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THE LQ CONTROLLER SYNTHESIS PROBLEM: AN OPERATOR CASE[†]

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The aim of this paper is to provide a new, direct approach to the classical LQ problem with an infinite time horizon. In our approach, the LQ problem is formulated as a parametric optimization problem of a special type, and then analysed by the methods presented in our earlier papers (Grabowski, 1990; 1993). The results simplify the well-known theory presented in (Curtain and Pritchard, 1978, Sec.4.4; Zabczyk, 1976).

1. Stabilizability and Detectability

In a Hilbert space H with the scalar product $\langle\cdot,\cdot\rangle$ we consider the following feedback system:

$$\begin{cases} \dot{x}(t) &= Ax(t) - BGx(t), \quad t \geq 0 \\ x(0) &= x_0 \\ y(t) &= Cx(t) \end{cases} \tag{1}$$

where $A: (D(A) \subset \mathrm{H}) \to \mathrm{H}$ is the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on H; $B \in \mathbf{L}(\mathrm{U},\mathrm{H})$, $C \in \mathbf{L}(\mathrm{H},\mathrm{Y})$ where U, Y are Hilbert spaces with scalar products $\langle\cdot,\cdot\rangle_{\mathrm{U}}$, $\langle\cdot,\cdot\rangle_{\mathrm{Y}}$, respectively; $x_0 \in \mathrm{H}$ is a fixed element of H, $G \in \mathbf{L}(\mathrm{H},\mathrm{U})$ is an operator parameter describing the linear feedback $u = -Gx$.

Consider also the set

$$\Gamma = \Big\{G \in \mathbf{L}(\mathrm{H},\mathrm{U}) \colon \ ||y||^2_{\mathrm{L}^2(0,\infty;\mathrm{Y})} + ||u||^2_{\mathrm{L}^2(0,\infty;\mathrm{U})} < \infty \quad \forall x_0 \in \mathrm{H}\Big\} \tag{2}$$

Definition 1. The semigroup $\{S(t)\}_{t\geq 0}$ is called *exponentially stable* (**EXS**), if there exist $M \geq 1$, $\alpha > 0$ such that

$$||S(t)||_{\mathbf{L}(\mathrm{H})} \leq Me^{-\alpha t} \qquad \forall t \geq 0$$

Definition 2. The pair (A, B) is called *stabilizable* if the set

$$\Omega = \Big\{G \in \mathbf{L}(\mathrm{H},\mathrm{U}) \colon \ \text{the semigroup generated by } A - BG \text{ is } \mathbf{EXS}\Big\} \tag{3}$$

is not empty.

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Lemma 1. *Let (A, B) be stabilizable. Then*

(i) *Ω is an open set, $\Omega \subset \Gamma$.*

(ii) *The mapping $\Omega \ni G \longmapsto H(G) \in \mathcal{S}$ is well-defined, where $\mathcal{S} \subset \mathbf{L}(\mathrm{H})$ denotes the positive cone of all self-adjoint nonnegative operators and $H(G)$ is a unique solution to the Lyapunov operator equation*

$$\big\langle (A - BG)x_1, Hx_2 \big\rangle + \big\langle x_1, H(A - BG)x_2 \big\rangle$$

$$= -\langle Cx_1, Cx_2 \rangle_{\mathrm{Y}} - \langle Gx_1, Gx_2 \rangle_{\mathrm{U}} \qquad \forall x_1, x_2 \in D(A) \tag{4}$$

Moreover,

$$\big\langle x_0, H(G)x_0 \big\rangle = \int_0^\infty \Big[\|Cx(t)\|_{\mathrm{Y}}^2 + \|Gx(t)\|_{\mathrm{U}}^2 \Big] \mathrm{d}t \tag{5}$$

(iii) *For every $x_0 \in \mathrm{H}$, the mapping*

$$\Omega \ni G \longmapsto \|y\|^2_{\mathrm{L}^2(0,\infty;\mathrm{Y})} + \|u\|^2_{\mathrm{L}^2(0,\infty;\mathrm{U})} = \big\langle x_0, H(G)x_0 \big\rangle \in [0, \infty)$$

is continuous.

Proof. (i) Clearly, $\Omega \subset \Gamma$. If $H \in \mathbf{L}(\mathrm{H})$ is such that $\|H\|$ is sufficiently small, then by the fundamental perturbation result (see (Pazy, 1983, Th.1.1, p.76)) the type of the semigroup generated by $A - BG - BH$ is negative provided that the same holds for the semigroup generated by $A - BG$. This establishes (i).

(ii) This follows from (Grabowski, 1990, Th.3, p.322, Th.4, p.323).

(iii) We recall the result from (Pazy, 1983, Cor.1.3, p.78)

$$\big\|S_{G+H}(t) - S_G(t)\big\| \le M\varphi(t) \quad \forall t \ge 0, \qquad \varphi(t) := e^{(\omega + M\|B\| \ \|H\|)t} - e^{\omega t}, \quad t \ge 0$$

for some $M \ge 1$, where $\{S_{G+H}(t)\}_{t \ge 0}$, $\{S_G(t)\}_{t \ge 0}$ are the semigroups generated by $A - BG - BH$ and $A - BG$, respectively, and ω is the type of $\{S_G(t)\}_{t \ge 0}$. But, for $G \in \Omega$ and a sufficiently small $\|H\|$, the function φ belongs to $\mathrm{L}^2(0, \infty)$, and its $\mathrm{L}^2(0, \infty)$ norm tends to 0 as $\|H\|$ tends to 0. Hence the mapping $\Omega \ni G \longmapsto CS_G(\cdot)x_0 \in \mathrm{L}^2(0, \infty; \mathrm{Y})$ is continuous. Only minor modifications are required to prove that the same holds for the mapping $\Omega \ni G \longmapsto GS_G(\cdot)x_0 \in \mathrm{L}^2(0, \infty; \mathrm{U})$. ∎

Definition 3. The pair (A, C) is called *detectable* if there exists $Q \in \mathbf{L}(\mathrm{Y}, \mathrm{H})$ such that the semigroup generated by $A + QC$ is **EXS**.

Lemma 2. *Let (A, B) be stabilizable. Assume additionally that the pair (A, C) is detectable. Then*

(i) $\Omega = \Gamma$

(ii) The mapping

$$J: \mathbf{L}(\mathrm{H}, \mathrm{U}) \ni \mathrm{G} \longmapsto \left\{ \begin{array}{ll} \|y\|^2_{\mathrm{L}^2(0,\infty;\mathrm{Y})} + \|u\|^2_{\mathrm{L}^2(0,\infty;\mathrm{U})}, & G \in \Omega \\ +\infty, & G \notin \Omega \end{array} \right\} \in [0, \infty]$$

is continuous.

Proof. (i) It is sufficient to prove that $\Omega \supset \Gamma$. We take $G \in \Gamma$ and represent the first two lines of (1) in the form

$$\left\{ \begin{array}{l} \dot{x}(t) = (A + QC)x(t) - \left[QCx(t) + BGx(t)\right] \\ x(0) = x_0 \end{array} \right.$$

with $Q \in \mathbf{L}(\mathrm{Y}, \mathrm{H})$ chosen in such a manner that the semigroup $\{T(t)\}_{t \geq 0}$ generated by $A + QC$ is **EXS**. The existence of Q is ensured by the detectability of (A, Q). Indeed, employing the variation-of-constants formula, we get

$$\|x(t)\| \leq \|T(t)x_0\| + \max\{\|Q\|, \|B\|\} \int_0^t \|T(t-\tau)\| \left[\|Cx(\tau)\|_{\mathrm{Y}} + \|Gx(\tau)\|_{\mathrm{U}}\right] \mathrm{d}\tau$$

By the definition of Γ, $Cx(\cdot) \in \mathrm{L}^2(0, \infty; \mathrm{Y})$, $Gx(\cdot) \in \mathrm{L}^2(0, \infty; \mathrm{U})$. Hence, from the basic properties of convolution, it follows that $\|x(\cdot)\| \in \mathrm{L}^2(0, \infty)$ for all $x_0 \in \mathrm{H}$. The last property is equivalent to the exponential stability of the semigroup generated by $A - BG$ (Pazy, 1983, Th.4.1, p.116) and thus $G \in \Omega$.

(ii) By (i) we have $J(G) = \infty$ on $\mathbf{L}(\mathrm{H}, \mathrm{U}) \setminus \Omega$ (we may assume that $\mathbf{L}(\mathrm{H}, \mathrm{U}) \setminus \Omega \neq \emptyset$ as otherwise the result to be proved follows from Lemma 1(iii)) and, to show the continuity of J, it suffices to prove that $J(G)$ tends to ∞ as G tends to $\partial\Omega$ from the inside. Take any $R > 0$ and let $\{G_k\}_{k \in \mathbb{N}}$ be a sequence in Ω with $G_k \longrightarrow G_\infty \in \partial\Omega$ as $k \to \infty$. We claim that, for almost all $k \in \mathbb{N}$, we have $J(G_k) \geq R$. Observe that the function

$$[0, \infty) \ni t \longmapsto \|y_\infty\|^2_{\mathrm{L}^2(0,t;\mathrm{Y})} + \|u_\infty\|^2_{\mathrm{L}^2(0,t;\mathrm{U})}$$
$$= \int_0^t \left[\|Cx_\infty(\tau)\|^2_{\mathrm{Y}} + \|G_\infty x_\infty(\tau)\|^2_{\mathrm{U}}\right] \mathrm{d}\tau$$

where x_∞, y_∞, u_∞ denote respectively the state, output, and control functions due to G_∞, is nondecreasing and tends to ∞ as $t \to \infty$. Hence there exists $T > 0$ such that

$$\int_0^T \left[\|Cx_\infty(t)\|^2_{\mathrm{Y}} + \|G_\infty x_\infty(t)\|^2_{\mathrm{U}}\right] \mathrm{d}t = 2R$$

The mapping $\mathbf{L}(\mathrm{H},\mathrm{U}) \ni G \longmapsto \|y\|^2_{\mathrm{L}^2(0,T;\mathrm{Y})} + \|u\|^2_{\mathrm{L}^2(0,T;\mathrm{U})} \in [0,\infty)$ is continuous. Indeed, from (Pazy, 1983, Cor.1.3, p.78), we know that

$$\left\|S_{G+H}(t) - S_G(t)\right\| \leq M\varphi(t) \qquad \forall t \geq 0$$

where $\{S_{G+H}(t)\}_{t\geq 0}$, $\{S_G(t)\}_{t\geq 0}$ are the semigroups generated by $A - BG - BH$ and $A - BG$ respectively, and ω is the type of $\{S_G(t)\}_{t\geq 0}$. But the function φ belongs to $\mathrm{L}^2(0,T)$, and its $\mathrm{L}^2(0,T)$ norm tends to 0 as $\|H\|$ tends to 0. Hence the mappings

$$\mathbf{L}(\mathrm{H},\mathrm{U}) \ni G \longmapsto CS_G(\cdot)x_0 \in \mathrm{L}^2(0,T;\mathrm{Y})$$
$$\mathbf{L}(\mathrm{H},\mathrm{U}) \ni G \longmapsto GS_G(\cdot)x_0 \in \mathrm{L}^2(0,T;\mathrm{U})$$

are both continuous.

By the continuity of the mapping $\mathbf{L}(\mathrm{H},\mathrm{U}) \ni G \longmapsto \|y\|^2_{\mathrm{L}^2(0,T;\mathrm{Y})} + \|u\|^2_{\mathrm{L}^2(0,T;\mathrm{U})}$ just proved, for any $\varepsilon \in (0,R]$, we get

$$\left| \|y_\infty\|^2_{\mathrm{L}^2(0,T;\mathrm{Y})} + \|u_\infty\|^2_{\mathrm{L}^2(0,T;\mathrm{U})} - \|y_k\|^2_{\mathrm{L}^2(0,T;\mathrm{Y})} - \|u_k\|^2_{\mathrm{L}^2(0,T;\mathrm{U})}\right| \leq \varepsilon$$

where y_k and u_k denote respectively the output and control functions due to G_k, for almost all $k \in \mathbb{N}$. However, this implies that

$$J(G_k) = \|y_k\|^2_{\mathrm{L}^2(0,\infty;\mathrm{Y})} + \|u_k\|^2_{\mathrm{L}^2(0,\infty;\mathrm{U})} \geq \|y_k\|^2_{\mathrm{L}^2(0,T;\mathrm{Y})} + \|u_k\|^2_{\mathrm{L}^2(0,T;\mathrm{U})} \geq R$$

for almost all $k \in \mathbb{N}$, and the proof is complete. ■

2. The LQ Controller Synthesis Problem

Now we formulate the *parametric optimization problem* which consists in finding $G \in \Omega$ such that

$$\langle x_0, H(G)x_0\rangle = \min_{K\in\Omega} \langle x_0, H(K)x_0\rangle \qquad \forall x_0 \in \mathrm{H} \tag{6}$$

Theorem 1. *If (A,B) is stabilizable and (A,C) is detectable, then the problem (6) has a unique solution.*

Before starting the proof, let us remark that this is a well-known fundamental result concerning the LQ problem (see (Zabczyk, 1976) and (Curtain and Pritchard, 1978, Sec.4.4)), reformulated above as a parametric optimization problem. However, a new derivation of this result will be given. The main novelty, besides reformulation, is the simple explicit proof of convergence of the *Newton-Kleinman sequence* of stabilizing controllers.

Proof. Using (4), it is easy to show that, if $G \in \Omega$, then for each $F \in \mathbf{L}(\mathrm{H},\mathrm{U})$ such that $G + F \in \Omega$, the operator $\Delta = H(G+F) - H(G)$ is the unique bounded self-adjoint operator satisfying the operator equation

$$\langle (A - BG - BF)x_1, \Delta x_2\rangle + \langle x_1, \Delta(A - BG - BF)x_2\rangle = \langle x_1, [H(G)B - G^*]Fx_2\rangle$$
$$+ \langle [H(G)B - G^*Fx_1, x_2\rangle - \langle Fx_1, Fx_2\rangle_{\mathrm{U}} \qquad \forall x_1, x_2 \in D(A) \tag{7}$$

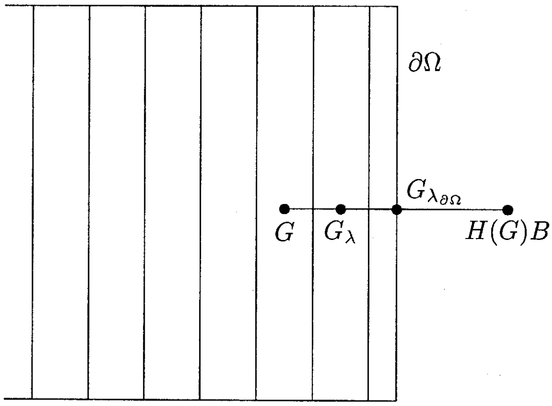

Fig. 1. An auxiliary diagram for the proof.

Now we show that the following implication holds:

$$G \in \Omega \Longrightarrow B^*H(G) \in \Omega \tag{8}$$

Suppose for a moment that, contrary to our claim, $B^*H(G) \notin \Omega$. Since Ω is an open set, there is $\lambda_{\partial\Omega} \in (0,1]$ such that (see Fig. 1)

$$G_\lambda = (1-\lambda)G + \lambda B^*H(G) \in \Omega \quad \text{for} \quad \lambda \in [0, \lambda_{\partial\Omega}) \quad \text{and} \quad G_{\lambda_{\partial\Omega}} \in \partial\Omega$$

Consequently, putting $F = G_\lambda - G = \lambda[B^*H(G) - G]$, $\lambda \in [0, \lambda_{\partial\Omega})$ in (7) we come to the conculsion that $\Delta = H(G_\lambda) - H(G)$ is a unique bounded, self-adjoint operator satisfying the operator equation

$$\begin{aligned}\langle (A - BG_\lambda)x_1, \Delta x_2\rangle + \langle x_1, \Delta(A - BG_\lambda)x_2\rangle \\ = (2\lambda - \lambda^2)\langle\, [H(G)B - G^*]\,[B^*H(G) - G]x_1, x_2\rangle\end{aligned}$$

for all $x_1, x_2 \in D(A)$ and all $\lambda \in [0, \lambda_{\partial\Omega})$. But $2\lambda - \lambda^2 \geq 0$ for $\lambda \in [0, \lambda_{\partial\Omega})$, and again by the results of (Grabowski, 1990, Th.3, p.322; Th.4, p.323), $(-\Delta) \geq 0$ (in the sense of quadratic forms). Hence the function

$$[0, \lambda_{\partial\Omega}) \ni \lambda \longmapsto \langle x_0, H(G_\lambda)x_0\rangle = \|y_\lambda\|^2_{L^2(0,\infty;Y)} + \|u_\lambda\|^2_{L^2(0,\infty;U)}$$

is bounded from above by $\langle x_0, H(G)x_0\rangle$, where $y_\lambda(t) = Cx_\lambda(t)$ and $u_\lambda(t) = Gx_\lambda(t)$, with x_λ denoting the solution of (1) with G replaced by G_λ. But, from Lemma 2(i), it follows that this function takes arbitrarily large values in a sufficiently small neighbourhood of $\lambda_{\partial\Omega}$. Hence our claim $B^*H(G) \notin \Omega$ leads to a contradiction, and thus (8) holds. By (8), the sequence $\{G_k\}_{k\in\mathbb{N}}$ given by

$$G_{k+1} = B^*H(G_k) \tag{9}$$

where G_1 is an arbitrary element of Ω, is well-defined and contained in Ω. Taking $G = G_k$, $F = G_{k+1} - G_k = B^*H(G_k) - G_k$ in (7), one obtains

$$\langle [A - BB^*H(G_k)]\, x_1, \Delta x_2 \rangle + \langle x_1, \Delta\, [A - BB^*H(G_k)]\, x_2 \rangle$$
$$= \langle x_1\, [H(G_k)B - G_k^*]\, [B^*H(G_k) - G_k]\, , x_2 \rangle \qquad \forall x_1, x_2 \in D(A), \quad \forall k \in \mathbb{N}$$

Applying once more the results from (Grabowski, 1990, Th.3, p.322; Th.4, p.323) we get $(-\Delta) \geq 0$. Thus the sequence of the terms

$$\langle x_0, H(G_k)x_0 \rangle = \|y_k\|^2_{\mathrm{L}^2(0,\infty;\mathrm{Y})} + \|u_k\|^2_{\mathrm{L}^2(0,\infty;\mathrm{U})}$$

is nonincreasing and bounded from below. Now, by standard arguments (Weidmann, 1980, Th.4.28, p.79) there exists $H_\infty \in \mathbf{L}(\mathrm{H})$, with $H_\infty = H_\infty^* \geq 0$, such that $H(G_k)x \longrightarrow H_\infty x$ as $k \to \infty$, for each $x \in \mathrm{H}$. Since $B^* \in \mathbf{L}(\mathrm{H}, \mathrm{U})$, we have

$$G_{k+1}x = B^*H(G_k)x \longrightarrow B^*H_\infty x = G_\infty x \qquad \forall x \in \mathrm{H} \tag{10}$$

By virtue of Lemma 2(ii),

$$\langle x_0, H(G_k)x_0 \rangle = \|y_k\|^2_{\mathrm{L}^2(0,\infty;\mathrm{Y})} + \|u_k\|^2_{\mathrm{L}^2(0,\infty;\mathrm{U})} \longrightarrow \|y_\infty\|^2_{\mathrm{L}^2(0,\infty;\mathrm{Y})}$$
$$+ \|u_\infty\|^2_{\mathrm{L}^2(0,\infty;\mathrm{U})} = \langle x_0, H_\infty x_0 \rangle < \infty$$

Hence $G_\infty \in \Omega$. Now we can apply Lemma 1(iii) to get

$$\langle x_0, H_\infty x_0 \rangle = \|y_\infty\|^2_{\mathrm{L}^2(0,\infty;\mathrm{Y})} + \|u_\infty\|^2_{\mathrm{L}^2(0,\infty;\mathrm{U})}$$
$$= \int_0^\infty \Big[\|Cx(t)\|^2_{\mathrm{Y}} + \|G_\infty x(t)\|^2_{\mathrm{U}} \Big]\, dt = \langle x_0, H(G_\infty)x_0 \rangle \qquad \forall x_0 \in \mathrm{H}$$

This means that H_∞ satisfies (4) with $G = G_\infty$, i.e.

$$\langle (A - BG_\infty)x_1, H_\infty x_2 \rangle + \langle x_1, H_\infty(A - BG_\infty)x_2 \rangle$$
$$= -\langle Cx_1, Cx_2 \rangle_{\mathrm{Y}} - \langle G_\infty x_1, G_\infty x_2 \rangle_{\mathrm{U}} \qquad \forall x_1, x_2 \in D(A) \tag{11}$$

Substituting $G = G_\infty$ in (7), for any $F \in \mathbf{L}(\mathrm{H}, \mathrm{U})$ such that $G_\infty + F \in \Omega$, we get

$$\langle (A - BG_\infty - BF)x_1, \Delta x_2 \rangle + \langle x_1, \Delta(A - BG_\infty - BF)x_2 \rangle$$
$$= -\langle Fx_1, Fx_2 \rangle_{\mathrm{U}} \qquad \forall x_1, x_2 \in D(A)$$

Recalling again the results from (Grabowski, 1990, Th.3, p.323; Th.4, p.322) we come to the inequality $H(G_\infty + F) \geq H(G_\infty)$, and thus G_∞ is a solution of (6). Moreover, from (11) and (Grabowski, 1990, Th.5, p.324) it follows that H_∞ is a *Hilbert-Schmidt operator* (**HS** operator) provided that G and C are finite-rank operators. ■

Remark 1. The infinite-dimensional version of the Kleinman algorithm was used for the first time in (Curtain and Rodman, 1990) to prove that (11) has a maximal bounded self-adjoint positive solution (being the limit of the Kleinman sequence), provided that (A, B) is only stabilizable.

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