generality, it is assumed that

$$\alpha(0, ..., 0, N_j, 0, ..., 0) = 1$$

Although this is a special case, the class of m-D polynomials having the above property is still very general.

Considering $f(z_1, ..., z_m)$ as an 1-D polynomial $f(z_1, ..., z_{j-1}, z_{j+1}, ..., z_m)$ with respect to one independent variable z_j $(j = 1, ..., m)$, then the remaining independent variables $z_1, ..., z_{j-1}, z_{j+1}, ..., z_m$ are considered as parameters. In other words, one gives constant values to $z_1, ..., z_{j-1}, z_{j+1}, ..., z_m$ and the above polynomial is factorized as 1-D polynomial with respect to z_j. The roots $p_i(\widetilde{z})$, where $\widetilde{z} \triangleq [z_1, ..., z_{j-1}, z_{j+1}, ..., z_m]^T$, obey the relation

$$f(z_1, ..., z_m) = \prod_{i=1}^{N_j} \left(z_j - p_i(\widetilde{z}) \right) \tag{6}$$

Now, the validity of (6), in the case that $p_i(\widetilde{z})$ is a polynomial of $\widetilde{z}$, will be examined.

Suppose that:

$$p_i(\widetilde{z}) = \sum_{i_1=0}^{L_1} \cdots \sum_{i_{j-1}=0}^{L_{j-1}} \sum_{i_{j+1}=0}^{L_{j+1}} \cdots \sum_{i_m=0}^{L_m} \theta(i_1, ..., i_{j-1}, i_{j+1}, ..., i_m) \times z_1^{i_1} ... z_{j-1}^{i_{j-1}} z_{j+1}^{i_{j+1}} ... z_m^{i_m} \tag{7}$$

where:

$$L_1 \leq N_1, ..., L_{j-1} \leq N_{j-1}, L_{j+1} \leq N_{j+1}, ..., L_m \leq N_m \tag{8}$$

Without loss of generality, one can assume that

$$L_1 = N_1, ..., L_{j-1} = N_{j-1}, L_{j+1} = N_{j+1}, ..., L_m = N_m \tag{9}$$

considering that $\theta(i_1, ..., i_{j-1}, i_{j+1}, ..., i_m) = 0$ for the remaining terms. The reason is that the exact values of $L_1, ..., L_{j-1}, L_{j+1}, ..., L_m$ are unknown. So,

$$p_i(\tilde{z}) = \sum_{i_1=0}^{N_1} \cdots \sum_{i_{j-1}=0}^{N_{j-1}} \sum_{i_{j+1}=0}^{N_{j+1}} \cdots \sum_{i_m=0}^{N_m} \theta(i_1, ..., i_{j-1}, i_{j+1}, ..., i_m)$$

$$\times z_1^{i_1} ... z_{j-1}^{i_{j-1}} z_{j+1}^{i_{j+1}} ... z_m^{i_m} \tag{10}$$

Clearly,in order to find $p_i(\tilde{z})$, by the Lagrange interpolation formula, the values at the points $(z_{1_{i_1}}, ..., z_{j-1_{i_{j-1}}}, z_{j+1_{i_{j+1}}}, ..., z_{m_{i_m}})$, are needed, where $0 \le i_1 \le N_1, ..., 0 \le i_{j-1} \le N_{j-1}, 0 \le i_{j+1} \le N_{j+1}, ..., 0 \le i_m \le N_m$.

If one gives these different values to $\tilde{z}$, the roots of $f(z_1, ..., z_m)$ are found, considered as an 1-D polynomial with respect to z_j. These roots are also the values of the supposed polynomials $p_i(z)$. Therefore these polynomials can be found by the Lagrange interpolation formula. Since the correct correspondence between one root and one $p_i(\tilde{z})$ is unknown, all the combinations, between them, should be tried.

Suppose that a polynomial $p_i(\tilde{z})$ has been constructed by the Lagrange interpolation formula. Then,the m-D polynomial $z_j - p_i(\tilde{z})$ is tested as a factor of $f(z_1, ..., z_m)$ by applying the following theorem.

Theorem 1. *A factor $z_j - p_i(z_1, ..., z_{j-1}, z_{j+1}, ..., z_m)$ is an m-D polynomial factor of $f(z_1, ..., z_m)$ if and only if*

$$f\Big(z_1, ..., z_{j-1}, p_i(z_1, ..., z_{j-1}, z_{j+1}, ..., z_m), z_{j+1}, ..., z_m\Big) \equiv 0 \tag{11}$$

Proof. The proof based on the 1-D division $f(z_1, ..., z_m) : (z_j - p_i(\tilde{z}))$, with respect to z_j, is omitted for the sake of brevity. ■

Remark. It is reminded that $p_i(\tilde{z})$ were assumed to be polynomials in (7). One should notice that if (11) holds, then $p_i(\tilde{z})$ are indeed polynomials. If not, $p_i(\tilde{z})$ are not polynomials and $f(z_1, ..., z_m)$ cannot be factored in polynomial factors according to (6). However, it may be factored in other forms.

Now, if one factor $z_j - p_i(\tilde{z})$ is found, the algorithmic division $f(z_1, ..., z_m) : (z_j - p_i(\tilde{z})) = q(z_1, ..., z_m)$ is carried out. The polynomial $q(z_1, ..., z_m)$ may be factorized with this or with other methods (Mastorakis and Theodorou, 1990;1992; Mastorakis *et al.*, 1991; Misra and Patel, 1990; Theodorou and Tzafestas, 1985).

4. Example

Consider the 2-D polynomial:

$$f(z_1, z_2) = z_1^2 + 2z_1z_2^2 + 2z_1 - 2z_2^3 + 3z_2^2 + 2z_2 \tag{12}$$

STEP 1. Clearly $N_1 = 2$ and $N_2 = 3$. The present method can be applied, because $a(N_1, i_2) = 0$ when $i_2 > 0$. Then, the variable z_2 is considered as a parameter.

STEP 2. Set $z_2 = 0$, then

$$f(z_1, 0) = z_1^2 + 2z_1 = z_1(z_1 + 2)$$

Roots: $0, -2$

Set $z_2 = 1$

$$f(z_1, 1) = z_1^2 + 2z_1 + 2z_1 - 2 + 3 + 2 = z_1^2 + 4z_1 + 3$$

Roots: $-1, -3$

Set $z_2 = -1$

$$f(z_1, -1) = z_1^2 + 2z_1 + 2z_1 + 2 + 3 - 2 = z_1^2 + 4z_1 + 3$$

Roots: $-1, -3$

Set $z_2 = 2$.

$$f(z_1, 2) = z_1^2 + 8z_1 + 2z_1 - 16 + 12 + 4 = z_1^2 + 10z_1$$

Roots: $-10, 0$

Now, one has 4 pairs of roots. So, one can construct 2^4 Lagrange polynomials having the form:

$$p(z_2) = \frac{(z_2-1)(z_2+1)(z_2-2)}{(0-1)(0+1)(0-2)}g(0) + \frac{(z_2-0)(z_2+1)(z_2-2)}{(1-0)(1+1)(1-2)}g(1)$$
$$+\frac{(z_2-0)(z_2-1)(z_2-2)}{(-1-0)(-1-1)(-1-2)}g(-1) + \frac{(z_2-0)(z_2-1)(z_2+1)}{(2-0)(2-1)(2+1)}g(2)$$

where $g(0) = 0$ or -2, $g(1) = -1$ or -3, $g(-1) = -1$ or -3, $g(2) = 0$ or -10. If one takes: $g(0) = -2$, $g(1) = -1$, $g(-1) = -3$, $g(2) = 0$, one finds $p(z_2) = 0 \cdot z_2^3 + 0 \cdot z_2^2 + z_2 - 2$. The polynomial $z_1 - p(z_2) = z_1 - (z_2 - 2)$ is a factor of $f(z_1, z_2)$ since

$$f\Big(p(z_2), z_2\Big) \equiv (z_2-2)^2 + 2(z_2-2)z_2^2 + 2(z_2-2) - 2z_2^3 + 3z_2^2 + 2z_2 \equiv 0$$

STEP 3. Now, the algorithmic division

$$f(z_1, z_2) : z_1 - z_2 + 2$$

is carried out.

This division is carried out considering z_2 as parameter

$$\begin{array}{l|l}
z_1^2 + 2(z_2^2+1)z_1 - 2z_2^3 + 3z_2^2 + 2z_2 & z_1 - (z_2 - 2) \\
\underline{-z_1^2 + (z_2-2)z_1} & \overline{z_1 + (2z_2^2 + z_2)} \\
(2z_2^2 + z_2)z_1 - 2z_2^3 + 3z_2^2 + 2z_2 & \\
\underline{-(2z_2^2 + z_2)z_1 + 2z_2^3 - 3z_2^2 - 2z_2} & \\
0 \qquad\qquad\qquad 0 &
\end{array}$$

Therefore

$$\cdot f(z_1, z_2) = (z_1 - z_2 + 2)(z_1 + 2z_2^2 + z_2) \tag{13}$$

Clearly, $z_1 - z_2 + 2$ is not further factorizable. Now, it is examined, if $z_1 + 2z_2^2 + z_2$ is factorizable.If yes, then the variable z must appear only in one factor: $z_1 + \alpha z_2 + \beta$. The other factor will be $cz_2 + d$. However $c_2 z + d$ cannot be a factor of $z_1 + 2z_2^2 + z_2$, because $z_1 + 2(-\frac{d}{c})^2 + 2(-\frac{d}{c}) \not\equiv 0$ $(\forall\, d, c)$. Thus the polynomial is not further factorizable.

An application of this example is the following.

Consider the 2-D system described by the transfer function:

$$G(z_1, z_2) = \frac{1}{f(z_1, z_2)} \doteq \frac{1}{(z_1 - z_2 + 2)(z_1 + 2z_2^2 + z_2)} \tag{14}$$

If the polynomial $f(z_1, z_2)$ has zeros inside the unit circle the system is unstable. For this reason the equation

$$f(z_1, z_2) = 0 \tag{15}$$

or

$$z_1^2 + 2z_1 z_2^2 + 2z_1 - 2z_2^3 + 3z_2^2 + 2z_2 = 0 \tag{16}$$

with $|z_1| \leq 1$, $|z_2| \leq 1$ is considered.

Since $f(z_1, z_2)$ is factorized, (16) is rewritten

$$z_1 - z_2 + 2 = 0 \qquad \text{or} \qquad z_1 + 2z_2^2 + z_2 = 0$$

The first of them has the obvious solution $z_1 = -1$, $z_2 = 1$ for which $|z_1| \leq 1$, $|z_2| \leq 1$. Therefore $f(z_1, z_2) = 0$ for $|z_1| \leq 1$, $|z_2| \leq 1$. Thus the considered system is unstable.

5. Conclusion

A new m-D factorization technique has been developed in this paper through the utilization of multidimensional Lagrange polynomials that were introduced for the first time here. The method is based on the computation of the roots of 1-D polynomials derived from the initial m-D polynomial. Actually, with the results of the present paper the class of factorizable m-D polynomials is widened considerably. Other works carried out by the authors concern the factorization of m-D polynomials through root perturbation or rational function simplification.

Appendix

Theorem A1. *If a multidimensional polynomial is equal to zero at the points* $z_{1i_1}, \ldots, z_{mi_m}$, *where* $0 \leq i_1 \leq N_1, \ldots, 0 \leq i_m \leq N_m$, *then this polynomial is the zero polynomial.*

Proof. Consider an m-D polynomial in the form:

$$h(z,..,z)=\sum_{j_1=0}^{N_1}\left(\sum_{j_2=0}^{N_2}\left(\dots\sum_{j_m=0}^{N_m}a(j_1,..,j_m)z_m^{j_m}\right)\dots\right)z_1^{j_1} \tag{A.1}$$

and take the following (N_1+1) values of $(z_1,...,z_m)$:

$$\begin{array}{llll} z_1=\zeta_{1,0}, & z_2=\zeta_{2,0}, & \dots, & z_m=\zeta_{m,0} \\ \vdots & \vdots & & \vdots \\ z_1=\zeta_{1,N_1}, & z_2=\zeta_{2,0}, & \dots, & z_m=\zeta_{m,0} \end{array}$$

If for all the above N_1+1 values of $(z_1,...,z_m)$ the polynomial $h(z_1,...,z_m)$ is zero, i.e.: $h(z_1,...,z_m)=0$, then

$$\sum_{j_2=0}^{N_2}\left(\dots\left(\sum_{j_m=0}^{N_m}a(j_1,..,j_m)z_m^{j_m}\right)\dots\right)z_2^{j_2}=0 \qquad (\forall j_1,\ j_1=0,...,N_1)$$

for $z_2=\zeta_{2,0},...,z_m=\zeta_{m,0}$.

Now, let the N_1+1 points

$$\begin{array}{lllll} z_1=\zeta_{1,0}, & z_2=\zeta_{2,1}, & z_3=\zeta_{3,0}, & \dots, & z_m=\zeta_{m,0} \\ \vdots & \vdots & \vdots & & \vdots \\ z_1=\zeta_{1,N_1}, & z_2=\zeta_{2,1}, & z_2=\zeta_{3,0}, & \dots, & z_m=\zeta_{m,0} \end{array}$$

If for all the above N_1+1 values of $z_1,...,z_m$ one has: $h(z_1,...,z_m)=0$, then

$$\sum_{j_2=0}^{N_2}\left(\dots\left(\sum_{j_m=0}^{N_m}a(j_1,..,j_m)z_m^{j_m}\right)\dots\right)z_2^{j_2}=0 \qquad (\forall j_1,\ j_1=0,...,N_1)$$

for $z_2=\zeta_{2,1},z_3=\zeta_{3,0},...,z_m=\zeta_{m,0}$.

Continuing in the same way, variating each time z_2, one finds

$$\sum_{j_2=0}^{N_2}\left(\dots\left(\sum_{j_m=0}^{N_m}a(j_1,..,j_m)z_m^{j_m}\right)\dots\right)z_2^{j_2}=0 \qquad (\forall j_1,\ j_1=0,...,N_1)$$

at the points

$$\begin{array}{llll} z_2=\zeta_{2,0}, & z_3=\zeta_{3,0}, & \dots, & z_m=\zeta_{m,0} \\ z_2=\zeta_{2,1}, & z_3=\zeta_{3,0}, & \dots, & z_m=\zeta_{m,0} \\ \vdots & \vdots & & \vdots \\ z_2=\zeta_{2,N_2}, & z_3=\zeta_{3,0}, & \dots, & z_m=\zeta_{m,0} \end{array}$$

Then, variating, in the same way, the variable z_3, it is found

$$\sum_{j_3=0}^{N_3}\left(\cdots\left(\sum_{j_m=0}^{N_m} a(j_1,..,j_m)z_m^{j_m}\right)\cdots\right)z_3^{j_3}=0 \qquad (\forall\, j_1, j_2,\ j_1=0,...,N_1,\ j_2=0,...,N_2)$$

for $z_3=\zeta_{3,0},...,\zeta_{3,N_3},...,z_m=\zeta_{m,0}$

Continuing this procedure, finally, one obtains:

$$a(j_1,...,j_m)=0, \qquad 0\le j_1\le N_1,...,0\le j_m\le N_m$$

■

Theorem A2. *If two multidimensional polynomials become equal at the points* $(z_{1_{i_1}},...,z_{m_{i_m}})$, *where* $0\le i_1\le N_1,...,0\le i_m\le N_m$, *then these polynomials are identically equal.*

Proof. Let $h_1(z_1,...,z_m)$ and $h_2(z_1,...,z_m)$ be polynomials of $z_1,...,z_m$. Suppose now that the polynomial:

$$h(z_1,...,z_m)=h_1(z_1,...,z_m)-h_2(z_1,...,z_m) \tag{A.2}$$

becomes equal to zero, at the points $(z_{1_{i_1}},...,z_{m_{i_m}})$, where $0\le i_1\le N_1,...,0\le i_m\le N_m$. Then, from theorem A1: $h(z_1,...,z_m)\equiv 0$ and so $h_1(z_1,...,z_m)\equiv h_2(z_1,...,z_m)$.

■

References

Antulas A.C. (1981): *A system-theoretic approach to the factorization theory of non-singular polynomial matrices.* — Int. J. Syst. Sci., pp.1005–1026.

Chakrabarti J.S. and Mitra S.K. (1977): *An algorithm for multivariable polynomial factorization.* — Proc. IEEE. Int. Symp. Circ. and Syst., pp.678–679.

Collins G.E. (1973): *Computer algebra of polynomials and functions.* — Am. Math., pp.80.

Dudgeon D.E. (1975): *The existence of cepstra for 2-D rational polynomials.* — IEEE Trans. Ac. Sp. Signal Process, v.2, pp.242–243.

Ekstrom M.P. and Woods J.W. (1976): *Two-dimensional spectral factorization with applications in recursive digital filtering.* — IEEE Trans. Ac. Sp. Signal Process, ASSP-24, v.2, pp.115–128.

Ekstrom M.P. and Twogood R.E. (1977): *Finite order recursive models for 2-D random fields.* — Proc. 20th Hidwest Symp. Circuit and System, pp.188–189.

Gorez R. (1981): *Matrix factorization and Chandrasekhar equations techniques in the design of linear quadratic optimal control systems.* — Int. J. Syst. Sci., pp.907–915.

Kaczorek T. (1985): *Two-Dimensional Linear Systems.* — Berlin: Springer-Verlag, Heidelberg.

Mastorakis N.E. and Theodorou N.J. (1990): *Operators' method for multidimensional polynomial factorization.* — Found. of Comp. and Decision Science, v.15, No.3–4, pp.159–172.

Mastorakis N.E. and Theodorou N.J. (1992): *State-space model factorization m-dimensions applications in stability theory.* — Found. of Comp. and Decision Science, v.17, No.1, pp.55–61.

Mastorakis N.E, Theodorou N.J. and Tzafestas S.G. (1992): *Factorization of m-D polynomials in linear m-D factors.* — Int. J. Syst. Sci., v.23, pp.1805–1824.

Mastorakis N.E, Tzafestas S.G. and Theodorou N.J. (1991): *A simple multidimensional polynomial factorization method.* — IMACS Annals on Computing and Applied Mathematics, v.10, pp.73–76.

Misra P. and Patel R.V. (1990): *Simple factorizability of 2-D polynomials.* — Int. Symp. *Circ. Syst.*, New Orleans, v.17, pp.1207–1210.

Musser D.E. (1975): *Multivariate polynomial factorization.* — J., Ass. Comp. Mach. (A.C.M).

Shieh L.S. and Clahin N. (1981): *A computer-aided method for the factorization of matrix polynomials.* — Int. J. Syst. Sci., v.12, pp.305–323.

Theodorou N.J. and Tzafestas S.G. (1985): *Reducibility and factorizability of multivariable polynomials: Overview and new results.* — Control Theory and Advanced Techn., v.1, pp.25–46.

Tzafestas S.G. (1986): *Multidimensional Systems, Techniques and Applications.* — New York: Marcel Dekker.

Youla D.C. and Pickel P.F. (1984): *The Quillen-Suslin theorem and the structure of n-D elementary polynomial matrices.* — IEEE Trans. Circ. Syst., v.6, pp.513–514.

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