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UNIT SLIDING MODE CONTROL IN INFINITE DIMENSIONAL SYSTEMS

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In contrast to the conventional component-wise design of sliding mode control, a new approach is developed for infinite-dimensional systems. The conventional approach is not applicable since, generally speaking, the infinite-dimensional control may not be represented in the component form as well as a sliding manifold. The concept of "unit control", previously introduced for finite-dimensional systems, does not depend on the dimension of control and is generalized for the dynamic processes governed by differential equations in Banach and Hilbert spaces. The design methods for heat and mechanical distributed processes are given.

1. Introduction

The conventional approach to sliding mode control design implies that each component of a control undergoes discontinuities on its own surface and as a result the sliding mode is enforced in their intersection referred to as a sliding manifold. However, this component-wise design idea may prove to be not applicable for infinite-dimensional systems since, generally speaking, neither the control nor the sliding manifold may be represented in the component form.

The so-called unit control previously introduced for finite-dimensional systems (Gutman, 1979; Gutman and Leitmann, 1976) is a discontinuous function of the system state too, but it undergoes discontinuities on the sliding manifold only while it is a continuous state function beyond the manifold. It is important that the unit control may be determined for any space with norm, which is common for infinite-dimensional dynamic processes governed by partial differential equations or, more general, equations in Banach and Hilbert spaces.

The objective of the paper is to develop sliding mode control design methods for infinite-dimensional systems based on the concept of "unit control". The methods are illustrated by designing controls for heat and mechanical distributed processes operating under uncertainty conditions.

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2. Unit Control for Finite-Dimensional Systems

The Lyapunov method approach implies the design of a control based on a Lyapunov function selected for a nominal (feedback or open loop) system. The objective is to find a control such that the time-derivative of the Lyapunov function is negative on the trajectories of the system with perturbations caused by uncertainties of a plant operator and environment conditions.

The roots of the above approach may be found in (Gutman, 1979; Gutman and Leitmann, 1976). The design idea may be explained for an affine system

$$\dot{x} = f(x,t) + B(x,t)u + h(x,t) \tag{1}$$

with state and control vectors $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, state-dependent vectors $f(x,t)$ and $h(x,t)$, and matrix $B(x,t) \in \mathbb{R}^{n\times m}$. The vector $h(x,t)$ represents the system uncertainties and its influence on the control process should be rejected.

The equation

$$\dot{x} = f(x,t) \tag{2}$$

represents an open loop nominal system which is assumed to be asymptotically stable with some known Lyapunov function:

$$V(x) > 0, \quad W_0 = \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{h=0,u=0} = \left\{\mathrm{grad}(V)\right\}^T f < 0 \tag{3}$$

The perturbation vector $h(x,t)$ is assumed to satisfy the matching conditions (Drazenovic, 1969)

$$h(x,t) \in \mathrm{span}\,(B)$$

or there exists a vector $\lambda(x,t) \in \mathbb{R}^m$ such that

$$h(x,t) = B(x,t)\lambda(x,t) \tag{4}$$

Here $\lambda(x,t)$ may be an unknown vector with known upper scalar estimate $\lambda_0(x,t)$

$$\left\|\lambda(x,t)\right\| < \lambda_0(x,t) \tag{5}$$

The time derivative of $V(x)$ on the trajectories of the perturbed system (1), (4) is of the form

$$W = \frac{\mathrm{d}V}{\mathrm{d}t} = W_0 + \left\{\mathrm{grad}(V)\right\}^T B(u+\lambda) < 0 \tag{6}$$

For the control depending on the upper estimate of the unknown disturbance

$$u = -\rho(x,t)\frac{B^T\,\mathrm{grad}(V)}{\left\|B^T\,\mathrm{grad}(V)\right\|} \tag{7}$$

with a scalar function

$$\rho(x,t) > \lambda_0(x,t)$$

and

$$\left\|B^T\{\operatorname{grad}(V)\}\right\|^2 = \left[\{\operatorname{grad}(V)\}^T B\right]\left[B^T \operatorname{grad}(V)\right]$$

the time derivative of the Lyapunov function $V(x)$

$$\begin{aligned} W &= W_0 - \rho(x,t)\left\|\{\operatorname{grad}(V)\}\right\| + \operatorname{grad}(V)^T B\lambda(x,t) \\ &< W_0 - \left\|B^T\{\operatorname{grad}(V)\}\right\|\left[\rho(x,t) - \lambda_0(x,t)\right] < 0 \end{aligned}$$

is negative. This means that the perturbed system with control (7) is asymptotically stable, too.

Two important features should be underlined for the system with control (7):

1. The control is a discontinuous function of the system state and undergoes discontinuities in the $(n-m)$-th dimensional manifold

$$s(x) = B^T \operatorname{grad}(V) = 0 \tag{8}$$

2. The disturbance $h(x,t)$ is rejected due to the enforcing sliding mode in the manifold $s(x) = 0$. Indeed, if the disturbance (4) is rejected, then control u is equal to $-\lambda(x,t)$, which is not, generally speaking, the case for the control (7) beyond the discontinuity manifold, $s(x) = B^T \operatorname{grad}(V) \neq 0$. This means that the sliding mode occurs in the manifold $s = 0$ and the equivalent value of control is equal to $-\lambda(x,t)$.

Note that the norm of control (7) with the gain $\rho(x,t) = 1$

$$\left\|\frac{B^T \operatorname{grad}(V)}{\left\|B^T \operatorname{grad}(V)\right\|}\right\|$$

is equal to 1 for any value of the state vector. This explains the term "unit control" for (7).

Later on, the unit control has been used directly without a Lyapunov function as the second stage of the conventional two-stage design procedure for sliding mode control: selection of a sliding manifold $s(x) = 0$ and enforcing a sliding mode in this manifold (Dorling and Zinober, 1986). Once the manifold $s(x) = 0$ has been selected in compliance with some performance criterion, the control is designed in the form (7):

$$u = -\rho(x,t)\frac{D^T s(x)}{\left\|D^T s(x)\right\|} \tag{9}$$

with $D = \{\partial s/\partial x\}B$, D is assumed to be nonsingular. The equation of a motion projection of the system (1) on the subspace s is of the form

$$\dot{s} = \left\{\frac{\partial s}{\partial x}\right\}(f+h) + Du \tag{10}$$

The conditions for the trajectories to converge to the manifold $s(x) = 0$ and the sliding mode to exist in this manifold may be derived based on the Lyapunov function

$$V = \frac{1}{2} s^T s > 0 \tag{11}$$

with the time derivative

$$\begin{aligned} \dot{V} &= s^T \left\{ \frac{\partial s}{\partial x} \right\} (f + h) - \rho(x,t) \| D^T s(x) \| \\ &< \| D^T s(x) \| \left[\left\| D^{-1} \left\{ \frac{\partial s}{\partial x} \right\} (f + h) \right\| - \rho(x,t) \right] \end{aligned} \tag{12}$$

For

$$\rho(x,t) > \left\| D^{-1} \left\{ \frac{\partial s}{\partial x} \right\} (f + h) \right\|$$

the value of $\dot{V}$ is negative and therefore the state will reach the manifold $s(x) = 0$ in a finite time interval for any initial conditions and then the sliding mode with the desired dynamics will occur. The boundedness of the interval preceding the sliding motion follows from the inequality resulting from (11), (12):

$$\dot{V} < -\gamma V^{1/2}, \qquad \gamma = \text{const} > 0$$

with the solution

$$V(t) < \left(-\frac{\gamma}{2} t + \sqrt{V_0} \right)^2, \qquad V_0 = V(0)$$

Since the solution vanishes after some $t_s < 2\sqrt{V_0}/\gamma$, the vector s vanishes so that the sliding mode starts after a finite time interval.

It is interesting to note the principal difference in the motions preceding the sliding mode in $s(x) = 0$ for the conventional component-wise control and unit control design methods. For the conventional method the control undergoes discontinuities if any of the vector components changes sign, while the unit control is a continuous state function until the manifold $s(x) = 0$ is reached. Due to this difference the unit control has proved to be an appropriate tool to design sliding mode control in infinite-dimensional systems with control and function $s(x)$ as elements of functional spaces which are not (or even cannot be) represented in a component-wise form (Utkin and Orlov, 1990).

3. Unit Control of Parabolic Systems

We start with an example of the dynamical system

$$\dot{x} = u, \qquad x(0) = x^0 \in H \tag{13}$$

in a Hilbert space H forced by the unit control

$$u = \frac{-x}{\|x\|} \tag{14}$$

which undergoes discontinuities in the trivial manifold $x = 0$. The example, although extremely simple, motivates the theoretical investigation and illustrates that discontinuous infinite-dimensional systems can be driven along discontinuity manifolds for a finite time interval. In the infinite-dimensional case, the motion along the discontinuity manifold will also be referred to as a sliding mode. Since $\|x\| = \sqrt{(x,x)}$ in the Hilbert space, where $(\cdot,\cdot)$ stands for the inner product, we have

$$\frac{1}{2}\frac{\mathrm{d}\|x\|^2}{dt} = \big(x(t), \dot{x}(t)\big) = -\big\|x(t)\big\|$$

and, therefore, $\|x(t)\| = (\|x^0\| - t)$ for $t \le \|x^0\|$. Thus in the infinite-dimensional system (13) starting from the time moment $t = \|x^0\|$, there appears a sliding mode on the discontinuity manifold $x = 0$.

In the subsequent sections we extend the unit control approach to the infinite-dimensional systems described by the differential equation

$$\dot{x} = -Ax + f(x,t) + bu(x,t) + h(x,t), \qquad x(0) = x^0 \tag{15}$$

in a reflexive Banach space X. From now on, $x(t)$ and $u(x,t)$ are abstract functions with values in reflexive Banach spaces X and U, respectively; operator functions $f(x,t)$ and $h(x,t)$ take their values in X; $b \in L(U,X)$, where $L(U,X)$ is the space of linear continuous operators mapping U into X; A is a linear sectorial operator. Recall (Henry, 1981) that a linear sectorial operator is defined as follows.

Definition 1. A linear closed operator A with dense domain $\mathcal{D}(A)$ is a *linear sectorial operator* iff for some $\varphi \in (0, \pi/2)$, $M \ge 1$, and $a \in \mathbb{R}^1$ the sector

$$S_{a,\varphi} = \Big\{\lambda : \varphi \le \big|\arg(\lambda - a)\big| \le \pi,\ \lambda \ne a\Big\}$$

is in the resolvent set $\rho(A) = \{\lambda \in \mathbb{C} : \|(\lambda I - A)^{-1}\| < \infty\}$ of the operator A (I is the identity operator) and

$$\frac{\big\|(\lambda I - A)^{-1}\big\| \le M}{|\lambda - a|} \quad \text{for all } \lambda \in S_{a,\varphi}$$

For the sectorial operator $A_1 = A + \lambda_0 I$, $\lambda_0 \ge a$, whose spectrum lies in the right half-plane, we introduce fractional degrees to be used subsequently.

Definition 2. (Henry, 1981) The operator A_1^{ε} is the identity operator for $\varepsilon = 0$,

$$A_1^{\varepsilon} = \frac{\sin \pi\varepsilon}{\pi}\int_0^{\infty} \lambda^{\varepsilon}(\lambda I + A_1)^{-1}\,\mathrm{d}\lambda \quad \text{for } \varepsilon \in (-1, 0)$$

A_1^{ε} is the inverse operator A_1^{-1} for $\varepsilon = -1$; A_1^{ε} is the inverse operator $(A_1^{-\varepsilon})^{-1}$ of $A_1^{-\varepsilon}$ for $\varepsilon \in (0,1)$.

It should be noted that the above definition ensures the group property $A_1^{\varepsilon_1+\varepsilon_2} = A_1^{\varepsilon_1}A_1^{\varepsilon_2}$ for $\varepsilon_1, \varepsilon_2, \varepsilon_1 + \varepsilon_2 \in [-1, 1]$, which enables us to extend the definition for all $\varepsilon \in \mathbb{R}^1$.

The equation

$$\dot{y} = -Ay + f(y,t) \tag{16}$$

represents an open-loop nominal system which is assumed to be asymptotically stable with an a-priori-known Lyapunov functional:

$$V(y) > 0, \quad W_0(y,t) = \frac{\mathrm{d}V}{\mathrm{d}t} = \{\operatorname{grad} V(y)\}^T[-Ay + f(y,t)] < 0 \tag{17}$$

for all $y \in X$, $t > 0$. The function $f(x,t)$ is assumed to locally satisfy the Hölder condition in t and the Lipschitz condition in $x \in X^\varepsilon$ with some $\varepsilon \in [0,1)$, where X^ε is the Banach space $\mathcal{D}(A_1^\varepsilon)$ with the graph norm $\|x\|_\varepsilon = \|A_1^\varepsilon x\|$, i.e.

$$\|f(x,t) - f(z,s)\| \le L_0(x,z,s,t)\Big(\|x-z\|_\varepsilon + |t-s|^h\Big)$$

for some continuous function $L_0(x,z,t,s) > 0$ and constant $h > 0$, and all $s,t \ge 0$, $x,y \in X^\varepsilon$. We denote by $\mathcal{L}ip(L_0,\varepsilon,h)$ the class of such functions $f(x,t)$.

The operator function $h(x,t)$ represents the system uncertainties and their influence on the control process should be rejected. This function is assumed to be of the same class $\mathcal{L}ip(L_0,\varepsilon,h)$ and satisfy the matching condition

$$h(x,t) = b\phi(x,t), \qquad \phi(x,t) \in \mathcal{L}ip(L_0,\varepsilon,h) \tag{18}$$

where the uncertain function $\phi(x,t)$ has an a-priori-known upper scalar estimate $N(x,t)$, i.e.

$$\|\phi(x,t)\| < N(x,t), \qquad x \in X, \qquad t \ge 0 \tag{19}$$

The above assumptions guarantee the unforced initial-value problem (15) with $u(x,t) \equiv 0$ to have a unique strong solution $x(t)$ for all $x^0 \in X^\varepsilon$ which is locally defined as follows (see e.g. Henry, 1981).

Definition 3. A continuous $\mathcal{D}(A)$-valued function $x(t)$ defined on some interval $t \in [0,T)$ is said to be the *strong solution* to the unforced ($u \equiv 0$) system (15) iff $x(0) = x^0$, $x(t)$ is differentiable in the state space for $t \in (0,T)$ and satisfies the differential equation (15) under $u \equiv 0$ for all $t \in (0,T)$.

We remark that the development here is confined to the investigation of the strong solution to the initial-value problem, although all the results can be generalized to the case when the solution to the problem is defined in a mild sense as a solution to the corresponding integral equation. Moreover, according to (Krasnoselskii *et al.*, 1976), such a relaxation of the solution concept allows us to extend the class of the admissible functions $f(x,t)$ and $h(x,t)$ by adding functions integrable in t.

In order to apply the above-mentioned Lyapunov redesign method to infinite-dimensional systems, let us calculate the time derivative of the Lyapunov functional $V(x)$ along the trajectories of the perturbed system (15), (18):

$$\begin{aligned} W(x,t) &= \frac{\mathrm{d}V}{\mathrm{d}t} = W_0(x,t) + \Big\langle \{\operatorname{grad} V(x)\}^T, b\big[u(x,t) + \phi(x,t)\big]\Big\rangle \\ &= W_0(x,t) + \Big\langle b^*\{\operatorname{grad} V(x)\}^T, \big[u(x,t) + \phi(x,t)\big]\Big\rangle \end{aligned} \tag{20}$$

where b^* is the dual operator and $\langle\cdot,\cdot\rangle$ is the dual product. Now, in accordance with the Hahn-Banach theorem (Dunford and Schwartz, 1958), let us fix a unit control $a(x,t) \in U$ such that

$$\|a(x,t)\| = 1 \text{ for all } x \in X : b^*\{\operatorname{grad} V(x)\}^T \neq 0$$

$$\left\langle b^*\{\operatorname{grad} V(x)\}^T, a(x,t)\right\rangle = \|b^*\{\operatorname{grad} V(x)\}^T\| \tag{21}$$

To design a robust control law stabilizing the perturbed system (15), it remains to construct the control signal

$$u(x,t) = -N(x,t)a(x,t) \tag{22}$$

which, due to (17) and (19), ensures the time derivative (20) of the Lyapunov functional $V(x)$ to be negative for all $x \neq 0$, $t \geq 0$. Hence (Henry, 1981) the perturbed system (15) forced by the control signal (22) is asymptotically stable.

Thus, the unit control signal (21) which undergoes discontinuities on the manifold $b^*\{\operatorname{grad} V(x)\}^T = 0$ allows us to synthesize the stabilizing control law (22) robust with respect to the external disturbances. It should be noted, however, that the proposed control algorithm cannot be used directly because it implies the unit control $a(x,t)$ to be known explicitly. However, if X and U are Hilbert spaces, the above synthesis procedure determines the feedback (22) in the explicit form. Indeed, if this is the case, the unit control is designed as follows:

$$a(x,t) = \frac{b^* \operatorname{grad} V(x)}{\|b^* \operatorname{grad} V(x)\|} \tag{23}$$

Summarizing, the following theorem has been proven.

Theorem 1. *The uncertain infinite-dimensional system (15) is asymptotically stabilizable by the discontinuous controller (21), (22) which imposes the robustness property with respect to the matching external disturbances (18) on the closed loop system. If X and U are Hilbert spaces, then the unit control (21) is designed in the explicit form (23).*

4. Unit Control of Heat Processes

To exemplify our theoretical results, let us consider a distributed parameter system described by the parabolic partial differential equation

$$\begin{cases} \rho(x)\dfrac{\partial Q}{\partial t} = \dfrac{\partial}{\partial x}\left[k(x)\dfrac{\partial Q}{\partial x}\right] - q(x)Q + u(x,t) + f(x,t), & 0 < x < 1, \quad t > 0 \\ \dfrac{\partial Q}{\partial x}(0,t) = \dfrac{\partial Q}{\partial x}(1,t) = 0, & t \geq 0 \\ Q(x,0) = Q_0(x), & 0 \leq x \leq 1 \end{cases} \tag{24}$$

Equation (24) describes the heat propagation in a one-dimensional rod insulated at both ends, where $Q(x,t)$ is the value of the temperature field at $x \in [0,1]$ for $t \geq 0$, $f(x,t)$ denotes an external disturbance, $\rho(x) \geq \rho_0 > 0$ is a heat capacity coefficient, $k(x) \geq k_0 > 0$ stands for a heat conductivity coefficient, $q(x) \geq 0$ is a heat exchange coefficient, $Q_0(x)$ is an initial state. All the functions $\rho(x)$, $q(x)$, $k(x)$, $Q_0(x)$, $f(x,t)$ are assumed to be smooth.

For the sake of simplicity, the development here is confined to the processes with one-dimensional spatial variable, but the extension to the case of several spatial variables is straightforward. We will focus our attention on the controller synthesis under the assumption that the structure of the plant is known and only the plant parameters and external disturbances are unknown. The range of the external disturbance $f(x,t)$ is assumed to be bounded:

$$\|f(\cdot,t)\|_{L_2(0,1)} = \sqrt{\int_0^1 f^2(x,t)\,\mathrm{d}x} \leq L_f, \qquad L_f = \mathrm{const}$$

Distributed sensing and actuation are also assumed to be available.

It is required to synthesize a control signal $u(x,t)$ to guarantee the quadratic convergence

$$\lim_{t\to\infty} \int_0^1 [Q(x,t)]^2\,\mathrm{d}x = 0 \tag{25}$$

of the plant state.

The idea of the control synthesis is based on deliberate introduction of a sliding motion along the manifold

$$Q = 0 \tag{26}$$

In order to synthesize this type of motion, we propose the following control algorithm:

$$u(Q,t) = -M\frac{Q}{\|Q\|_{L_2(0,1)}} \tag{27}$$

where $M = M_1 + L_f$, $M_1 > 0$. The norm of the control signal (27) with $M = 1$ is equal to 1 for each value of the state vector beyond the discontinuity manifold (26) and, therefore, it is a unit control. The unit control law (27) turns out to solve the stabilization problem (25) for a finite time moment.

Theorem 2. *Consider the heat process (24) with the assumptions above. Then the unit control law (27) solves the stabilization problem in a finite time* $T < \rho_0 M_1^{-1}\|Q_0\|_{L_2(0,1)}$, *i.e.*

$$\|Q(\cdot,t)\|_{L_2(0,1)} \equiv 0, \qquad t \geq T \tag{28}$$

where

$$\rho_0 > \max_{0\leq x\leq 1} \rho(x)$$

is an upper estimate of the heat capacity coefficient.

Proof. According to (Friedman, 1969), for all initial conditions which are beyond the discontinuity manifold (26) there exists a unique local solution of the parabolic system (24). Differentiating $V(t) = (1/2)\int_0^1 \rho(x)[Q(x,t)]^2\,\mathrm{d}x$ with respect to t along the trajectories of (24), employing integration by parts, and utilizing the control law (27) yield the inequality

$$\begin{aligned}\dot V(t) &= \int_0^1 \rho Q\dot Q\,\mathrm{d}x = \int_0^1 Q\Big\{(kQ')' - q\Delta Q + u + f\Big\}\,\mathrm{d}x\\ &= -\int_0^1 q(Q)^2\,\mathrm{d}x - \int_0^1 k(Q')^2\,\mathrm{d}x + \int_0^1 Q\Big\{-M\frac{Q}{\|Q\|_{L_2}} + f\Big\}\,\mathrm{d}x\\ &\le -M_1\sqrt{\int_0^1 (Q)^2\,\mathrm{d}x} \le -M_1\sqrt{2\rho_0^{-1}V(t)}\end{aligned}$$

which gives rise to (28). In order to reproduce this conclusion, one should note that the solution $V(t)$ to the latter inequality is majored for all $t \ge 0$ by the solution to the differential equation $\dot V_0(t) = -M_1\sqrt{2\rho_0^{-1}V_0(t)}$ initialized with the same condition $V_0(0) = V(0)$. Since $V_0(t) = 0$ for all $t \ge T$, $V(t)$ vanishes after the finite time moment T, which completes the proof. ∎

Thus, starting with a finite time instant, the discontinuous control law (27) enforces the desired system motion in the discontinuity manifold (26) regardless of external disturbances and parameter variations, and hence it imposes useful robustness properties of the closed-loop system.

The next two sections illustrate a conventional way of sliding mode control design developed for finite-dimensional systems and consisting of two stages: selection of a discontinuity manifold with the desired dynamics of the sliding mode and then selection of a discontinuous control enforcing the sliding motion in this manifold.

5. Unit Control of Coupled Thermal Fields

In this section, we continue to study stabilization of heat processes and consider the control problem for thermal fields of the plants governed by the partial differential equation

$$\begin{cases} \dfrac{\partial Q}{\partial t} = \dfrac{\partial^2 Q}{\partial x^2} + DQ + Fu(x,t), & 0 < x < 1, \quad t > 0\\[2mm] \dfrac{\partial Q}{\partial x}(0,t) = \dfrac{\partial Q}{\partial x}(1,t) = 0, & t \ge 0\\[2mm] Q(x,0) = Q_0(x), & 0 \le x \le 1\end{cases} \tag{29}$$

where $Q(x,t) \in \mathbb{R}^n$, $u(x,t) \in \mathbb{R}^m$ for all $x \in \mathbb{R}^1, t \ge 0$. The constant matrices D and F of appropriate dimensions are assumed to be controllable. From a physical viewpoint, the problem consists in heating n similar plants by using m distributed sources. The matrix D characterizes the heat exchange with the environment and between the plants.

Let the control signal drive the system (29) to the manifold

$$S(Q) = cQ = c_1 Q_1 + c_2 Q_2 = 0 \tag{30}$$

where

$$Q_1 \in \mathbb{R}^{n-m}, \quad Q_2 \in \mathbb{R}^m, \quad \det c_2 \neq 0, \quad \det(cF) \neq 0$$

Then the state equation can be represented in terms of Q_1 and S as follows:

$$\frac{\partial Q_1}{\partial t} = \frac{\partial^2 Q_1}{\partial x^2} + D_{11} Q_1 + D_{12} S + F_1 u(x,t) \tag{31}$$

$$\frac{\partial S}{\partial t} = \frac{\partial^2 S}{\partial x^2} + D_{21} Q_1 + D_{22} S + cFu(x,t) \tag{32}$$

Due to the equivalent control method, the system motion on the manifold (30) is governed by the equation

$$\frac{\partial Q_1}{\partial t} = \frac{\partial^2 Q_1}{\partial x^2} + RQ_1, \quad R = D_{11} - F_1(cF)^{-1} \tag{33}$$

which is obtained by substitution of the equivalent control value

$$u_{\mathrm{eq}} = -(cF)^{-1} D_{21} Q_1$$

resulted from (31), (32) with $S(x,t)$ identically equal to zero. The applicability of the equivalent control method to parabolic systems is verified in (Orlov, 1997).

Following the aforementioned design procedure, in the first step one needs to choose a discontinuity manifold (30) to ensure the prescribed properties of the motion in the sliding mode. It is well-known that for the finite-dimensional system

$$\dot{Q} = DQ + Fu$$

the equation of the sliding mode along the manifold $S = cQ = 0$ takes the form $\dot{Q}_1 = RQ_1$ by virtue of the equivalent control technique. A matrix c for the controllable system may be chosen such that $\det(cF) \neq 0$ and the eigenvalues of the matrix R take up the desired values with negative real parts $\operatorname{Re}\lambda\{R\} < 0$ (see (Utkin, 1992) for details). Based on this fact, the required rates of L_2-convergence

$$\lim_{t\to\infty} \left\| Q_1(\cdot,t) \right\|_{L_2(0,1)} = 0 \tag{34}$$

of the state of the distributed parameter system (31) may also be imposed. In order to demonstrate the desired rates of convergence (34), let us introduce the Lyapunov functional

$$V(t) = \int_0^1 Q_1^T(x,t) W Q_1(x,t)\,\mathrm{d}x \tag{35}$$

where W is the positive definite solution of the Lyapunov equation $R^TW+WR=-I$, I being the identity matrix of an appropriate dimension, and find the time derivative of the functional along the trajectories of (31):

$$\dot V(t)=-2\int_0^1\Big(\frac{\partial Q_1}{\partial x}\Big)^T W\Big(\frac{\partial Q_1}{\partial x}\Big)\,\mathrm{d}x-\int_0^1 Q_1^TQ_1\,\mathrm{d}x \tag{36}$$

Denoting by $\lambda_{\max}$ the maximal eigenvalue of the matrix W and bearing in mind that $Q_1^TWQ_1\le\lambda_{\max}Q_1^TQ_1$, we obtain

$$\dot V(t)\le-\lambda_{\max}^{-1}V(t) \tag{37}$$

Since $W=\int_0^\infty\exp\{R^Tt\}\exp\{Rt\}\,\mathrm{d}t$ in accordance with (Kwakernaak and Sivan, 1972), by choosing the eigenvalues of R with negative real parts sufficiently large in magnitude the value of $\lambda_{\max}$ may be made as small as desired. Thus an appropriate choice of c ensures that $V(t)\to0$ as $t\to\infty$, as well as (34) at desired rates.

In the second step, a discontinuous control is designed to drive the system state to the manifold $S=0$. We demonstrate that the unit control

$$u(Q)=-\frac{M(Q)S_1(Q)}{\big\|S_1(Q(\cdot,t))\big\|_{L_2(0,1)}} \tag{38}$$

where

$$S_1(Q)=(cF)^{-1}S(Q),\qquad M(Q)=M_0\big\|Q(\cdot,t)\big\|_{L_2(0,1)}$$
$$M_0=\big\|(cF)^{-1}cD\big\|+M_1,\qquad M_1>0 \tag{39}$$

guarantees that in the closed-loop system (29), (38) starting from a finite time moment there appears a sliding motion on the manifold $S_1(Q)=S(Q)=0$. Indeed, differentiating the Lyapunov functional

$$V_1(t)=\int_0^1 S_1^T\big(Q(x,t)\big)S_1\big(Q(x,t)\big)\,\mathrm{d}x$$

along the trajectories (29), employing integration by parts, applying boundary conditions and utilizing the control law (38), (39) yield

$$\begin{aligned}\frac12\dot V_1(t)&=\int_0^1 S_1^T\dot S_1\,\mathrm{d}x=\int_0^1 S_1^T\left[\frac{\partial^2S_1}{\partial x^2}+(cF)^{-1}cDQ+u\right]\mathrm{d}x\\&=-\int_0^1\Big(\frac{\partial S_1}{\partial x}\Big)^T\frac{\partial S_1}{\partial x}\,\mathrm{d}x\\&\quad-\int_0^1 S_1^T\left[\frac{M(Q)S_1(Q)}{\big\|S_1(Q(\cdot,t))\big\|_{L_2(0,1)}}-(cF)^{-1}cDQ\right]\mathrm{d}x\le-M_1\sqrt{V_1(t)}\end{aligned} \tag{40}$$

The solution to the latter inequality has been shown to vanish after the finite time $T\le V_1(0)/M_1$. Therefore, starting from T, the unit control signal (38) enforces the system motion in the sliding mode along the manifold (30) and $T \to 0$ as $M_1 \to \infty$. Thus the unit control approach leads to a decoupling system design and ensures the desired rate of transient decay.

Summarizing , the following result has been shown.

Theorem 3. *Consider the coupled thermal fields (29) with the assumptions above. Then the unit control law (38) drives the system (29) to the discontinuity manifold (30) in the finite time* $T \le V_1(0)/M_1$ *and, starting from* T*, the sliding motion of the controlled system is governed by the sliding mode equation (33) with the desired rate of transient decay.*

6. Unit Control of Distributed Oscillators

In this section, we consider the distributed parameter system described by the following hyperbolic partial differential equation:

$$\begin{cases} p(x)\dfrac{\partial^2\Theta}{\partial t^2} = \dfrac{\partial}{\partial x}\left[\kappa(x)\dfrac{\partial\Theta}{\partial x}\right] - h(x)\Theta - \alpha(x)\dfrac{\partial\Theta}{\partial t} & \\ \qquad + v(x,t) + g(x,t), & 0 < x < 1,\ t > 0 \\ \Theta(0,t) = \Theta(1,t) = 0, & t \ge 0 \\ \Theta(x,0) = \Theta_0(x), \qquad \dfrac{\partial\Theta}{\partial t}(x,0) = \Theta_1(x), & 0 \le x \le 1 \end{cases} \tag{41}$$

Equation (41) describes the oscillations of a string with fixed ends where the state vector consists of the location $\Theta(x,t)$ and the velocity $\dot{\Theta}(x,t)$ of the string at $x \in [0,1]$ and $t \ge 0$, $p(x) \ge p_0 > 0$ is a density coefficient, $\kappa(x) \ge \kappa_0 > 0$ stands for an elasticity coefficient, $h(x) \ge 0$ denotes a restoring stiffness coefficient, $\alpha(x) \ge 0$ is a dissipation coefficient, $\Theta_0(x)$ and $\Theta_1(x)$ are respectively an initial state and an initial velocity, $g(x,t)$ is an external disturbance. All the functions $p(x)$, $h(x)$, $\alpha(x)$, $\kappa(x)$, $\Theta_0(x)$, $\Theta_1(x)$, $g(x,t)$ are assumed to be smooth.

For simplicity, we restrict our investigation to the processes with one-dimensional spatial variable, but the extension to the case of several spatial variables is straightforward. It is assumed that the structure of the plant is known and only plant parameters and external disturbances are unknown. The range of the external disturbance $g(x,t)$ is assumed to be bounded:

$$\|g(\cdot,t)\|_{L_2(0,1)} = \sqrt{\int_0^1 g^2(x,t)\,\mathrm{d}x} \le L_g, \qquad L_g = \text{const}$$

Distributed sensing and actuation are also assumed to be available.

It is required to synthesize the control signal $v(x,t)$ to guarantee the quadratic convergence

$$\lim_{t\to\infty}\int_0^1 \left\{[\Theta(x,t)]^2 + \left[\frac{\partial\Theta(x,t)}{\partial t}\right]^2 + \left[\frac{\partial\Theta(x,t)}{\partial x}\right]^2\right\} \mathrm{d}x = 0 \tag{42}$$

of the system state and its time and spatial derivatives.

Letting the discontinuity manifold

$$S(\Theta,\dot{\Theta}) = s\Theta(x,t) + \dot{\Theta}(x,t) \tag{43}$$

where

$$\begin{cases} s > 0, \quad L(t) = L_1 + L_2(\Theta,\dot{\Theta}), \quad L_1 > 0 \\ L_2(\Theta,\dot{\Theta}) = L_g + N_1|\Theta| + N_2|\dot{\Theta}| \\ N_1 > \max\limits_{0\le x\le 1} \big|h(x)\big|, \quad N_2 > \max\limits_{0\le x\le 1} \big|sp(x) - \alpha(x)\big| \end{cases} \tag{44}$$

we shall demonstrate that the unit control algorithm

$$v(\Theta,\dot{\Theta},t) = -L(t)\frac{S}{\|S\|_{L_2(0,1)}} \tag{45}$$

solves the stabilization problem (42).

Theorem 4. *Consider the distributed mechanical oscillator (41) with the assumptions above. Then the relation (42) holds with the unit control law (43)–(45).*

Proof. According to (Friedman, 1969), for all initial conditions which are beyond the discontinuity manifold (43), there exists a unique local solution of the hyperbolic system (41). Using the Lyapunov functional

$$V(t) = \frac{1}{2}\int_0^1 \Big\{p(x)\,[S(x,t)]^2 + \kappa(x)\Big[\frac{\partial\Theta(x,t)}{\partial x}\Big]^2\Big\}\,\mathrm{d}x \tag{46}$$

one can prove that the solution to (41) is well-posed for all $T \ge 0$ and asymptotically stable in the L_2-sense (42). Indeed, differentiating V with respect to t along the trajectories of (41), employing integration by parts and utilizing the control law (43)–(45) yield

$$\begin{aligned} \dot{V}(t) &= -\int_0^1 \Big\{s\kappa\Big[\frac{\partial\Theta}{\partial x}\Big]^2 + sh[\Theta]^2 + (\alpha - ps)\Big[\frac{\partial\Delta\Theta}{\partial t}\Big]^2\Big\}\,\mathrm{d}x \\ &\quad - \int_0^1 S\Big\{\frac{L(t)S}{\|S\|_{L_2(0,1)}} + f\Big\}\,\mathrm{d}x \\ &\le -\int_0^1 s\kappa\Big[\frac{\partial\Theta}{\partial x}\Big]^2\mathrm{d}x - L_1\sqrt{\int_0^1 S^2\,\mathrm{d}x} < 0 \quad \text{if} \quad V(t) \ne 0 \end{aligned} \tag{47}$$

which implies the uniform boundedness $V(t) \le V(0) < \infty$ of the Lyapunov functional for all $t \ge 0$, and consequently, the uniform L_2-boundedness of the solutions of (41) and their time and spatial derivatives, as well as their asymptotic stability (42). This completes the proof. ■

It is easy to see that the proposed control law utilizes the unit control signal $L(t)S/\|S\|_{L_2(0,1)}$, similar to that for heat processes, and guarantees the desired inequality (47) for the time derivative of the Lyapunov functional, regardless of external disturbances and parameter variations. Therefore, the control algorithm (45) imposes useful robustness properties on the closed-loop system.

It is of interest to note the principal difference in the system dynamics of heat processes and distributed oscillators enforced by the unit control signals. For the distributed oscillator (41) the discontinuous control signal (45) drives the system to the discontinuity manifold (43) asymptotically and guarantees the asymptotic stability (42) only, while for the heat process (24) the unit control signal (27) drives the system to the discontinuity manifold (26) for a finite time moment and solves the stabilization problem (25) for a finite time moment as well.

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