

A NEW MULTIRATE LQ OPTIMAL REGULATOR FOR LINEAR TIME-INVARIANT SYSTEMS AND ITS STABILITY ROBUSTNESS PROPERTIES

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In the present paper, the LQ optimal regulation problem for continuous-time systems is solved by using a new class of multirate controllers, called Two-Point-Multirate Controllers (TPMRCs). In such a type of controllers, the control is constrained to a certain piecewise constant signal, while each of the controlled plant outputs is detected many times over a fundamental sampling period T_0 . The proposed control strategy is readily applicable in the cases where the state variables of the controlled plant are not available for feedback, since TPMRCs provide the ability to reconstruct exactly the action of static state feedback controllers from input-output data, without resorting to state estimators, and without introducing high-order exogenous dynamics in the control loop. On the basis on this strategy, the original problem is reduced to a discrete-time LQ regulation problem for the performance index with cross-product terms (LQRCPT), for which a fictitious static state-feedback controller is needed to be computed. Moreover, the stability robustness properties of the TPMRC-based LQ regulator are analysed. In particular, guaranteed stability margins for TPMRC-based LQ optimal regulators are derived on the basis of a fundamental spectral factorization equality called the Modified Return Difference Equality. The suggested guaranteed stability margins are expressed directly in terms of the singular values of the elementary cost and system matrices associated with the equivalent discrete-time LQRCPT optimal design. Sufficient conditions to guarantee the suggested stability margins are established. Finally, the connection between the suggested stability margins and the selection of cost weighting matrices is presented.

1. Introduction

Multirate sampling schemes have long been a focus of interest to many control designers. There are several reasons to use such a sampling scheme in digital control systems. First of all, in complex multivariable control systems, it is often unrealistic (or sometimes impossible) to sample all the physical signals uniformly at one single rate. In such situations, we are forced to use multirate sampling. Furthermore, in

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general, we get better performance if we can sample and hold faster. But faster A/D and D/A conversions mean a higher cost in implementation. For signals with different bandwidths, better trade-offs between the performance and the implementation cost can be obtained using A/D and D/A converters at different rates. On the other hand, multirate controllers are in general time-varying. Thus multirate control systems can achieve what singlerate ones cannot, e.g. gain improvement, simultaneous stabilization and decentralized control. Finally, multirate controllers are normally more complex than singlerate ones. But they often are finite-dimensional and periodic in a certain sense and hence can be implemented on microprocessors via difference equations with finitely-many coefficients. Therefore, like singlerate controllers, multirate controllers do not violate the finite memory constraint in microprocessors.

The study of multirate systems has its origins in the late 1950s (Jury and Mullin, 1959; Kalman and Bertram, 1959; Kranc, 1957). Recent interests are focused on analysis issues (Boykin and Frazier, 1975; Glasson, 1983; Kalman and Bertram, 1959; Meyer and Burrus, 1975), stability issues (Araki and Yamamoto, 1986), stabilization and pole assignment (Araki and Hagiwara, 1986; Colaneri *et al.*, 1990; Hagiwara and Araki, 1988; Meyer, 1990a; 1990b; Ravi *et al.*, 1990), LQG/LQR designs (Al-Rahmani and Franklin, 1990; 1992; Apostolakis and Jordan, 1991; Berg *et al.*, 1988; Chen and Francis, 1991; Meyer, 1992), H^∞ -control (Arvanitis and Paraskevopoulos, 1995a; Chen and Qiu, 1994; Voulgaris and Bamieh, 1993), decentralized control (Sezer and Siljak, 1990), adaptive designs (Arvanitis, 1995a; 1995b; 1996a; 1996b; 1996c; 1996d; 1996e; Arvanitis and Kalogeropoulos, 1997; Arvanitis and Paraskevopoulos, 1995b), and others (Arvanitis and Kalogeropoulos, 1998a; 1998b; Arvanitis and Paraskevopoulos, 1993). In particular, in their excellent works, Al-Rahmani and Franklin (1990; 1992) introduce an optimal multirate design for solving the continuous optimal LQ regulation problem for linear periodic and time-invariant systems. A main feature of this approach is that the control is constrained to a certain piecewise constant signal. The solution of the continuous LQ regulation problem is then obtained by feeding back the states of the plant under control, which are sampled on the basis of a fundamental sampling period T_0 . Although this approach has several distinct advantages when compared with singlerate ones (e.g. better response characteristics, smaller LQ regulation costs and more flexibility in sampling rate selection), its main drawback is the need for the complete availability of system states which, in general, are rarely known. It is worth noticing that, in the case where only output measurements are available, in order to apply the technique reported in (Al-Rahmani and Franklin, 1990; 1992), one can use, instead of the sampled system states, their estimates obtained from any classical state-estimation method. However, estimator-based controllers have their own disadvantages. First of all, the number of states of the estimator and that of the system must be, in most cases, the same and the estimator must run on-line. When the controlled system is of high order, this implies high computation rates in the controller. On the other hand, whereas continuous state feedback methods are able to meet the robustness objective, it was shown (Doyle, 1978) that the introduction of a state estimator negates this advantage. Even though some level of robustness may be recovered, the robustness using dynamic compensators and estimator-based controllers is still an open research topic. In conclusion, the technique presented in

(Al-Rahmani and Franklin, 1990; 1992) has limited usefulness in the cases where the state variables of the system under control are not available.

In the present paper, the LQ optimal regulation problem for continuous-time systems is solved using an alternative approach which is based on a control strategy that is essentially a combination of the control strategies reported in (Arvanitis, 1996b; 1996e; Al-Rahmani and Franklin, 1990; 1992). We refer to this novel control strategy as a Two-Point-Multirate Controller (abbreviated here as TPMRC). TPMRCs constitute a rather typical multirate control strategy where the control (actuators) updates are performed at different rates than the output samples (see (Apostolakis and Jordan, 1991; Araki and Yamamoto, 1986; Berg *et al.*, 1988) for alternative multirate schemes of this type). However, the main difference between TPMRCs and other multirate techniques results from the ability of TPMRCs to transform the original multirate design into a simplified single-rate-like design, for an equivalent discrete time-invariant state-space model of the multirate digital control system, whose dimensions are considerably lower than the dimensions of the respective state-space models utilized by other well-known multirate techniques (Apostolakis and Jordan, 1991; Araki and Yamamoto, 1986; Berg *et al.*, 1988). Furthermore, under mild conditions, TPMRCs provide the ability of the exact reconstruction of the action of a static state feedback controller, without resorting to the design of state estimators, and without introducing high-order exogenous dynamics in the control loop. On the basis of the proposed approach, the original LQ regulation problem is reduced to an associated discrete-time LQ regulation problem for the performance index with cross-product terms (LQRCPT), for which a fictitious static state feedback controller is needed to be computed. The parameters of the TPMRC sought can easily be obtained on the basis of this fictitious controller. Thus, in the case where the state variables of the control plant are not available, the present technique essentially resorts to the computation of simple gain controllers, rather than to the computation of state observers when compared with known techniques. Finally, the designed TPMRC-based LQ regulators can possess any prescribed degree of stability, since there is a possibility to choose the transition matrices of the controllers arbitrarily.

Moreover, some stability robustness properties of the TPMRC-based LQ regulator are investigated. In particular, our concern is to analyse how sensitive the stable modes of the closed-loop system will be under small variations of the plant parameters and, in particular, whether these modes will remain inside the unit circle for such variations. To this end, guaranteed stability margins for the TPMRC-based LQ regulator are derived as measures of its stability robustness. These margins are expressed explicitly in terms of the singular values of the elementary cost and system matrices associated with the state-space description of the afore-mentioned equivalent discrete-time LQRCPT design. The suggested guaranteed stability margins are obtained on the basis of lower bounds on the minimum singular value of the return difference matrix of the regulator's feedback loop. This return difference matrix obeys a fundamental spectral factorization equality, called the Modified Return Difference Equality, which is instrumental for our investigation. Furthermore, sufficient conditions to guarantee the suggested stability margins are established. Finally, a connection between the suggested stability margins and the selection of the cost-weighting matrices is

presented. It is worth noticing that our investigation on guaranteed stability margins of the TPMRC-based LQ regulator is focused on a broad variety of important particular cases for the matrices of the state-space description of the equivalent LQR-CPT design. The reason for such a type of investigation is due to the fact that, in our case, it is very difficult (if not impossible) to obtain a universal lower bound for the minimum singular value of the regulator's return difference matrix, as was the case for the continuous-time regulator (Lehtomaki *et al.*, 1981). This difficulty results from the entanglement of the solution to the Riccati equation (which is associated with the equivalent LQ-CPT design), as well as of the matrices representing the aforementioned cross-product terms of the equivalent performance index on the left- and right-hand sides of the Modified Return Difference Equality, respectively. From this point of view, our analysis appears to have some analogy to the analysis reported in (Chung *et al.*, 1994) for the continuous-time LQR-CPT. However, our results are completely different when compared with those reported in (Chung *et al.*, 1994) because of fundamental differences between the Return Difference Equalities fulfilled by continuous and discrete-time LQ regulators.

2. Preliminaries and Problem Formulation

Throughout this paper, we consider continuous-time, linear, time-invariant, multi-input, multi-output (MIMO) systems described in a state space by the following equations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \quad (1)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state, $\mathbf{u}(t) \in \mathbb{R}^m$ is the input and $\mathbf{y}(t) \in \mathbb{R}^p$ is the output of the system and where all the matrices have real entries and appropriate dimensions. It is further assumed that system (1) is controllable (stabilizable) and observable (detectable).

The following definitions will be useful in the sequel.

Definition 1. Let \mathbf{c}_i^T , $i = 1, 2, \dots, p$ be the i -th row of the matrix \mathbf{C} . For an observable matrix pair (\mathbf{A}, \mathbf{C}) , a collection of p integers $\{n_1, n_2, \dots, n_p\}$ is called an *observability index vector* of the pair (\mathbf{A}, \mathbf{C}) if the following relationships hold simultaneously:

$$\sum_{i=1}^p n_i = n, \quad \text{rank} \left[\mathbf{c}_1 \cdots (\mathbf{A}^T)^{n_1-1} \mathbf{c}_1 \cdots \mathbf{c}_p \cdots (\mathbf{A}^T)^{n_p-1} \mathbf{c}_p \right] = n$$

Definition 2. The *generalized reachability Grammian* of order N on the interval $[0, T_0]$ is defined by

$$\mathbf{W}_N(T_0, 0) = T_N^{-1} \sum_{\mu=0}^{N-1} \Delta_\mu \Delta_\mu^T \quad (2)$$

for

$$T_N = \frac{T_0}{N}, \quad \Delta_\mu \triangleq \widehat{A}_N^{N-\mu-1} \widehat{B}_{T_N} \tag{3}$$

and where, in (3),

$$\widehat{A}_N = \exp(AT_N), \quad \widehat{B}_{T_N} = \int_0^{T_N} \exp(A\tau) B \, d\tau \tag{4}$$

Now, define

$$p_N = \text{rank } W_N(T_0, 0)$$

Since $W_N(T_0, 0) \geq 0$, we can always find (perhaps not uniquely) an $n \times p_N$ full-rank matrix B_N such that

$$W_N(T_0, 0) = B_N B_N^T \tag{5}$$

It is worth noticing at this point that B_N can be computed as follows:

- a) If $W_N(T_0, 0)$ is positive definite, then B_N can be obtained from the Cholesky factorization of $W_N(T_0, 0)$. That is, if \widehat{S} is an upper-triangular full-rank matrix obtained from the Cholesky factorization of $W_N(T_0, 0)$ (i.e. \widehat{S} is such that $\widehat{S}^T \widehat{S} = W_N(T_0, 0)$) and U_S is a unitary matrix (i.e. $U_S^T U_S = I$), then

$$B_N = \widehat{S}^T U_S^T \tag{6}$$

- b) If $W_N(T_0, 0)$ is positive semi-definite, then we can proceed as follows: Let a Singular Value Decomposition of $W_N(T_0, 0)$ be defined as

$$W_N(T_0, 0) = U \begin{bmatrix} \Sigma^* & 0 \\ 0 & 0 \end{bmatrix} V^T$$

where $U \in \mathbb{R}^{n \times n}$, $V \in \mathbb{R}^{n \times n}$ are unitary matrices and $\Sigma^* \in \mathbb{R}^{p_N \times p_N}$ is defined by $\Sigma^* = \text{diag} \{ \sigma_1, \sigma_2, \dots, \sigma_{p_N} \}$, where σ_j , $j = 1, 2, \dots, p_N$ are the nonzero singular values of $W_N(T_0, 0)$. Since, by definition, $W_N(T_0, 0)$ is a symmetric matrix, we have $U = V$. Let M_{Σ^*} be an upper-triangular full-rank matrix obtained from the Cholesky factorization of Σ^* (i.e. $M_{\Sigma^*}^T M_{\Sigma^*} = \Sigma^*$). Now, let $M_W \in \mathbb{R}^{p_N \times n}$ be constructed as

$$M_W = \begin{bmatrix} M_{\Sigma^*} & 0_{p_N \times (n-p_N)} \end{bmatrix}$$

Then

$$M_W^T M_W = \begin{bmatrix} \Sigma^* & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$W_N(T_0, 0) = UM_W^T M_W U^T$$

Consequently,

$$B_N = UM_W^T V^T \tag{7}$$

In the sequel, we set

$$\Phi = \exp(AT_0)$$

Let B_N^r be the $n \times p_N$ full-rank matrix defined as the right pseudoinverse of B_N^T , i.e.

$$B_N^r = B_N (B_N^T B_N)^{-1} \tag{8}$$

Consider now applying the multirate control strategy depicted in Fig. 1 to system (1). In particular, we assume that all the samplers start simultaneously at $t = 0$. The hold circuits H_0 and H_N are zero order holds with holding times T_0 and T_N , respectively. The inputs of the plant are constrained to the following piecewise constant controls:

$$u(kT_0 + \mu T_N + \zeta) = T_N^{-1} \Delta_\mu^T B_N^r \hat{u}(kT_0), \quad \hat{u}(kT_0) \in \mathbb{R}^{p_N} \tag{9}$$

for $t = kT_0 + \mu T_N$, $\mu = 0, 1, \dots, N - 1$, $k \geq 0$ and $\zeta \in [0, T_N)$.

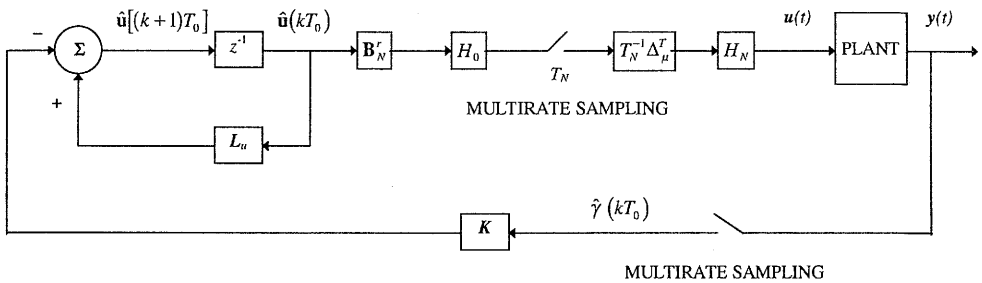


Fig. 1. Control of linear systems Two-Point Multirate Controllers.

The i -th plant output $y_i(t)$ is detected at every $T_i = T_0/M_i$ such that

$$y_i(kT_0 + \rho T_i) = c_i^T x(kT_0 + \rho T_i), \quad \rho = 0, 1, \dots, M_i - 1 \tag{10}$$

where $M_i \in \mathbb{Z}^+$, $i = 1, 2, \dots, p$ are the so-called output multiplicities of the sampling.

It is worth noticing that, in general, $M_i \neq N$. That is, multirate sampling of the plant inputs and outputs may be performed at a different rate.

The sampled values of the plant outputs obtained over $[kT_0, (k+1)T_0)$ are stored in the M^* -dimensional column vector $\hat{\gamma}(kT_0)$ of the form

$$\hat{\gamma}(kT_0) = \left[y_1(kT_0) \cdots y_1(kT_0 + (M_1 - 1)T_1) \cdots y_p(kT_0) \cdots y_p(kT_0 + (M_p - 1)T_p) \right]^T$$

where

$$M^* = \sum_{i=1}^p M_i$$

The vector $\hat{\gamma}(kT_0)$ is used in the discrete dynamic control law of the form

$$\hat{u}[(k+1)T_0] = L_u \hat{u}(kT_0) - K \hat{\gamma}(kT_0) \tag{11}$$

where $L_u \in \mathbb{R}^{pN \times pN}$, $K \in \mathbb{R}^{pN \times M^*}$.

In the sequel, the multirate control strategy described above will be referred to as the Two-Point-Multirate Controller (abbreviated here as TPMRC).

The multirate optimal scheme suggested in this paper is based on solving the continuous-time LQ regulation problem with the control strategy of Fig. 1. More precisely, the control objective is to find an optimal $u(t)$ constrained by (9) and (11), which, given the initial value $y(0)$, minimizes the cost function

$$J = \frac{1}{2} \int_0^\infty \left[y^T(t) Q y(t) + u^T(t) \Gamma u(t) \right] dt \tag{12}$$

Note that $Q \in \mathbb{R}^{p \times p}$ and $\Gamma \in \mathbb{R}^{m \times m}$ are symmetric matrices with $Q \geq 0$ and $\Gamma > 0$, while $(A, C^T Q C)$ is an observable(detectable) pair.

3. LQ Optimal Regulation Using TPMRCs

In this section, we present a solution to the afore-mentioned LQ optimal regulation problem via TPMRCs. To this end, observe first that since $u(t)$ is a function of $\hat{u}(kT_0)$, the problem under consideration is essentially that of finding an optimal $\hat{u}_{opt}(kT_0) \in \mathbb{R}^{pN}$, $k \geq 0$ which minimizes (12). Moreover, since $\hat{u}(kT_0)$ obeys (11), the LQ optimal regulation problem considered here can be reduced to the determination of the optimal gains L_u and K which minimize (12). These optimal gains can be determined by using the following procedure:

Observe first that the following relationship holds (see (Al-Rahmani and Franklin, 1990) for its derivation):

$$x(kT_0 + \mu T_N + \zeta) = \exp\{A(\mu T_N + \zeta)\} x(kT_0) + B_N^*(\mu, \zeta) \hat{u}(kT_0) \tag{13}$$

where

$$B_N^*(\mu, \zeta) = T_N^{-1} \left\{ \exp(A\zeta) V_\mu^T + \hat{B}_\zeta \Delta_\mu^T \right\} B_N^r \tag{14}$$

In (14), the matrices V_μ and \widehat{B}_ζ are defined as follows:

$$V_\mu = \widehat{A}_N^{N-\mu} \Theta_\mu(T_N) \Theta_\mu^T(T_N), \quad \widehat{B}_\zeta = \int_0^\zeta \exp(A\lambda) B \, d\lambda \tag{15}$$

where

$$\Theta_\mu(T_N) = [\widehat{B}_{T_N} \quad \widehat{A}_N \widehat{B}_{T_N} \quad \dots \quad \widehat{A}_N^{\mu-1} \widehat{B}_{T_N}]$$

It is pointed out that

$$B_N^*(N-1, T_N) = B_N$$

Therefore, at the sampling instants $t = kT_0$, we can easily obtain

$$x[(k+1)T_0] = \Phi x(kT_0) + B_N \widehat{u}(kT_0) \tag{16}$$

Let also note that, at every $t = kT_0 + \rho T_i$, $\rho = 0, 1, \dots, M_i - 1$, we have

$$x(kT_0 + \rho T_i) = \widehat{A}_i^\rho x(kT_0) + B_{M_i}^*(\rho) \widehat{u}(kT_0) \tag{17}$$

where

$$\widehat{A}_i = \exp(AT_i), \quad B_{M_i}^*(\rho) = T_N^{-1} E_{M_i}(\rho) B_N^r \tag{18}$$

and

$$E_{M_i}(\rho) = \sum_{j=0}^{a(i,\rho)-1} \int_{jT_N}^{(j+1)T_N} \exp\left\{A\left(\rho \frac{N}{M_i} T_N - \xi\right)\right\} B \, d\xi \Delta_j^T + \int_{a(i,\rho)T_N}^{\rho T_N N/M_i} \exp\left\{A\left(\rho \frac{N}{M_i} T_N - \xi\right)\right\} B \, d\xi \Delta_{a(i,\rho)}^T \tag{19}$$

with

$$a(i, \rho) = \text{INT}_S \left(\rho \frac{N}{M_i} \right)$$

$\text{INT}_S(\nu)$ being the greatest integer that is less than or equal to $\nu \in \mathbb{R}^+$.

Now, define the following matrices:

$$\widetilde{Q}_N = \int_0^{T_0} \exp(A^T \xi) C^T Q C \exp(A \xi) \, d\xi \equiv \sum_{\mu=0}^{N-1} (\widehat{A}_N^\mu)^T \Xi(T_N) \widehat{A}_N^\mu \tag{20}$$

$$\widetilde{G}_N = T_N^{-1} \left\{ \sum_{\mu=0}^{N-1} (\widehat{A}_N^\mu)^T \left[\Xi(T_N) V_\mu^T + \Lambda(T_N) \Delta_\mu^T \right] \right\} B_N^r \tag{21}$$

$$\widetilde{\Gamma}_N = T_N^{-2} (B_N^r)^T \left\{ \sum_{\mu=0}^{N-1} [V_\mu \quad \Delta_\mu] \begin{bmatrix} \Xi(T_N) & \Lambda(T_N) \\ \Lambda^T(T_N) & N(T_N) + T_N \Gamma \end{bmatrix} \begin{bmatrix} V_\mu^T \\ \Delta_\mu^T \end{bmatrix} \right\} B_N^r \tag{22}$$

where

$$\Xi(T_N) = \int_0^{T_N} \exp(\mathbf{A}^T \xi) \mathbf{C}^T \mathbf{Q} \mathbf{C} \exp(\mathbf{A} \xi) d\xi \tag{23}$$

$$\Lambda(T_N) = \int_0^{T_N} \exp(\mathbf{A}^T \xi) \mathbf{C}^T \mathbf{Q} \mathbf{C} \widehat{\mathbf{B}}_\xi d\xi \tag{24}$$

$$\mathbf{N}(T_N) = \int_0^{T_N} \widehat{\mathbf{B}}_\xi^T \mathbf{C}^T \mathbf{Q} \mathbf{C} \widehat{\mathbf{B}}_\xi d\xi \tag{25}$$

It is pointed out that the matrices $\Xi(T_N)$, $\Lambda(T_N)$ and $\mathbf{N}(T_N)$ (and consequently $\widetilde{\mathbf{Q}}_N$, $\widetilde{\mathbf{G}}_N$ and $\widetilde{\mathbf{\Gamma}}$) can be easily computed on the basis of the algorithm reported in (Van Loan, 1978).

Now, substituting (1), (9) and (13) in (12) and taking into account (20)–(25), we finally obtain

$$J = \frac{1}{2} \sum_{k=0}^{\infty} \begin{bmatrix} \mathbf{x}^T(kT_0) & \widehat{\mathbf{u}}^T(kT_0) \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{Q}}_N & \widetilde{\mathbf{G}}_N \\ \widetilde{\mathbf{G}}_N^T & \widetilde{\mathbf{\Gamma}}_N \end{bmatrix} \begin{bmatrix} \mathbf{x}(kT_0) \\ \widehat{\mathbf{u}}(kT_0) \end{bmatrix} \tag{26}$$

From the previous analysis, it becomes clear that the original optimal LQ regulation problem has been reduced to an associated LQ optimal regulation problem for the performance index with cross-product terms (abbreviated here as LQRCPT), namely the problem of finding a TPMRC of the form (9), (11), which minimizes the performance index (26) subject to the dynamic constraints defined by (16).

In the sequel, the nature of the control law (11) will be explained. To this end, we establish the following fundamental theorems.

Theorem 1. *The following basic formula of the multirate sampling mechanism holds:*

$$\mathbf{H} \mathbf{x} \left[(k + 1)T_0 \right] = \widehat{\gamma}(kT_0) - \mathbf{D} \widehat{\mathbf{u}}(kT_0), \quad k \geq 0 \tag{27}$$

where the matrices $\mathbf{H} \in \mathbb{R}^{M^* \times n}$ and $\mathbf{D} \in \mathbb{R}^{M^* \times pN_0}$ are defined as

$$\mathbf{H} = \begin{bmatrix} \mathbf{c}_1^T (\widehat{\mathbf{A}}_1^{M_1})^{-1} \\ \vdots \\ \mathbf{c}_1^T \widehat{\mathbf{A}}_1^{-1} \\ \vdots \\ \mathbf{c}_p^T (\widehat{\mathbf{A}}_p^{M_p})^{-1} \\ \vdots \\ \mathbf{c}_p^T \widehat{\mathbf{A}}_p^{-1} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \mathbf{c}_1^T \widehat{\mathbf{B}}_{1,0} \\ \vdots \\ \mathbf{c}_1^T \widehat{\mathbf{B}}_{1,M_1-1} \\ \vdots \\ \mathbf{c}_p^T \widehat{\mathbf{B}}_{p,0} \\ \vdots \\ \mathbf{c}_p^T \widehat{\mathbf{B}}_{p,M_p-1} \end{bmatrix} \tag{28}$$

and

$$\widehat{B}_{i,\rho} = B_{M_i}^*(\rho) - \widehat{A}_i^{\rho-M_i} B_N, \quad i = 1, 2, \dots, p, \quad \rho = 0, 1, \dots, M_i - 1 \quad (29)$$

Proof. Solving (16) for $x(kT_0)$ and substituting its solution in (17), we obtain

$$x(kT_0 + \rho T_i) = \widehat{A}_i^{\rho-M_i} x[(k+1)T_0] + \widehat{B}_{i,\rho} \widehat{u}(kT_0) \quad (30)$$

where we have made use of the fact that $\Phi \equiv \widehat{A}_i^{M_i}$. Introducing (30) in (10), yields

$$y_i(kT_0 + \rho T_i) = c_i^T \widehat{A}_i^{\rho-M_i} x[(k+1)T_0] + c_i^T \widehat{B}_{i,\rho} \widehat{u}(kT_0) \quad (31)$$

for $\rho = 0, 1, \dots, M_i - 1$. Moving the terms containing $x[(k+1)T_0]$ to the left-hand side, moving the terms of $y_i(kT_0 + \rho T_i)$ to the right and expressing the equations for $\rho = 0, 1, \dots, M_i - 1$ in a matrix form, we finally obtain (27). ■

Theorem 2. (Hagiwara and Araki, 1988) *Let $n_i, i = 1, 2, \dots, p$ be positive integers which comprise an observability index vector of the observable pair (A, C) . If $M_i, i = 1, 2, \dots, p$ are chosen such that $M_i \geq n_i$, then the matrix H has full column rank.*

Theorem 3. *Let (A, C) be an observable pair and suppose that $M_i \geq n_i, i = 1, 2, \dots, p$. Then, for almost every sampling period T_0 , we can make the control law (11) equivalent to any static state feedback control law of the form*

$$\widehat{u}(kT_0) = -F x(kT_0) \quad \text{for } k \geq 1 \quad (32)$$

by suitably choosing the controller pair (K, L_u) such that

$$KH = F, \quad L_u = KD \quad (33)$$

Proof. Pre-multiplying (27) by K , we obtain

$$KHx[(k+1)T_0] = K\widehat{\gamma}(kT_0) - KD\widehat{u}(kT_0), \quad k \geq 0$$

Therefore the control law (11) becomes equivalent to the state feedback law (32) if, for the matrix K , the first of eqns. (33) holds and if we evaluate L_u by the second of (33). From Theorem 2, the matrix H has full column rank if we select $N_i \geq n_i, i = 1, 2, \dots, p$, so for almost every T_0 there exists a matrix K , which fulfils (33). ■

Theorem 4. *Let (A, C) be an observable pair and suppose that for some $M_i = n_i^*, i = 1, 2, \dots, p$ such that $M^* \geq n + p_N$, the matrix $[H D]$ has full column rank. Then, for almost every sampling period T_0 , there exists a matrix K such that*

$$K[H D] = [F L_u] \quad (34)$$

where F is an arbitrarily specified matrix corresponding to any desired state feedback and L_u is an arbitrarily specified matrix corresponding to the desired state transition matrix of the controller (11) itself.

Proof. From (33) and for L_u having a prespecified value we obtain (34). If for some $M_i = n_i^*$, $i = 1, 2, \dots, p$ such that $M^* \geq n + p_N$ the matrix $[H \ D]$ has full column rank, then (34) is solvable with respect to K for almost every sampling period T_0 . ■

Remark 1. In Theorems 3 and 4, the term ‘for almost every sampling period T_0 ’ is used to express briefly the fact that the assertion fails only at isolated values of T_0 .

From the previous analysis, it becomes clear that we can equivalently realize any desired static state feedback matrix F by a dynamic controller of the form (11) which possesses any prescribed degree of stability, since we can choose the matrix L_u (which corresponds to the transition matrix of the controller itself) arbitrarily. The choice $L_u = 0$ is of course permitted, which leads to static TPMRCs of the form

$$\hat{u} \left[(k + 1)T_0 \right] = -K\hat{\gamma}(kT_0) \tag{35}$$

It becomes also clear that, in order to find a control law of the form (11) which minimizes the performance index (26), one has essentially to refer to an easier problem, i.e. to the design of a *fictitious* static state feedback law of the form (32), which has an equivalent action. The calculation of the matrix pair (K, L_u) is then performed by using either (33) or (34), after choosing a desired (usually stable) matrix L_u .

A state feedback law of the form (32) which minimizes the index (26) is well-known to be (Ogata, 1987)

$$F = \left(\tilde{\Gamma}_N + B_N^T P B_N \right)^{-1} \left(\tilde{G}_N + B_N^T P \Phi \right) \tag{36}$$

where P is the symmetric positive-definite solution of the following discrete algebraic Riccati equation:

$$P = \Phi^T P \Phi + \tilde{Q}_N - \left(\tilde{G}_N + \Phi^T P B_N \right) \left(\tilde{\Gamma}_N + B_N^T P B_N \right)^{-1} \left(\tilde{G}_N^T + B_N^T P \Phi \right) \tag{37}$$

Note that the solvability of (37) and the asymptotic stability of the corresponding closed-loop system are assured by the following lemmas whose proofs can be found in (Al-Rahmani and Franklin, 1990).

Lemma 1. *The matrix $\tilde{\Gamma}_N$ is positive definite if $\Gamma > 0$ or $C^T Q C > 0$.*

Lemma 2. *Define*

$$\hat{Q}_N = \tilde{Q}_N - \tilde{G}_N \tilde{\Gamma}_N^{-1} \tilde{G}_N^T$$

Then the matrix \hat{Q}_N is positive semi-definite.

Lemma 3. *There exists a unique positive definite solution P to (37) and the corresponding closed-loop system with the closed-loop system matrix*

$$\Phi_{cl} = \Phi - B_N F \equiv \Phi - B_N K H \quad (38)$$

is asymptotically stable if (and only if) (A, B) is controllable (stabilizable) and $(A, C^T Q C)$ is observable (detectable).

If a fictitious state feedback matrix F has been determined on the basis of (36), the TPMRC matrix pair (K, L_u) , in the case where L_u is not prespecified, can be obtained as follows:

Case I. ($M_i = n_i$) In this case, H is nonsingular. Therefore

$$K = \left(\tilde{\Gamma}_N + B_N^T P B_N \right)^{-1} \left(\tilde{G}_N + B_N^T P \Phi \right) H^{-1}$$

and

$$L_u = \left(\tilde{\Gamma}_N + B_N^T P B_N \right)^{-1} \left(\tilde{G}_N + B_N^T P \Phi \right) H^{-1} D \quad (39)$$

Case II. ($M_i > n_i$) We have

$$K = \left(\tilde{\Gamma}_N + B_N^T P B_N \right)^{-1} \left(\tilde{G}_N + B_N^T P \Phi \right) H^1 \quad (40)$$

and

$$L_u = \left(\tilde{\Gamma}_N + B_N^T P B_N \right)^{-1} \left(\tilde{G}_N + B_N^T P \Phi \right) H^1 D \quad (41)$$

where H^1 is the left pseudoinverse of H (i.e. $H^1 H = I$).

Similarly, in the case where L_u is desired to have a prespecified value, one can easily obtain

$$K = \left[\left(\tilde{\Gamma}_N + B_N^T P B_N \right)^{-1} \left(\tilde{G}_N + B_N^T P \Phi \right) L_u \right] \widehat{H}^1 \quad (42)$$

where \widehat{H}^1 is the left pseudoinverse of $[H D]$.

Although, under the mild conditions of controllability (stabilizability) and observability (detectability) of the controlled plant, which are summarized in Lemma 3, the TPMRC-based LQ regulator is stable for normal values of the plant parameters, it is not at all clear what its stability margins are and how sensitive its stability will be to small variations in these parameters. In this respect, our basic aim in the sequel is to address this important concern. Our investigation is based on the behaviour of the minimum singular value of the regulator's return difference matrix which obeys a fundamental spectral factorization equality exhibited in the next section.

4. The Modified Return Difference Equality

As was already mentioned, our purpose in the sequel is to study the stability robustness properties of the TPMRC-based LQ optimal regulator proposed in the previous section. In particular, we are mainly interested in establishing guaranteed stability margins for such regulators in the cases of perturbations of the loop transfer function matrix. The minimum inward and upward gain margins of the optimal regulator are defined, in general, to be positive scalars GM_{in} and GM_{up} for which a simultaneous insertion of the gains g_i , $i = 1, 2, \dots, m$ in the i -th feedback loop of the closed-loop regulator will not destabilize the closed-loop system if

$$GM_{in} \leq g_i \leq GM_{up}$$

Similarly, the guaranteed phase margin of the regulator is defined to be a scalar PM for which a simultaneous insertion of the phase factor $e^{j\phi}$, $i = 1, 2, \dots, m$ in the above i -th feedback loop will keep the closed loop stable if

$$|\phi_i| \leq PM$$

Guaranteed gain and phase margins for LQ regulators can, in general, be obtained on the basis of the following lemma which was established separately in (Arvanitis and Kalogeropoulos, 1998a; 1998b; Lehtomaki *et al.*, 1981; Safonov *et al.*, 1981; Shaked, 1986).

Lemma 4. *Consider a stable discrete-time feedback system with loop transfer function $T(z)$ and return difference $I + T(z)$ (where a negative feedback is assumed). Suppose that $T(z)$ undergoes an additive change $\Delta T(z)$ which preserves the number of unstable poles of $T(z)$ such that the actual plant loop transfer function matrix is $T_{act}(z) = T_{nom}(z) + \Delta T(z)$. Then the stability of the closed-loop system is preserved if*

$$\sigma_{\max}(\Delta T(z)) \leq \sigma_{\min}(I + T(z)) \quad \text{whenever} \quad |z| = 1 \quad (43)$$

where, for a matrix $M \in \mathbb{C}^{q \times r}$, $\sigma_{\max}(M)$ and $\sigma_{\min}(M)$ denote the maximum and minimum singular value of M , respectively. Furthermore, if $\exists \beta \in [0, 1]$ such that

$$\sigma_{\min}(I + T(z)) \geq \beta \quad \text{for} \quad |z| = 1$$

then the multivariable gain and phase margins (GM and PM, respectively) seen at the input terminal are

$$GM = \left(\frac{1}{1 + \beta}, \frac{1}{1 - \beta} \right), \quad PM = \pm \arccos \left(1 - \frac{\beta^2}{2} \right) \quad (44)$$

For any two real matrices M and V of appropriate dimensions, their maximum and minimum singular values fulfil the following properties:

$$\sigma_{\max}(MV) \leq \sigma_{\max}(M)\sigma_{\max}(V) \tag{45}$$

$$\sigma_{\min}(M)\sigma_{\min}(V) \leq \sigma_{\min}(MV) \tag{46}$$

$$\sigma_{\max}(M + V) \leq \sigma_{\max}(M) + \sigma_{\max}(V) \tag{47}$$

$$\sigma_{\max}(M^{-1}) = \sigma_{\min}^{-1}(M) \text{ for a nonsingular matrix } M \tag{48}$$

$$\sigma_{\min}(M - V) \leq \sigma_{\min}(M) - \sigma_{\min}(V) \text{ if } M \geq V \geq 0 \tag{49}$$

$$|\sigma_{\min}(M + V) - \sigma_{\min}(M)| \leq \sigma_{\max}(V) \tag{50}$$

Lemma 4 and (45)–(50) are very useful for the robustness analysis of TPMRC-based feedback loops that follows. It is worth noticing that the stability robustness analysis presented in this paper concerns the stable closed-loop system with the closed-loop matrix Φ_{cl} of the form (38).

We begin our stability robustness analysis by calculating the loop transfer function matrix $T(z)$ at the plant input $\hat{u}(kT_0)$. From (11), (16) and (27), we can easily obtain

$$X(z) = (zI - \Phi)^{-1} B_N \hat{U}(z) \tag{51}$$

$$\hat{J}(z) = zHX(z) + D\hat{U}(z) \tag{52}$$

$$\hat{U}(z) = -(zI - L_u)^{-1} K\hat{\Gamma}(z) \tag{53}$$

where $X(z)$, $\hat{U}(z)$ and $\hat{J}(z)$ are the usual \mathcal{Z} -transforms of $x(kT_0)$, $\hat{u}(kT_0)$ and $\hat{\gamma}(kT_0)$, respectively. Now, substituting (51) in (52), and then the resulting relation in (53), we can readily conclude that the loop transfer function matrix calculated at the plant input is given by

$$T(z) = (zI - L_u)^{-1} K [zH(zI - \Phi)^{-1} B_N + D] \tag{54}$$

where a negative feedback is assumed.

In the sequel, we are interested in establishing lower bounds on the minimum singular values of the matrix $I + T(z)$ and in deriving, on the basis of these bounds, guaranteed stability margins for the TPMRC-based LQ optimal regulator. Our investigation is based on the fundamental spectral factorization equality established in the following theorem whose proof is given in Appendix A.

Theorem 5. *The following equality holds for the TPMRC-based LQ optimal regulator:*

$$F^T(z^{-1})(I - zL_u^T)(\tilde{\Gamma}_N + B_N^T P B_N)(I - z^{-1}L_u)F(z) = \tilde{\Gamma}_N + S(z) + W(z) \tag{55}$$

where

$$F(z) = I + T(z)$$

$$S(z) = B_N^T (z^{-1}I - \Phi^T)^{-1} \tilde{Q}_N (zI - \Phi)^{-1} B_N$$

$$W(z) = B_N^T (z^{-1}I - \Phi^T)^{-1} \tilde{G}_N + \tilde{G}_N^T (zI - \Phi)^{-1} B_N$$

Equality (55) is instrumental for the calculation of guaranteed stability margins of TPMRC-based LQ regulators, since on the basis of this equality one can obtain lower bounds on the minimum singular value of the regulator's return difference matrix.

It is worth noticing at this point that, as was shown in (Lehtomaki *et al.*, 1981), in the case of the continuous-time LQ regulator, the minimum singular value of its return difference matrix is bounded from below in all the cases by 1. However, in the case of the TPMRC-based LQ regulator (as well as in all the cases of discrete-time LQ regulators (Shaked, 1986)), it is very difficult (if not impossible) to obtain such a simple and universal lower bound of the minimum singular value of the regulator's return difference matrix. This is mainly due to the entanglement, in the discrete-time case, of the solution P to the Riccati equation (37), to the left-hand side of the Modified Return Difference Equality (55) which, from this point of view, is quite different from the Return Difference Equality which is fulfilled by the continuous-time regulator (Lehtomaki *et al.*, 1981). One can say that a universal analysis of the robustness properties of the TPMRC-based LQ regulator (as well as of other types of discrete-time regulators) would probably be plausible if a universal upper bound for the maximum singular value of the solution to the discrete algebraic Riccati equation was available. However, up to now, such a universal upper bound has not been found. Furthermore, in our case, the cross-product terms which are involved on the right-hand side of (55) in the form of the matrix \tilde{G}_N cause additional difficulties which make our attempts to obtain such a universal bound ineffectual. The interested reader is referred to (Chung *et al.*, 1994) for a detailed analysis of analogous problems caused by cross-product terms in the case of the continuous-time LQ regulator. For these fundamental reasons, our investigation on guaranteed stability margins of the TPMRC-based LQ regulator that follows is focused on a broad variety of important particular cases for the matrices of the state-space description of the equivalent LQRCPT design. From this point of view, our analysis appears to have some analogy to that reported in (Chung *et al.*, 1994) for the continuous-time LQRCPT. However, our results are completely different from those in (Chung *et al.*, 1994) because of the afore-mentioned differences between the Return Difference Equalities fulfilled by the continuous and discrete-time LQ regulators.

5. Guaranteed Stability Margins for TPMRC-Based LQ Regulators

The Modified Return Difference Equality (55) is used in this section in order to derive guaranteed stability margins for the TPMRC-based LQ regulator. To this end, we first establish the following theorem whose proof is given in Appendix B.

Theorem 6. Suppose that $\mathbf{T}(z)$ undergoes a stable additive perturbation. Suppose also that

$$\mathbf{S}(z) + \mathbf{W}(z) \geq \mathbf{0} \quad (56)$$

and that the matrix

$$\widehat{\Gamma}_N = \mathbf{B}_N \widetilde{\Gamma}_N^{-1} \mathbf{B}_N^T$$

is nonsingular. Then the minimum singular value of $\mathbf{I} + \mathbf{T}(z)$ is bounded from below, for $|z| = 1$, by α of the form

$$\alpha = \lambda^{-1} \left[\frac{\sigma_{\min}(\widetilde{\Gamma}_N)}{\sigma_{\max}(\widetilde{\Gamma}_N) + \sigma_{\max}^2(\mathbf{B}_N)\phi} \right]^{1/2} \quad (57)$$

where

$$\lambda = 1 + \sigma_{\max}(\mathbf{L}_u)$$

$$\phi = \frac{\psi}{\sigma_{\min}^2(\widetilde{\Gamma}_N)\sigma_{\min}^2(\mathbf{B}_N)}$$

$$\begin{aligned} \psi = & \left[\sigma_{\min}(\widetilde{\Gamma}_N)\sigma_{\max}(\Phi) + \sigma_{\max}(\mathbf{B}_N)\sigma_{\max}(\widetilde{\mathbf{G}}_N) \right]^2 \sigma_{\max}(\widetilde{\Gamma}_N) \\ & + \sigma_{\min}(\widetilde{\Gamma}_N)\sigma_{\min}^2(\mathbf{B}_N) \left[\sigma_{\min}(\widetilde{\Gamma}_N)\sigma_{\max}(\widetilde{\mathbf{Q}}_N) + \sigma_{\max}^2(\widetilde{\mathbf{G}}_N) \right] \end{aligned}$$

The guaranteed gain and phase margins of the TPMRC-based LQ regulator are then given by

$$\text{GM}_\alpha = (1 \pm \alpha)^{-1}, \quad \text{PM}_\alpha = \pm \arccos \left(1 - \frac{\alpha^2}{2} \right) \quad (58)$$

An alternative result which is a direct consequence of Theorem 6 is given in the following proposition.

Proposition 1. Suppose that $\mathbf{T}(z)$ undergoes a stable additive perturbation. Suppose also that

$$\widetilde{\mathbf{Q}}_N > \mathbf{0}, \quad \mathbf{W}(z) \geq \mathbf{0} \quad (59)$$

and $\det \widehat{\Gamma}_N \neq 0$. Then the lower bound of the form (57) for $\sigma_{\min}(\mathbf{I} + \mathbf{T}(z))$, as well as the stability margins of the form (58) are guaranteed for the TPMRC-based LQ optimal regulator.

Proof. If (59) holds, then $\mathbf{S}(z) \geq \mathbf{0}$. This relation, together with the fact that $\mathbf{W}(z) \geq \mathbf{0}$, implies (56) and the result then follows from Theorem 6. ■

In the sequel, our aim is to establish guaranteed stability margins for the TPMRC-based LQ optimal regulator in the cases where (56) or (59), do not hold. To this end, we first establish the following theorem whose proof is given in Appendix C.

Theorem 7. *Suppose that $T(z)$ undergoes a stable additive perturbation. Suppose also that*

$$\tilde{Q}_N - I \geq 0 \tag{60}$$

$$\tilde{\Gamma}_N \geq \tilde{G}_N^T \tilde{G}_N \geq 0 \tag{61}$$

and that $\det \tilde{\Gamma}_N \neq 0$. Then the minimum singular value of $I + T(z)$ is bounded from below for $|z| = 1$ by

$$b = \lambda^{-1} \left[\frac{\sigma_{\min}(\tilde{\Gamma}_N - \tilde{G}_N^T \tilde{G}_N)}{\sigma_{\max}(\tilde{\Gamma}_N) + \sigma_{\max}^2(\mathbf{B}_N)\phi} \right]^{1/2}$$

The guaranteed gain and phase margins of the TPMRC-based LQ optimal regulator are then given by

$$GM_b = (1 \pm b)^{-1}, \quad PM_b = \pm \arccos \left(1 - \frac{b^2}{2} \right) \tag{62}$$

In the case where it is impossible to obtain the lower bound b due to the fact that inequalities (60) and/or (61) do not hold, the following theorem whose proof is given in Appendix D provides us guaranteed stability margins for the TPMRC-based LQ optimal regulator.

Theorem 8. *Suppose that $T(z)$ undergoes a stable additive perturbation. Suppose also that*

$$\hat{Q}_N \equiv \tilde{Q}_N - \tilde{G}_N \tilde{\Gamma}_N^{-1} \tilde{G}_N^T \geq 0 \tag{63}$$

and that

$$\sigma_{\min} \left(I + \tilde{\Gamma}_N^{-1} \tilde{G}_N^T (zI - \Phi)^{-1} \mathbf{B}_N \right) \geq q, \quad q \in [0, 1] \tag{64}$$

Finally, suppose that $\det \hat{\Gamma}_N \neq 0$. Then the minimum singular value of $I + T(z)$ is bounded from below for $|z| = 1$ by γ , where

$$\gamma = \lambda^{-1} \left[\frac{q^2 \sigma_{\min}(\hat{\Gamma}_N)}{\sigma_{\max}(\hat{\Gamma}_N) + \sigma_{\max}^2(\mathbf{B}_N)\phi} \right]^{1/2}$$

The guaranteed gain and phase margins of the TPMRC-based LQ optimal regulator are then given by

$$GM_\gamma = (1 \pm \gamma)^{-1}, \quad PM_\gamma = \pm \arccos \left(1 - \frac{\gamma^2}{2} \right) \tag{65}$$

Notice that the parameter q can be easily calculated because the matrix $\mathbf{I} + \tilde{\Gamma}_N^{-1} \tilde{\mathbf{G}}_N^T (z\mathbf{I} - \Phi)^{-1} \mathbf{B}_N$ does not require the computation of \mathbf{P} .

In the particular case where Φ is asymptotically stable, we are able to establish the following theorem whose proof is given in Appendix E.

Theorem 9. *Suppose that $\mathbf{T}(z)$ undergoes a stable additive perturbation. Suppose also that Φ is an asymptotically stable matrix and that, for some N , M_i , $i = 1, 2, \dots, p$ and T_0 , the matrix $\tilde{\mathbf{G}}_N$ of the form (21) can also be expressed as*

$$\tilde{\mathbf{G}}_N \triangleq \Phi^T \Sigma \mathbf{B}_N \quad (66)$$

where Σ is the positive semi-definite solution to the following discrete Lyapunov equation:

$$\Sigma = \Phi^T \Sigma \Phi + \tilde{\mathbf{Q}}_N \quad (67)$$

Assume that

$$\tilde{\mathbf{Q}}_N > \mathbf{0}, \quad \tilde{\Gamma}_N \geq \mathbf{B}_N^T \Sigma \mathbf{B}_N \geq \mathbf{0}, \quad \det \hat{\Gamma}_N \neq 0$$

Then

$$\sigma_{\min}(\mathbf{I} + \mathbf{T}(z)) \geq d \text{ for } |z| = 1$$

where

$$d = \lambda^{-1} \left[\frac{\sigma_{\min}(\tilde{\Gamma}_N - \mathbf{B}_N^T \Sigma \mathbf{B}_N)}{\sigma_{\max}(\tilde{\Gamma}_N) + \sigma_{\max}^2(\mathbf{B}_N) \phi} \right]^{1/2}$$

The guaranteed gain and phase margins of the TPMRC-based LQ optimal regulator are then given by

$$\text{GM}_d = (1 \pm d)^{-1}, \quad \text{PM}_d = \pm \arccos \left(1 - \frac{d^2}{2} \right) \quad (68)$$

Clearly, the bound d for $\sigma_{\min}(\mathbf{I} + \mathbf{T}(z))$ depends on the solution Σ to the discrete Lyapunov equation (67). In order to obtain guaranteed stability margins for the TPMRC-based LQ optimal regulator which are independent of Σ , we present the following theorem.

Theorem 10. *Suppose that $\mathbf{T}(z)$ undergoes a stable additive perturbation. Suppose that the assumptions of Theorem 9 are satisfied. Assume in addition that either*

$$\sigma_{\max}(\Phi) < 1 \text{ (Case I)} \quad (69)$$

or

$$\text{The eigenvalues of } \Phi \text{ are distinct and } \sigma_{\max}(\Phi) \geq 1 \text{ (Case II)} \quad (70)$$

and that

$$\chi \leq \frac{\sigma_{\min}(\tilde{\Gamma}_N)}{\sigma_{\max}^2(\mathbf{B}_N)}$$

where

$$\chi = \begin{cases} \sigma_{\max}(\tilde{\mathbf{Q}}_N) [1 - \sigma_{\max}^2(\Phi)]^{-1} & \text{for Case I} \\ \sigma_{\max}(\tilde{\mathbf{Q}}_N) [1 - \nu_{\max}^2(\Phi)]^{-1} & \text{for Case II} \end{cases} \quad (71)$$

and where, in general, $\nu_{\max}(\mathbf{M})$ denotes the maximum absolute value of the eigenvalues of a matrix \mathbf{M} . Then

$$\sigma_{\min}(\mathbf{I} + \mathbf{T}(z_0)) \geq e$$

where

$$e = \lambda^{-1} \left[\frac{\sigma_{\min}(\tilde{\Gamma}_N) - \sigma_{\max}^2(\mathbf{B}_N)\chi}{\sigma_{\max}(\tilde{\Gamma}_N) + \sigma_{\max}^2(\mathbf{B}_N)\phi} \right]^{1/2} \quad (72)$$

The guaranteed gain and phase margins of the TPMRC-based LQ optimal regulator are then given by

$$\text{GM}_e = (1 \pm e)^{-1}, \quad \text{PM}_e = \pm \arccos \left(1 - \frac{e^2}{2} \right) \quad (73)$$

Proof. From (49) we have

$$\begin{aligned} \sigma_{\min}(\tilde{\Gamma}_N - \mathbf{B}_N^T \Sigma \mathbf{B}_N) &\geq \sigma_{\min}(\tilde{\Gamma}_N) - \sigma_{\max}(\mathbf{B}_N^T \Sigma \mathbf{B}_N) \\ &\geq \sigma_{\min}(\tilde{\Gamma}_N) - \sigma_{\max}^2(\mathbf{B}_N) \sigma_{\max}(\Sigma) \end{aligned}$$

Note that in afore-mentioned Cases I and II, according to the results in (Shaked, 1986), we have

$$\sigma_{\max}(\Sigma) \leq \sigma_{\max}(\tilde{\mathbf{Q}}_N) [1 - \sigma_{\max}^2(\Phi)]^{-1} \quad \text{for Case I}$$

$$\sigma_{\max}(\Sigma) \leq \sigma_{\max}(\tilde{\mathbf{Q}}_N) [1 - \nu_{\max}^2(\Phi)]^{-1} \quad \text{for Case II}$$

Therefore

$$\sigma_{\min}(\tilde{\Gamma}_N - \mathbf{B}_N^T \Sigma \mathbf{B}_N) \geq \sigma_{\min}(\tilde{\Gamma}_N) - \sigma_{\max}^2(\mathbf{B}_N) \chi$$

and on the basis of Theorem 9 we readily obtain (72). Furthermore, if (71) is satisfied, then $e \in [0, 1]$. Relation (73) then follows from (44). ■

It is worth noticing at this point that the stability margins provided by Theorem 10 are tighter than those provided by Theorem 9. However, as was already

mentioned, these margins can be computed without resorting to the solution to the Lyapunov equation (67).

In the case where Φ is asymptotically stable, we are also able to obtain the following alternative result.

Theorem 11. *Suppose that $T(z)$ undergoes a stable additive perturbation. Suppose that Φ is an asymptotically stable matrix and that, for some N , M_i , $i = 1, 2, \dots, p$ and T_0 , the matrix \tilde{G}_N of the form (21), can also be expressed as*

$$\tilde{G}_N \hat{=} -\Phi^T \Sigma B_N \quad (74)$$

where Σ is the solution to (67). Assume that $\tilde{Q}_N > \mathbf{0}$ and $\det \hat{\Gamma}_N \neq 0$. Then the lower bound of the form (57) for the minimum singular value of $I + T(z)$, as well as the gain and phase margins of the form (58) are guaranteed for the TPMRC-based LQ optimal regulator.

Proof. Following an argument similar to that presented in the proof of Theorem 9 for the case where \tilde{G}_N can be expressed by (74), we can easily conclude that

$$F^T(z^{-1})(I - zL_u^T)(\tilde{\Gamma}_N + B_N^T P B_N)(I - z^{-1}L_u)F(z) = \tilde{\Gamma}_N + B_N^T \Sigma B_N$$

Since $\Sigma \geq \mathbf{0}$, we have

$$F^T(z^{-1})(I - zL_u^T)(\tilde{\Gamma}_N + B_N^T P B_N)(I - z^{-1}L_u)F(z) \geq \tilde{\Gamma}_N$$

Consequently,

$$\sigma_{\min}(I + T(z)) \geq \alpha$$

and our claim follows from Theorem 6. ■

Theorems 6–11, provide us guaranteed stability margins for the TPMRC-based LQ regulator in the case where $\det \hat{\Gamma}_N \neq 0$. For a singular matrix $\hat{\Gamma}_N$, these theorems do not work. In this case, a way to derive guaranteed stability margins for TPMRC-based LQ regulators is to use a procedure similar to that proposed in (Shaked, 1986). More precisely, the following theorem whose proof is given in Appendix F can be established.

Theorem 12. *Suppose that $T(z)$ undergoes a stable additive perturbation. Let the matrix $\hat{\Phi}$ be defined as*

$$\hat{\Phi} = \Phi - B_N \tilde{\Gamma}_N^{-1} \tilde{G}_N^T$$

Suppose that the conditions of either Theorem 6 or Proposition 1 are satisfied, except for the non-singularity of $\hat{\Gamma}_N$. Assume that one of the following statements holds separately:

- (a) *The matrix $\hat{\Phi}$ is asymptotically stable and $\sigma_{\max}(\hat{\Phi}) < 1$ (Case I).*

- (b) The matrix $\widehat{\Phi}$ is asymptotically stable, its eigenvalues are all distinct and $\sigma_{\max}(\widehat{\Phi}) \geq 1$ (Case II).
- (c) The eigenvalues of $\widehat{\Phi}$ are all distinct, none of them lies on the unit circle, but some of them lie outside this circle. It is further assumed that none of the latter is a reciprocal of the remaining eigenvalues (Case III).

Then the minimum singular value of $I + T(z)$ is bounded from below, for $|z| = 1$, by h , where

$$h = \lambda^{-1} \left[\frac{\sigma_{\min}(\widetilde{\Gamma}_N)}{\sigma_{\max}(\widetilde{\Gamma}_N) + \sigma_{\max}^2(\mathbf{B}_N)\vartheta} \right]^{1/2} \tag{75}$$

with

$$\vartheta = \begin{cases} \left[\sigma_{\max}(\widetilde{\mathbf{Q}}_N) + \frac{\sigma_{\max}^2(\widetilde{\mathbf{G}}_N)}{\sigma_{\min}(\widetilde{\Gamma}_N)} \right] \left[1 - \sigma_{\max}^2(\Phi - \mathbf{B}_N \widetilde{\Gamma}_N^{-1} \widetilde{\mathbf{G}}_N^T) \right]^{-1} & \text{for Case I} \\ \left[\sigma_{\max}(\widetilde{\mathbf{Q}}_N) + \frac{\sigma_{\max}^2(\widetilde{\mathbf{G}}_N)}{\sigma_{\min}(\widetilde{\Gamma}_N)} \right] \left[1 - \nu_{\max}^2(\Phi - \mathbf{B}_N \widetilde{\Gamma}_N^{-1} \widetilde{\mathbf{G}}_N^T) \right]^{-1} & \text{for Case II} \\ \frac{\sigma_{\max}(\widetilde{\mathbf{Q}}_N) + \frac{\sigma_{\max}^2(\widetilde{\mathbf{G}}_N)}{\sigma_{\min}(\widetilde{\Gamma}_N)} + \omega^2}{1 - \widehat{\rho}^2} & \text{for Case III} \end{cases}$$

and

$$\omega = \frac{\sigma_{\max}(\widehat{\mathbf{V}}_{\text{out}})\sigma_{\max}(\widehat{\mathbf{V}}_{\text{out}}\mathbf{B}_N)\sigma_{\max}(\widetilde{\Gamma}_N) \left[\nu_{\max}^2(\widehat{\Phi}) - 1 \right]}{\sigma_{\min}(\widehat{\Phi}_{\text{out}})\sigma_{\min}^{1/2}(\widetilde{\Gamma}_N)\sigma_{\min}^2(\mathbf{B}_N^T \widehat{\mathbf{V}}_{\text{out}}^+)}$$

In the above relation $\widehat{\Phi}_{\text{out}}$ and $\widehat{\mathbf{V}}_{\text{out}}$ are the diagonal Jordan block of the eigenvalues of $\widehat{\Phi}$ lying outside the unit circle and the matrix of their corresponding eigenrows, respectively, and $\widehat{\mathbf{V}}_{\text{out}}^+$ is the conjugate transpose of $\widehat{\mathbf{V}}_{\text{out}}$. Finally, the scalar $\widehat{\rho}$ in Case III is defined as

$$\widehat{\rho} \triangleq \max \left\{ \rho^*, 1/\rho^{**} \right\}$$

where ρ^* is the largest absolute value of the eigenvalues of $\widehat{\Phi}$ lying inside the unit circle and ρ^{**} is the smallest absolute value of the remaining eigenvalues of $\widehat{\Phi}$.

The guaranteed gain and phase margins of the TPMRC-based LQ optimal regulator are then given by

$$\text{GM}_h = (1 \pm h)^{-1}, \quad \text{PM}_h = \pm \arccos \left(1 - \frac{h^2}{2} \right) \tag{76}$$

Of course, Theorem 12 operates only in the case where relations (56) or (59) hold. In the opposite case, guaranteed stability margins for the TPMRC-based LQ

regulator can be derived on the basis of the following two propositions which are direct consequences of Theorems 7, 8 and 12.

Proposition 2. *Suppose that $T(z)$ undergoes a stable additive perturbation and that the conditions of Theorem 7 are satisfied except for the non-singularity of $\hat{\Gamma}_N$. Suppose also that the conditions of Theorem 12 are simultaneously satisfied, except for (56) or (59). Then the minimum singular value of $I + T(z)$ is bounded from below, for $|z| = 1$, by g , where*

$$g = \lambda^{-1} \left[\frac{\sigma_{\min}(\tilde{\Gamma}_N - \tilde{G}_N^T \tilde{G}_N)}{\sigma_{\max}(\tilde{\Gamma}_N) + \sigma_{\max}^2(B_N)\vartheta} \right]^{1/2}$$

The guaranteed gain and phase margins of the TPMRC-based LQ optimal regulator are then given by

$$GM_g = (1 \pm g)^{-1}, \quad PM_g = \pm \arccos \left(1 - \frac{g^2}{2} \right)$$

Proposition 3. *Suppose that $T(z)$ undergoes a stable additive perturbation and that the conditions of Theorem 8 are satisfied except for the non-singularity of $\hat{\Gamma}_N$. Suppose also that the conditions of Theorem 12 are simultaneously satisfied, except for (56) or (59). Then the minimum singular value of $I + T(z)$ is bounded from below, for $|z| = 1$, by m , where*

$$m = \lambda^{-1} \left[\frac{q^2 \sigma_{\min}(\tilde{\Gamma}_N)}{\sigma_{\max}(\tilde{\Gamma}_N) + \sigma_{\max}^2(B_N)\vartheta} \right]^{1/2}$$

The guaranteed gain and phase margins of the TPMRC-based LQ optimal regulator are then given by

$$GM_m = (1 \pm m)^{-1}, \quad PM_m = \pm \arccos \left(1 - \frac{m^2}{2} \right)$$

Moreover, in the particular case where $\hat{\Gamma}_N$ is singular and Φ is an asymptotically stable matrix, the following three propositions which are direct consequences of Theorems 9–12 can be easily established.

Proposition 4. *Suppose that $T(z)$ undergoes a stable additive perturbation and that the conditions of Theorem 9 are satisfied except for the non-singularity of $\hat{\Gamma}_N$. Suppose also that the conditions of Theorem 12 are simultaneously satisfied, except for (56) or (59). Then the minimum singular value of $I + T(z)$ is bounded from below, for $|z| = 1$, by τ , where*

$$\tau = \lambda^{-1} \left[\frac{\sigma_{\min}(\tilde{\Gamma}_N - B_N^T \Sigma B_N)}{\sigma_{\max}(\tilde{\Gamma}_N) + \sigma_{\max}^2(B_N)\vartheta} \right]^{1/2}$$

The guaranteed gain and phase margins of the TPMRC-based LQ optimal regulator are then given by

$$GM_\tau = (1 \pm \tau)^{-1}, \quad PM_\tau = \pm \arccos\left(1 - \frac{\tau^2}{2}\right)$$

Proposition 5. Suppose that $T(z)$ undergoes a stable additive perturbation and that the conditions of Theorem 10 are satisfied except for the non-singularity of $\widehat{\Gamma}_N$. Suppose also that the conditions of Theorem 12 are simultaneously satisfied, except for (56) or (59). Then the minimum singular value of $I + T(z)$ is bounded from below, for $|z| = 1$, by f , where

$$f = \lambda^{-1} \left[\frac{\sigma_{\min}(\widetilde{\Gamma}_N) - \sigma_{\max}^2(B_N)\chi}{\sigma_{\max}(\widetilde{\Gamma}_N) + \sigma_{\max}^2(B_N)\vartheta} \right]$$

The guaranteed gain and phase margins of the TPMRC-based LQ optimal regulator are then given by

$$GM_f = (1 \pm f)^{-1}, \quad PM_f = \pm \arccos\left(1 - \frac{f^2}{2}\right)$$

Proposition 6. Suppose that $T(z)$ undergoes a stable additive perturbation and that the conditions of Theorem 11 are satisfied, except for the non-singularity of $\widehat{\Gamma}_N$. Suppose also that conditions of Theorem 12 are simultaneously satisfied, except for (56) or (59). Then the lower bound of the form (75) for the minimum singular value of $I + T(z)$, as well as the gain and phase margins of the form (76), are satisfied for the TPMRC-based LQ optimal regulator.

An alternative way to calculate guaranteed stability margins for the TPMRC-based LQ optimal regulator in the case where matrix $\widehat{\Gamma}_N$ is singular is provided by the following theorem whose proof is given in Appendix G.

Theorem 13. Suppose that $T(z)$ undergoes a stable additive perturbation and that the conditions of either Theorem 6 or Proposition 1 are satisfied, except for the non-singularity of $\widehat{\Gamma}_N$. Suppose also that

$$\widehat{Q}_N > 0 \tag{77}$$

and

$$\sigma_{\max}^2(\widehat{\Phi}) < 1 + \sigma_{\min}^2(B_{N,\Gamma})\eta \tag{78}$$

where

$$B_{N,\Gamma} = B_N \widetilde{\Gamma}_N^{-1/2}$$

and

$$\eta = \nu_{\max} \left[\widehat{\Phi}^T \left[\widehat{Q}_N^{-1} + \widehat{\Gamma}_N \right]^{-1} \widehat{\Phi} + \widehat{Q}_N \right] \tag{79}$$

Then the minimum singular value of $\mathbf{I} + \mathbf{T}(z)$ is bounded from below, for $|z| = 1$, by $\hat{\alpha}$, where

$$\hat{\alpha} = \lambda^{-1} \left[\frac{\sigma_{\min}(\tilde{\Gamma}_N)}{\sigma_{\max}(\tilde{\Gamma}_N) + \sigma_{\max}^2(\mathbf{B}_N)\mu} \right]^{1/2} \tag{80}$$

with

$$\begin{aligned} \mu = & \frac{\nu_{\max}(\hat{\mathbf{Q}}_N)}{1 + \sigma_{\min}^2(\mathbf{B}_{N,\Gamma})\eta - \sigma_{\max}^2(\hat{\Phi})} \left[\sigma_{\max}(\Phi) + \sigma_{\min}^{-1}(\tilde{\Gamma}_N)\sigma_{\max}(\mathbf{B}_N)\sigma_{\max}(\tilde{\mathbf{G}}_N) \right]^2 \\ & + \left[\sigma_{\max}(\tilde{\mathbf{Q}}_N + \sigma_{\min}^{-1}(\tilde{\Gamma}_N)\sigma_{\max}^2(\tilde{\mathbf{G}}_N)) \right] \end{aligned} \tag{81}$$

The guaranteed gain and phase margins of the TPMRC-based LQ optimal regulator are then given by

$$\text{GM}_{\hat{\alpha}} = (1 \pm \hat{\alpha})^{-1}, \quad \text{PM}_{\hat{\alpha}} = \pm \arccos \left(1 - \frac{\hat{\alpha}^2}{2} \right) \tag{82}$$

In the case where (56) or (59) do not hold, guaranteed stability margins for the TPMRC-based LQ regulator can be derived on the basis of the following two propositions which are direct consequences of Theorems 7, 8 and 13.

Proposition 7. *Suppose that $\mathbf{T}(z)$ undergoes a stable additive perturbation and that the conditions of Theorem 7 are satisfied except for the non-singularity of $\hat{\Gamma}_N$. Suppose also that the conditions of Theorem 13 are simultaneously satisfied, except for (56) or (59). Then the minimum singular value of $\mathbf{I} + \mathbf{T}(z)$ is bounded from below, for $|z| = 1$, by \hat{b} , where*

$$\hat{b} = \left[\frac{\sigma_{\min}(\tilde{\Gamma}_N - \tilde{\mathbf{G}}_N^T \tilde{\mathbf{G}}_N)}{\sigma_{\max}(\tilde{\Gamma}_N) + \sigma_{\max}^2(\mathbf{b}_N)\mu} \right]^{1/2} \tag{83}$$

with μ given by (79). The guaranteed gain and phase margins of the TPMRC-based LQ optimal regulator are then given by

$$\text{GM}_{\hat{b}} = (1 \pm \hat{b})^{-1}, \quad \text{PM}_{\hat{b}} = \pm \arccos \left(1 - \frac{\hat{b}^2}{2} \right) \tag{84}$$

Proposition 8. *Suppose that $\mathbf{T}(z)$ undergoes an stable additive perturbation and that the conditions of Theorem 8 are satisfied except for the non-singularity of $\hat{\Gamma}_N$. Suppose also that the conditions of Theorem 13 are simultaneously satisfied, except for (56) or (59). Then the minimum singular value of $\mathbf{I} + \mathbf{T}(z)$ is bounded from below, for $|z| = 1$, by $\hat{\gamma}$, where*

$$\hat{\gamma} = \left[\frac{q^2 \sigma_{\min}(\tilde{\Gamma}_N)}{\sigma_{\max}(\tilde{\Gamma}_N) + \sigma_{\max}^2(\mathbf{B}_N)\mu} \right]^{1/2} \tag{85}$$

The guaranteed gain and phase margins of TPMRC-based LQ optimal regulator are then given by

$$GM_{\hat{\gamma}} = (1 \pm \hat{\gamma})^{-1}, \quad PM_{\hat{\gamma}} = \pm \arccos \left(1 - \frac{\hat{\gamma}^2}{2} \right) \tag{86}$$

Moreover, in the case where $\hat{\Gamma}_N$ is singular and Φ is asymptotically stable, the following three propositions which are direct consequences of Theorems 9–11 and 13, can be established.

Proposition 9. *Suppose that $T(z)$ undergoes a stable additive perturbation and that the conditions of Theorem 9 are satisfied except for the non-singularity of $\hat{\Gamma}_N$. Suppose also that the conditions of Theorem 13 are simultaneously satisfied, except for (56) or (59). Then the minimum singular value of $I + T(z)$ is bounded from below, for $|z| = 1$, by \hat{d} , where*

$$\hat{d} = \left[\frac{\sigma_{\min}(\tilde{\Gamma}_N - B_N^T \Sigma B_N)}{\sigma_{\max}(\tilde{\Gamma}_N) + \sigma_{\max}^2(B_N)\mu} \right]^{1/2} \tag{87}$$

The guaranteed gain and phase margins of the TPMRC-based LQ optimal regulator are then given by

$$GM_{\hat{d}} = (1 \pm \hat{d})^{-1}, \quad PM_{\hat{d}} = \pm \arccos \left(1 - \frac{\hat{d}^2}{2} \right) \tag{88}$$

Proposition 10. *Suppose that $T(z)$ undergoes a stable additive perturbation and that the conditions of Theorem 10 are satisfied except for the non-singularity of $\hat{\Gamma}_N$. Suppose also that the conditions of Theorem 13 are simultaneously satisfied, except for (56) or (59). Then the minimum singular value of $I + T(z)$ is bounded from below, for $|z| = 1$, by \hat{e} , where*

$$\hat{e} = \left[\frac{\sigma_{\min}(\tilde{\Gamma}_N) - \sigma_{\max}^2(B_N)\chi}{\sigma_{\max}(\tilde{\Gamma}_N) + \sigma_{\max}^2(B_N)\mu} \right]^{1/2} \tag{89}$$

The guaranteed gain and phase margins of the TPMRC-based LQ optimal regulator are then given by

$$GM_{\hat{e}} = (1 \pm \hat{e})^{-1}, \quad PM_{\hat{e}} = \pm \arccos \left(1 - \frac{\hat{e}^2}{2} \right) \tag{90}$$

Proposition 11. *Suppose that $T(z)$ undergoes a stable additive perturbation and that the conditions of Theorem 11 are satisfied except for the non-singularity of $\hat{\Gamma}_N$. Suppose also that the conditions of Theorem 13 are simultaneously satisfied, except for (56) or (59). Then the lower bound of the form (80) for the minimum singular value*

of $I + T(z)$, as well as the gain and phase margins of the form (82), are satisfied for the TPMRC-based LQ optimal regulator.

Remark 2. From the previous analysis it is easily recognized that the entanglement of the parameter λ in the form of the suggested gain and phase margins implies that the stability margins of the dynamic TPMRC-based LQ optimal regulator change according to the matrix L_u and can be set at satisfactory values by simply adjusting the TPMRC dynamics. Since the form of L_u (and consequently the designed TPMRC) depends on the selection of the output multiplicities of the sampling M_i , $i = 1, 2, \dots, p$, for a particular sampling period the suggested stability margins are functions of the output multiplicities. Furthermore, it is pointed out that here guaranteed stability margins for both the static and the dynamic TPMRC-based LQ regulator are obtained on the basis of the matrices $B_N, \tilde{\Gamma}_N, \tilde{Q}_N$ and \tilde{G}_N whose forms depend on the particular choice of N . Therefore the suggested stability margins are greatly affected by the selection of the input multiplicity of the sampling N .

Remark 3. In the case where the conditions of Theorems 6 and 12 are simultaneously satisfied, it would be very interesting to clarify whether the guaranteed stability margins provided by relations (58) are superior when compared with the stability margins provided by (76).

To settle this concern, suppose first that $\det \hat{\Gamma}_N \neq 0$ and that simultaneously $\hat{\Phi}$ is asymptotically stable with $\sigma_{\max}(\hat{\Phi}) < 1$ (Case I of Theorem 12). In this case, it is obvious that for h to be less than or equal to α , it suffices that $\vartheta \geq \phi$, or equivalently

$$\left[\sigma_{\max}(\tilde{Q}_N) + \frac{\sigma_{\max}^2(\tilde{G}_N)}{\sigma_{\min}(\tilde{\Gamma}_N)} \right] \left[1 - \sigma_{\max}^2(\Phi - B_N \tilde{\Gamma}_N^{-1} \tilde{G}_N^T) \right]^{-1} \geq \frac{\psi}{\sigma_{\min}^2(\tilde{\Gamma}_N) \sigma_{\min}^2(B_N)} \tag{91}$$

This implies that the stability margins of the TPMRC-based LQ optimal regulator provided by (76) are inferior when compared with the stability margins provided by (58).

In the case where $\det \hat{\Gamma}_N \neq 0$ and the conditions related to Case II of Theorem 12 are simultaneously satisfied, we can similarly conclude that the stability margins provided by (76) are inferior when compared with the stability margins provided by (58) if the following relationship holds:

$$\left[\sigma_{\max}(\tilde{Q}_N + \frac{\sigma_{\max}^2(\tilde{G}_N)}{\sigma_{\min}(\tilde{\Gamma}_N)}) \right] \left[1 - \nu_{\max}^2(\Phi - B_N \tilde{\Gamma}_N^{-1} \tilde{G}_N^T) \right]^{-1} \geq \frac{\psi}{\sigma_{\min}^2(\tilde{\Gamma}_N) \sigma_{\min}^2(B_N)} \tag{92}$$

Finally, following a similar argument, we can easily conclude that for the case where $\det \hat{\Gamma}_N \neq 0$ and the conditions related to Case III of Theorem 12 are simulta-

neously satisfied, the stability margins provided by (76) are inferior when compared with the stability margins provided by (58) if the following relationship holds:

$$\frac{\sigma_{\max}(\tilde{Q}_N) + \frac{\sigma_{\max}^2(\tilde{G}_N)}{\sigma_{\min}(\tilde{\Gamma}_N)} + \omega^2}{1 - \hat{\rho}^2} \geq \frac{\psi}{\frac{\sigma_{\min}^2(\tilde{\Gamma}_N)\sigma_{\min}^2(B_N)}{\sigma_{\min}^2(\tilde{\Gamma}_N)\sigma_{\min}^2(B_N)}} \quad (93)$$

Inequalities (91)–(93) can also be used in order to compare the stability margins involving the parameters ϕ and θ , i.e. those provided by Theorems 7–11 and Propositions 2–6, respectively. Similar inequalities can be easily obtained for the comparison of the stability margins estimated through Theorems 6–11 and 13, and Propositions 7–11, as well as for the comparison of the stability margins estimated through Theorem 12 and Propositions 2–6, and Theorem 13 and Propositions 7–11, provided that the conditions of the respective theorems and propositions are simultaneously satisfied.

Remark 4. It is worth noticing that, although here our interest has mainly been focused on guaranteed stability margins of the TPMRC-based LQ optimal regulator, under unstructured additive perturbations, similar results for the stability robustness properties of this regulator can be obtained in the cases where other types of perturbations occur, e.g. unstructured multiplicative perturbations. In this particular case, stability margins of the TPMRC-based LQ optimal regulator can be easily obtained by investigating the minimum singular value of the regulator inverse return difference matrix $I + T^{-1}(z)$ and by taking into account the fact that the following fundamental inequality holds (Nuzman and Sandell, 1979)

$$\sigma_{\min} [I + T^{-1}(z)] \geq \frac{\sigma_{\min}(I + T(z))}{1 + \sigma_{\min}(I + T(z))}$$

while (Safonov *et al.*, 1981)

$$\begin{aligned} \text{GM} &= \inf_{|z|=1} \left\{ \frac{1}{1 + \sigma_{\min}(I + T^{-1}(z))} \right\} \\ \text{PM} &= \inf_{|z|=1} \arccos \left\{ 1 - \frac{\sigma_{\min}^2(I + T^{-1}(z))}{2} \right\} \end{aligned}$$

6. Illustrative Examples

To demonstrate the efficiency of the proposed results, we present the following illustrative examples.

Example 1. Consider the controllable and observable system of the form (1), with

$$\mathbf{A} = \begin{bmatrix} 2 & -4 & 3 & 1 \\ 2 & -1 & 2 & 2 \\ -2 & 2 & -3 & -2 \\ 1 & -2 & 3 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & -2 \\ 2 & -5 \\ 3 & 3 \\ 1 & 15 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let the basic sampling period be $T_0 = 1$ sec and the uniform input multiplicity of the sampling be $N = 30$. Consider now the problem of designing a TPMRC-based LQ optimal regulator in order to minimize the performance index of the form (10) with $\mathbf{Q} = \text{diag}\{0.1, 0.1\}$ and $\mathbf{\Gamma} = \begin{bmatrix} 20 & 1 \\ 1 & 20 \end{bmatrix}$. Observe that since an observability index vector for the pair (\mathbf{A}, \mathbf{B}) is given by $(n_1, n_2) = (3, 1)$, we can choose the output multiplicities of the sampling M_1, M_2 as $(M_1, M_2) = (4, 6)$. With this choice we have

$$\mathbf{W}_N(T_0, 0) = \begin{bmatrix} 26.6378 & -4.1317 & -12.9048 & 1.7457 \\ -4.1317 & 7.4215 & 1.6351 & 0.1548 \\ -12.9048 & 1.6351 & 13.9294 & -5.8354 \\ 1.7457 & 0.1548 & -5.8354 & 48.2236 \end{bmatrix}$$

$$\mathbf{B}_N = \begin{bmatrix} 5.1612 & 0 & 0 & 0 \\ -0.8005 & 2.6040 & 0 & 0 \\ -2.5004 & -0.1408 & 2.7673 & 0 \\ 0.3382 & 0.1634 & -1.7948 & 6.6979 \end{bmatrix}$$

$$\tilde{\mathbf{Q}}_N = \begin{bmatrix} 0.0202 & 0.0253 & 0.0347 & 0.0034 \\ 0.0253 & 0.1006 & 0.0940 & 0.0057 \\ 0.0347 & 0.0940 & 0.1079 & 0.0153 \\ 0.0034 & 0.0057 & 0.0153 & 0.0168 \end{bmatrix}$$

$$\tilde{\mathbf{\Gamma}}_N = \begin{bmatrix} 21.9072 & 0.0176 & 0.7429 & -0.1393 \\ 0.0176 & 20.8510 & -0.6461 & 0.7544 \\ 0.7429 & -0.6461 & 20.8972 & -0.0288 \\ -0.1393 & 0.7544 & -0.0288 & 20.8403 \end{bmatrix}$$

$$\tilde{\mathbf{G}}_N = \begin{bmatrix} -0.0591 & -0.0056 & 0.1395 & -0.0187 \\ -0.1261 & 0.0982 & 0.1213 & 0.0795 \\ -0.2329 & 0.0526 & 0.1400 & 0.0306 \\ -0.0765 & -0.0245 & -0.0210 & 0.0033 \end{bmatrix}$$

The optimal fictitious state feedback is given by

$$F = \begin{bmatrix} -0.0080 & -0.0138 & -0.0226 & -0.0046 \\ 0.0087 & 0.0087 & 0.0155 & -0.0031 \\ 0.0180 & 0.0052 & 0.0184 & -0.0058 \\ -0.0010 & 0.0117 & 0.0071 & 0.0025 \end{bmatrix}$$

Therefore, evaluating (40) and (41), the TPMRC gains K and L_u are given by

$$K = 10^{-4} \times \begin{bmatrix} -4 & 10 & -29 & -84 & 53 & -20 & -26 & 5 & 44 & 72 \\ -14 & -34 & 8 & 71 & -68 & 22 & 32 & -2 & -48 & -83 \\ -39 & -71 & -17 & 69 & -107 & 20 & 41 & -1 & -62 & -111 \\ 10 & -2 & 22 & 54 & -27 & 18 & 21 & 1 & -24 & -41 \end{bmatrix}$$

$$L_u = \begin{bmatrix} 0.0101 & 0.0719 & -0.0149 & 0.0507 \\ -0.0275 & -0.0678 & 0.0158 & -0.0059 \\ -0.0631 & -0.0757 & 0.0125 & -0.0007 \\ 0.0021 & -0.0525 & 0.0219 & -0.0274 \end{bmatrix}$$

Note that the eigenvalues of L_u are $\lambda_{1,2} = 0.0032 \pm j0.0204$ and $\lambda_{3,4} = -0.0395 \pm j0.0134$, and they lie inside the unit circle. This means that the designed TPMRC-based LQ optimal regulator is stable.

In the case where it is desired to design a static TPMRC-based LQ optimal regulator, it is easy to check that with the same choice of M_1, M_2 as above, the matrix $[H \ D]$ has full column rank. Evaluating (42), we obtain

$$K = 10^{-4} \times \begin{bmatrix} -100 & 104 & 272 & -392 & -71 & 203 & -90 & -71 & 219 & -154 \\ 126 & -258 & -11 & 180 & -44 & 95 & -160 & -87 & 216 & -147 \\ 182 & -441 & 124 & 78 & -167 & 337 & -336 & -261 & 462 & -285 \\ 48 & -91 & -67 & 198 & 11 & -33 & -1 & -9 & -5 & 42 \end{bmatrix}$$

Let us now compute guaranteed stability margins for the afore-mentioned TPMRC-based LQ optimal regulator. To this end, observe first that, in our case, $\hat{\Gamma}_N$ is nonsingular, since

$$\hat{\Gamma}_N = \begin{bmatrix} 1.2175 & -0.1902 & -0.6130 & 0.1054 \\ -0.1902 & 0.3556 & 0.0888 & -0.0333 \\ -0.6130 & 0.0888 & 0.6752 & -0.2891 \\ 0.1054 & -0.0333 & -0.2891 & 2.3152 \end{bmatrix}$$

Note also that (56) and (63) hold and that (64) holds for $q = 0.9671$. On the other hand, (60) does not hold since, as can be easily checked, the matrix $\tilde{Q}_N - I$ is not

positive semi-definite. The matrices Φ and $\hat{\Phi}$ are both asymptotically stable with distinct eigenvalues and with $\sigma_{\max}(\Phi) = 2.0432 > 1$ (Case II of Theorem 10) and $\sigma_{\max}(\hat{\Phi}) = 1.9639 > 1$ (Case II of Theorem 12), respectively. However, the matrix \tilde{G}_N cannot be expressed either in the form (66) or in the form (74). Finally, in the present case, (78) does not hold. That is, in order to compute guaranteed stability margins for the designed TPMRC-based LQ optimal regulators, only the bounds provided by Theorems 6, 8, 12 and Proposition 3 can be used. Applying these results for the case of static TPMRC-based LQ optimal regulators, we obtain the bounds $\alpha = 0.1370$, $\gamma = 0.1325$, $h = 0.6726$ and $m = 0.6505$ for $\sigma_{\min}(\mathbf{I} + \Omega(z))$ and $|z| = 1$. These bounds are depicted in Fig. 2, together with the plots of $\sigma_{\min}(\mathbf{I} + \Omega(z))$ (solid line) and $\sigma_{\min}(\mathbf{I} + \tilde{\Gamma}_N^{-1} \tilde{G}_N^T(z\mathbf{I} - \Phi)^{-1} \mathbf{B}_N)$ (dotted line). In the case of a dynamic TPMRC-based LQ optimal regulator, since $\sigma_{\max}(\mathbf{L}_u) = 0.1545$, we obtain the bounds $\alpha = 0.1187$, $\gamma = 0.1148$, $h = 0.5826$ and $m = 0.5634$ for $\sigma_{\min}(\mathbf{I} + \mathbf{T}(z))$ as $|z| = 1$. These bounds are depicted in Fig. 3, together with the plots of $\sigma_{\min}(\mathbf{I} + \mathbf{T}(z))$. Note that, in the present case, $\phi = 20.6436$ while $\vartheta = 0.4262$. Hence, according to Remark 3 the margins provided by Theorem 12 and Proposition 3 are sharper than the margins provided by Theorems 6 and 8.

In Figs. 4 and 5, for the choice $(M_1, M_2) = (4, 6)$, the variations of the bounds α and h , respectively, versus the variation of the input multiplicity of the sampling N , are given for the case of the static TPMRC-based LQ optimal regulator. Finally, for the case of the dynamic TPMRC-based LQ optimal regulator and for the choice $N = 30$, the variations of the afore-mentioned bounds versus the output multiplicities of the sampling M_1 and M_2 are depicted in Figs. 6 and 7. ♦

Example 2. Consider the third-order controllable and observable system of the form (1) with

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & -\pi^2 - 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 1 & -2 \\ 5 & 1 & -1 \end{bmatrix}$$

Let the basic sampling period be $T_0 = 1$ sec and the uniform input multiplicity of the sampling be $N = 45$. It is worth noticing that, according to the results in (Al-Rahmani and Franklin, 1992), for the above system single-rate sampling with period $T_0 = 1$ sec cannot be used because the respective discrete-system is not controllable. In this respect, in the sequel, we try to design a TPMRC-based LQ optimal regulator in order to minimize the performance index of the form (10) with $\mathbf{Q} = \text{diag}\{0.5, 0.5\}$ and $\mathbf{\Gamma} = \text{diag}\{8, 8\}$. To this end, since an observability index vector for the pair (\mathbf{A}, \mathbf{B}) is given by $(n_1, n_2) = (1, 2)$, we can chose the output multiplicities of the sampling M_1, M_2 as $(M_1, M_2) = (6, 5)$. With this choice we have

$$\mathbf{W}_N(T_0, 0) = \begin{bmatrix} 0.4323 & 0 & 0 \\ 0 & 0.2160 & 0.00002 \\ 0 & 0.00002 & 0.0199 \end{bmatrix}$$

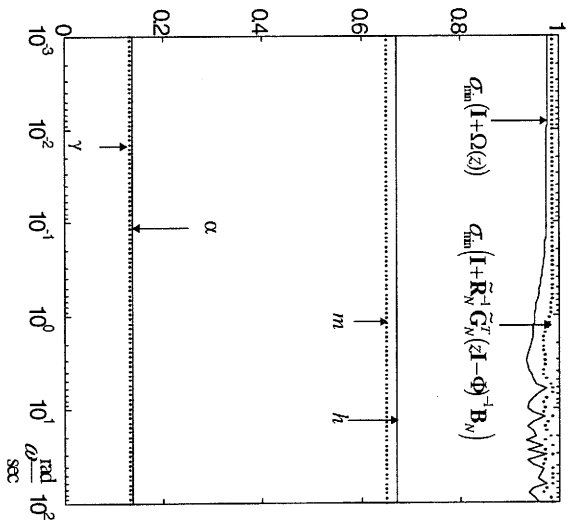


Fig. 2. Plots of $\sigma_{\min}(\mathbf{I} + \Omega(z))$, $\sigma_{\min}(\mathbf{I} + \tilde{\mathbf{R}}_N^{-1} \tilde{\mathbf{G}}_N^T(z\mathbf{I} - \Phi)^{-1} \mathbf{B}_N)$, α , γ , h and m of Example 1 in the case of static TPMRCs.

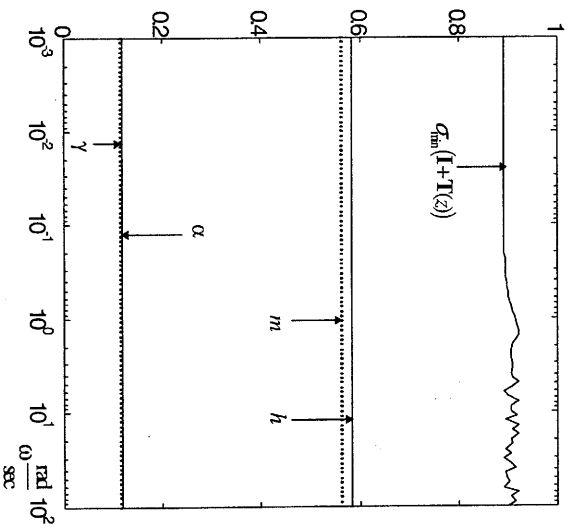


Fig. 3. Plots of $\sigma_{\min}(\mathbf{I} + \mathbf{T}(z))$, α , γ , h and m of Example 1 in the case of dynamic TPMRCs.

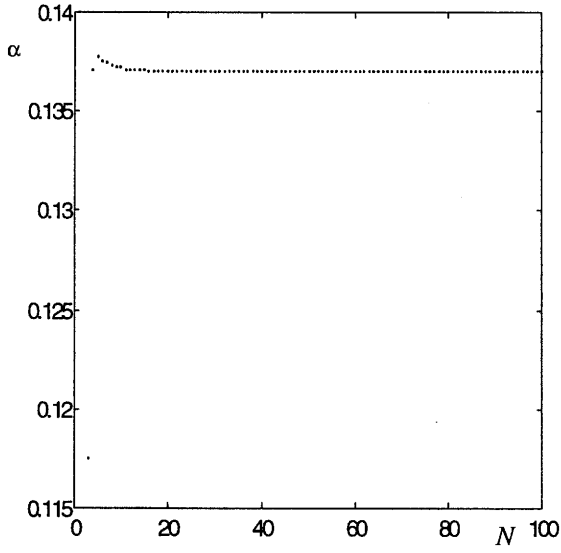


Fig. 4. Bound α versus N of Example 1 and for $(M_1, M_2) = (4, 6)$ in the case of static TPMRCs.

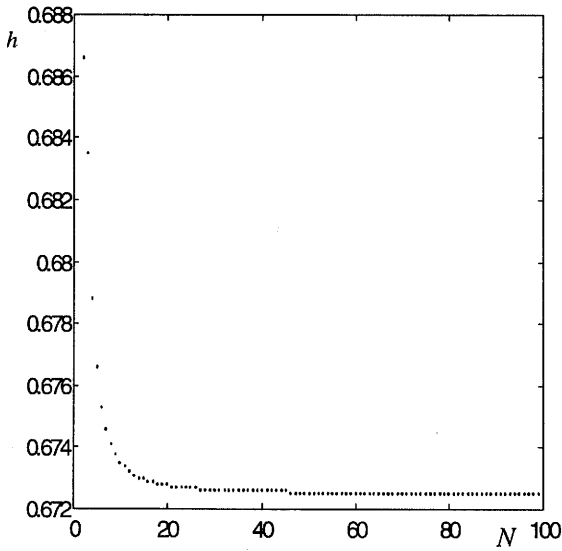


Fig. 5. Bound h versus N of Example 1 and for $(M_1, M_2) = (4, 6)$ in the case of static TPMRCs.

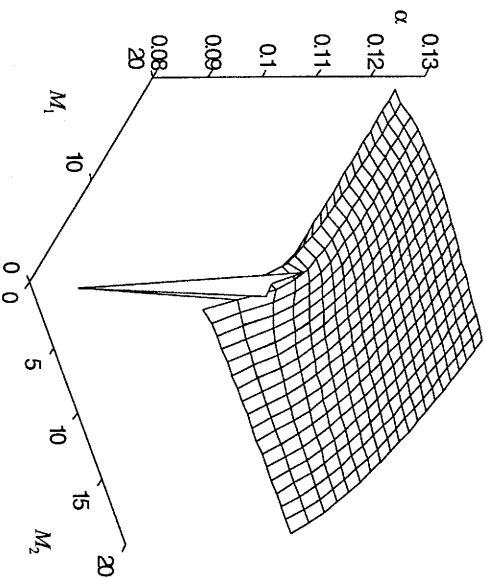


Fig. 6. Bound α versus M_1 , M_2 of Example 1 and for $N = 30$ in the case of dynamic TPMRCs.

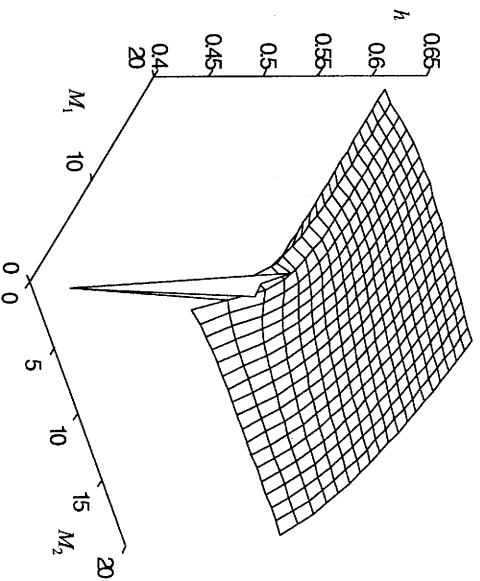


Fig. 7. Bound h versus M_1 , M_2 of Example 1 and for $N = 30$ in the case of dynamic TPMRCs.

$$\mathbf{B}_N = \begin{bmatrix} 0.6575 & 0 & 0 \\ 0 & 0.4648 & 0 \\ 0 & 0.00004 & 0.1410 \end{bmatrix}, \quad \tilde{\mathbf{Q}}_N = \begin{bmatrix} 5.6203 & -0.0409 & -3.5289 \\ -0.0409 & 0.2659 & 0.0995 \\ -3.5289 & 0.0995 & 4.3863 \end{bmatrix}$$

$$\tilde{\mathbf{\Gamma}}_N = \begin{bmatrix} 9.6553 & 0.1414 & 0.0617 \\ 0.1414 & 8.0422 & -0.0076 \\ 0.0617 & -0.0076 & 8.0123 \end{bmatrix}, \quad \tilde{\mathbf{G}}_N = \begin{bmatrix} 2.0647 & 0.0120 & 0.1422 \\ -0.3571 & -0.0614 & -0.0194 \\ -1.0599 & 0.1035 & -0.1634 \end{bmatrix}$$

The optimal fictitious state feedback is given by

$$\mathbf{F} = \begin{bmatrix} 0.2813 & -0.0308 & -0.0285 \\ -0.0028 & -0.0131 & 0.0111 \\ 0.0058 & -0.0039 & -0.0531 \end{bmatrix}$$

Then, evaluating (40) and (41), the TPMRC gains \mathbf{K} and \mathbf{L}_u are given by

$$\mathbf{K} = 10^{-4} \times \begin{bmatrix} -17 & -55 & -62 & -47 & -20 & 7 & 125 & 59 & 32 & 34 & 48 \\ 17 & 11 & 4 & -3 & -7 & -9 & 5 & 2 & -4 & -9 & -11 \\ -11 & -20 & -21 & -16 & -7 & 1 & 15 & 0.6 & -4 & -1 & 6 \end{bmatrix}$$

$$\mathbf{L}_u = \begin{bmatrix} -0.1634 & 0.0156 & 0.0043 \\ -0.0035 & 0.0053 & -0.0019 \\ -0.0041 & 0.0024 & 0.0069 \end{bmatrix}$$

The eigenvalues of \mathbf{L}_u are $\lambda_1 = -0.1630$ and $\lambda_{3,4} = 0.0059 \pm j0.0017$, and they lie inside the unit circle. This means that the designed TPMRC-based LQ optimal regulator is stable.

In the case where it is desired to design a static TPMRC-based LQ optimal regulator, it is easy to check that with the same choice of M_1, M_2 as above $[\mathbf{H} \ \mathbf{D}]$ has full column rank. Evaluating (42), we obtain

$$\mathbf{K} = 10^{-4} \times \begin{bmatrix} -281 & 437 & 32 & -773 & -1024 & 48 & -665 & 613 & 407 & -89 & 708 \\ -5 & -13 & 3 & 7 & -35 & -134 & -19 & -7 & 36 & 47 & -23 \\ -240 & 198 & 62 & -274 & -328 & 278 & -191 & 329 & 107 & -241 & 78 \end{bmatrix}$$

Let us now compute guaranteed stability margins for the afore-mentioned TPMRC-based LQ optimal regulator. To this end, observe first that here $\hat{\mathbf{\Gamma}}_N$ is nonsingular, since

$$\hat{\mathbf{\Gamma}}_N = \begin{bmatrix} 0.0448 & -0.0006 & -0.0001 \\ -0.0006 & 0.0269 & 0.00001 \\ 0.0001 & 0.00001 & 0.0025 \end{bmatrix}$$

Note also that (56) and (63) hold and that (64) holds for $q = 0.8951$. On the other hand, (60) does not hold since, as can be easily checked, $\tilde{Q}_N - \mathbf{I}$ is not positive semi-definite. The matrices Φ and $\hat{\Phi}$ are both asymptotically stable with $\sigma_{\max}(\Phi) = 0.3679 < 1$ (Case I of Theorem 10) and $\sigma_{\max}(\hat{\Phi}) = 0.3798 < 1$ (Case I of Theorem 12). However, \tilde{G}_N cannot be expressed either in the form (66) or in the form (74). Finally, in the present case (77) and (78) are satisfied. That is, in our case, in order to compute guaranteed stability margins for the designed TP-MRC-based LQ optimal regulators, only the bounds provided by Theorems 6, 8, 12 and 13 and Propositions 3 and 8 can be used. Applying these results for the case of static TP-MRC-based LQ optimal regulators, we obtain the bounds $\alpha = 0.3163$, $\gamma = 0.2831$, $h = 0.7466$, $m = 0.6683$, $\hat{\alpha} = 0.7321$ and $\hat{\gamma} = 0.6553$ for $\sigma_{\min}(\mathbf{I} + \Omega(z))$ as $|z| = 1$. These bounds are depicted in Fig. 8, together with the plots of $\sigma_{\min}(\mathbf{I} + \Omega(z))$ (solid line) and $\sigma_{\min}(\mathbf{I} + \tilde{\Gamma}_N^{-1} \tilde{G}_N(z) \mathbf{I} - \Phi)^{-1} \mathbf{B}_N$ (dotted line). In the case of a dynamic TP-MRC-based LQ optimal regulator, since $\sigma_{\max}(\mathbf{L}_u) = 0.1643$, we obtain the bounds $\alpha = 0.2717$, $\gamma = 0.2432$, $h = 0.6412$, $m = 0.5740$, $\hat{\alpha} = 0.6287$ and $\hat{\gamma} = 0.5627$ for $\sigma_{\min}(\mathbf{I} + \mathbf{T}(z))$ as $|z| = 1$. These bounds are depicted in Fig. 9, together with the plots of $\sigma_{\min}(\mathbf{I} + \mathbf{T}(z))$. Note that, in the present case, $\phi = 162.6548$, while $\vartheta = 10.8458$ and $\mu = 12.1788$. Hence, according to Remark 3, the margins provided by Theorem 12 and Proposition 3 are sharper than the margins provided by Theorems 6, 8, 13 and Proposition 8.

In Figs. 10–12, for the choice $(M_1, M_2) = (6, 5)$, the variations of the bounds α , h and $\hat{\alpha}$, respectively, are given versus the variation of the input multiplicity of the sampling N , for the case of the static TP-MRC-based LQ optimal regulator. Finally, for the case of the dynamic TP-MRC-based LQ optimal regulator and for the choice $N = 45$, the variations of the afore-mentioned bounds with respect to the output multiplicities of the sampling M_1 and M_2 are depicted in Figs. 13–15. ♦

Example 3. Consider the case where

$$\mathbf{A} = \begin{bmatrix} -1 & -0.5 \\ -1.5 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1.5 & -1 \\ 3 & 3 \end{bmatrix}, \quad \mathbf{C} = \mathbf{I}_{2 \times 2}$$

$$T_0 = 0.1 \text{ sec}, \quad \mathbf{Q} = \begin{bmatrix} 20 & 1 \\ 1 & 20 \end{bmatrix}, \quad \mathbf{\Gamma} = \text{diag} \{20, 20\}$$

Let the uniform input multiplicity of the sampling be $N = 2$. We want to design a TP-MRC-based LQ optimal regulator in order to minimize the performance index of the form (10). To this end, since an observability index vector for the pair (\mathbf{A}, \mathbf{B}) is given by $(n_1, n_2) = (1, 1)$, we can choose the output multiplicities of the sampling M_1, M_2 as $(M_1, M_2) = (2, 3)$. With this choice we have

$$\mathbf{W}_N(T_0, 0) = \begin{bmatrix} 0.2898 & 0.0754 \\ 0.0754 & 1.6171 \end{bmatrix}$$

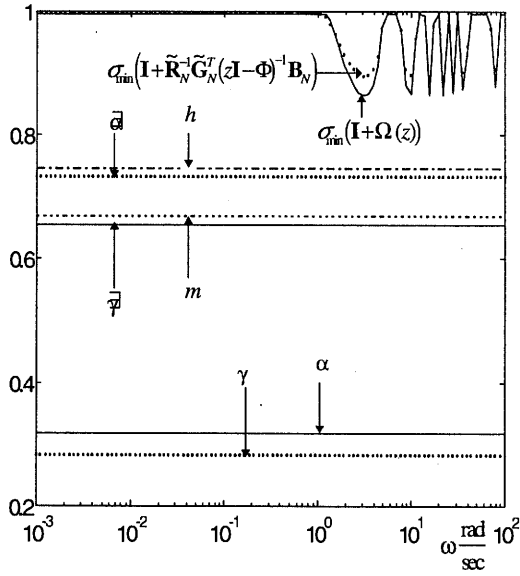


Fig. 8. Plots of $\sigma_{\min}(\mathbf{I} + \Omega(z))$, $\sigma_{\min}(\mathbf{I} + \tilde{\mathbf{R}}_N^{-1} \tilde{\mathbf{G}}_N^T (z\mathbf{I} - \Phi)^{-1} \mathbf{B}_N)$, α , γ , h , m $\hat{\alpha}$ and $\hat{\gamma}$ of Example 2 in the case of static TPMRCs.

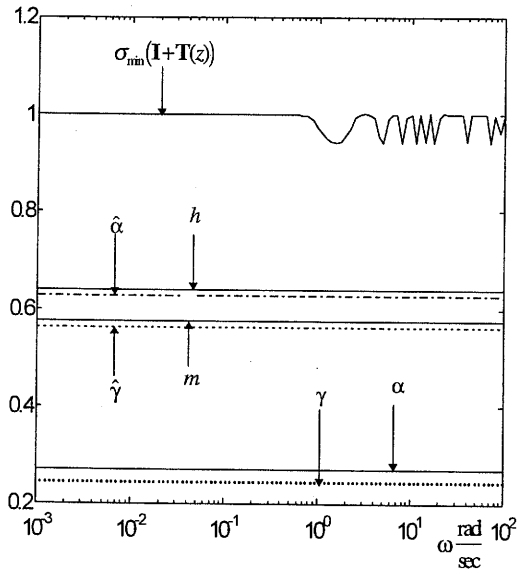


Fig. 9. Plots of $\sigma_{\min}(\mathbf{I} + \mathbf{T}(z))$, α , γ , h , m $\hat{\alpha}$ and $\hat{\gamma}$ of Example 2, in the case of dynamic TPMRCs.

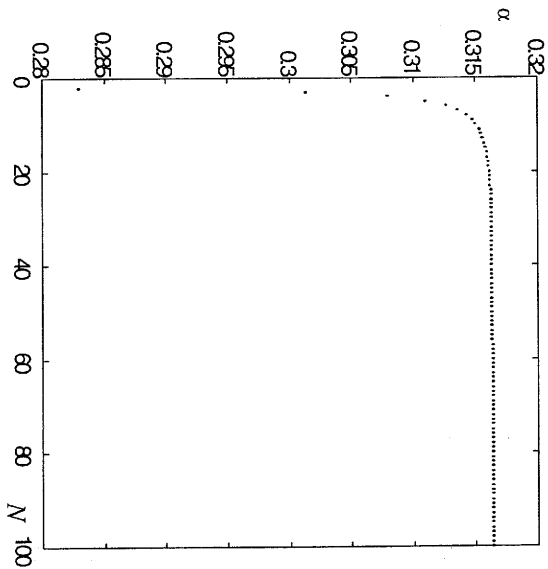


Fig. 10. Bound α versus N of Example 2 for $(M_1, M_2) = (6, 5)$ in the case of static TPMRCs.

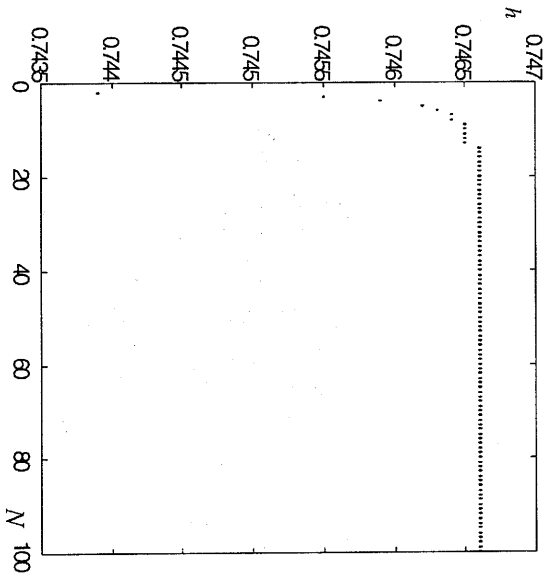


Fig. 11. Bound h versus N of Example 2 for $(M_1, M_2) = (6, 5)$ in the case of static TPMRCs.

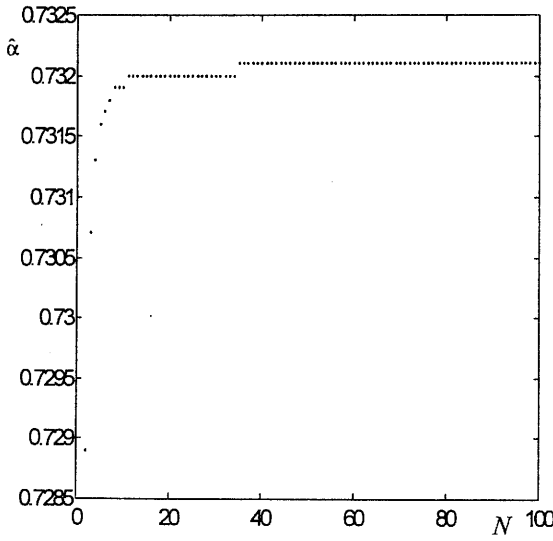


Fig. 12. Bound $\hat{\alpha}$ versus N of Example 2 and for $(M_1, M_2) = (6, 5)$ in the case of static TPMRCs.

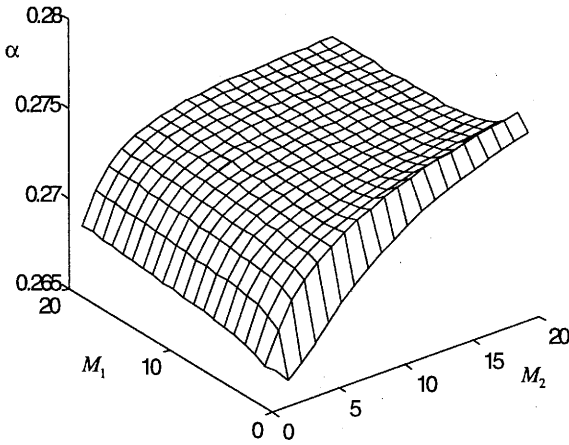


Fig. 13. Bound α versus M_1, M_2 of Example 2 and for $N = 45$ in the case of dynamic TPMRCs.

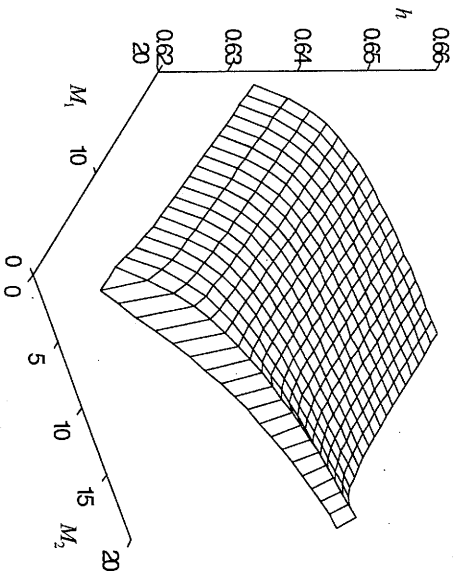


Fig. 14. Bound h versus M_1 , M_2 of Example 2 for $N = 45$ in the case of dynamic TPMRCs.

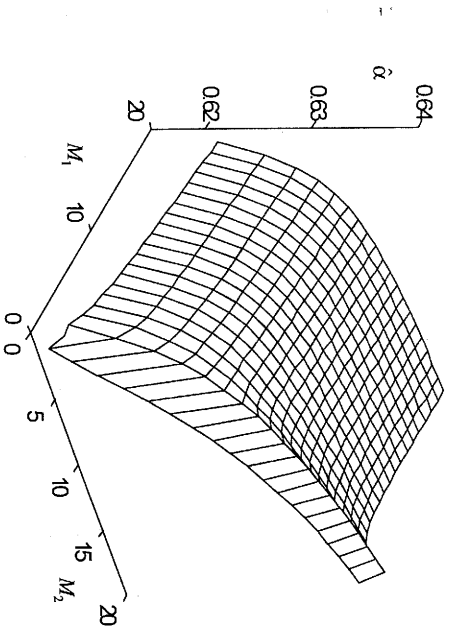


Fig. 15. Bound $\hat{\alpha}$ versus M_1 , M_2 of Example 2 for $N = 45$ in the case of dynamic TPMRCs.

$$B_N = \begin{bmatrix} 0.5383 & 0 \\ 0.14 & 1.2639 \end{bmatrix}, \quad \tilde{Q}_N = \begin{bmatrix} 1.8168 & -0.0846 \\ -0.0846 & 1.8140 \end{bmatrix}$$

$$\tilde{\Gamma}_N = \begin{bmatrix} 20.2107 & 0.1383 \\ 0.1383 & 21.0693 \end{bmatrix}, \quad \tilde{G}_N = \begin{bmatrix} 0.4974 & -0.0529 \\ 0.1337 & 1.88 \end{bmatrix}$$

The optimal fictitious state feedback is given by

$$F = \begin{bmatrix} 0.1801 & -0.0124 \\ -0.0939 & 0.2140 \end{bmatrix}$$

Evaluating (40) and (41), the TPMRC gains K and L_u are given by

$$K = \begin{bmatrix} 0.0860 & 0.0819 & -0.0021 & -0.0062 & -0.0100 \\ -0.0541 & -0.0531 & 0.0675 & 0.0680 & 0.0684 \end{bmatrix}$$

$$L_u = \begin{bmatrix} -0.0744 & 0.0049 \\ 0.0153 & -0.1840 \end{bmatrix}$$

The designed TPMRC-based LQ regulator is stable since the eigenvalues of L_u , which are $\lambda_1 = -0.0737$ and $\lambda_2 = -0.1846$, lie inside the unit circle.

If it is desired to design a static TPMRC-based LQ optimal regulator, it is easy to check that with the same choice of M_1 and M_2 as above, the matrix $[H \ D]$ has full column rank. Evaluating (42), we obtain

$$K = \begin{bmatrix} -0.1808 & 0.3614 & 0.0113 & -0.0071 & -0.0163 \\ 0.0896 & -0.1826 & -0.1426 & 0.0737 & 0.2834 \end{bmatrix}$$

Let us now compute guaranteed stability margins for the afore-mentioned TPMRC-based LQ optimal regulator. To this end, observe first that here $\hat{\Gamma}_N$ is nonsingular, since

$$\hat{\Gamma}_N = \begin{bmatrix} 0.0143 & 0.0035 \\ 0.0035 & 0.0767 \end{bmatrix}$$

Note also that (56) and (63) hold and that (64) holds for $q = 0.962$. As can be easily checked, (60) and (61) also hold. The matrices Φ and $\hat{\Phi}$ are both asymptotically stable with $\sigma_{\max}(\Phi) = 0.99 < 1$ (Case I of Theorem 10) and $\sigma_{\max}(\hat{\Phi}) = 0.9635 < 1$ (Case I of Theorem 12). However, \tilde{G}_N cannot be expressed either in the form (66) or in the form (74). Finally, in the present case, inequalities (77) and (78) are satisfied. That is, in our case in order to compute guaranteed stability margins for the designed TPMRC-based LQ optimal regulators, the bounds provided by Theorems 6–8, 12 and 13, and Propositions 2, 3, 7 and 8 can be used. Applying these results for the case

of static TPMRC-based LQ optimal regulators, we obtain the bounds $\alpha = 0.3522$, $b = 0.3475$, $\gamma = 0.3388$, $h = 0.5542$, $g = 0.5468$, $m = 0.5332$, $\hat{\alpha} = 0.6163$, $\hat{b} = 0.6081$ and $\hat{\gamma} = 0.5928$ for $\sigma_{\min}(\mathbf{I} + \mathbf{Q}(z))$ as $|z| = 1$. In the case of a dynamic TPMRC-based LQ optimal regulator, since $\sigma_{\max}(\mathbf{L}_u) = 0.1850$, we obtain the bounds $\alpha = 0.2972$, $b = 0.2932$, $\gamma = 0.2859$, $h = 0.4677$, $g = 0.4615$, $m = 0.4499$, $\hat{\alpha} = 0.5201$, $\hat{b} = 0.5131$ and $\hat{\gamma} = 0.5003$ for $\sigma_{\min}(\mathbf{I} + \mathbf{T}(z))$ as $|z| = 1$. Note that, in the present case, $\phi = 87.3864$ while $\vartheta = 27.5298$ and $\mu = 19.7791$. Hence, according to Remark 3, the margins provided by Theorem 13 and Propositions 7 and 8 are sharper than the margins provided by Theorems 6–8 and 12, and Propositions 2 and 3. ♦

Example 4. Let

$$\mathbf{A} = \begin{bmatrix} 6 & -0.5 \\ -1.5 & -1 \end{bmatrix}$$

Let all the other matrices, the sampling period and the input multiplicity of the sampling be as in Example 3. Since an observability index vector for the pair (\mathbf{A}, \mathbf{B}) is given by $(n_1, n_2) = (1, 1)$, we can chose the output multiplicities of the sampling M_1 and M_2 as $(M_1, M_2) = (3, 4)$. With this choice we have

$$\mathbf{W}_N(T_0, 0) = \begin{bmatrix} 0.6141 & 0.1020 \\ 0.1020 & 1.6138 \end{bmatrix}$$

$$\mathbf{B}_N = \begin{bmatrix} 0.7837 & 0 \\ 0.1302 & 1.2637 \end{bmatrix}, \quad \tilde{\mathbf{Q}}_N = \begin{bmatrix} 3.8719 & -0.1286 \\ -0.1286 & 1.8148 \end{bmatrix}$$

$$\tilde{\mathbf{T}}_N = \begin{bmatrix} 20.4027 & 0.1551 \\ 0.1551 & 21.0644 \end{bmatrix}, \quad \tilde{\mathbf{G}}_N = \begin{bmatrix} 1.1215 & -0.0611 \\ 0.1476 & 1.1844 \end{bmatrix}$$

The optimal fictitious state feedback is given by

$$\mathbf{F} = \begin{bmatrix} 1.6686 & -0.0984 \\ -0.2391 & 0.2157 \end{bmatrix}$$

Evaluating (40) and (41), the TPMRC gains \mathbf{K} and \mathbf{L}_u are given by

$$\mathbf{K} = \begin{bmatrix} 0.6460 & 0.7893 & 0.9649 & 0.0038 & -0.0222 & -0.0516 & -0.0852 \\ -0.0975 & -0.1201 & -0.1476 & 0.0482 & 0.0509 & 0.0542 & 0.0582 \end{bmatrix}$$

$$\mathbf{L}_u = \begin{bmatrix} -0.6414 & 0.0024 \\ 0.0705 & -0.1667 \end{bmatrix}$$

The designed TPMRC-based LQ regulator is stable since the eigenvalues of \mathbf{L}_u , which are $\lambda_1 = -0.1663$ and $\lambda_2 = -0.6418$, lie inside the unit circle.

If it is desired to design a static TPMRC-based LQ optimal regulator, it is easy to check that with the same choice of M_1 and M_2 as above, the matrix $[H \ D]$ has full column rank. Evaluating (42), we obtain

$$K = \begin{bmatrix} -1.1520 & 0.5437 & 2.3689 & 0.0113 & 0.0250 & -0.0112 & -0.1289 \\ 0.1744 & -0.0775 & -0.3483 & -0.1020 & -0.0029 & 0.1010 & 0.2206 \end{bmatrix}$$

Let us compute now guaranteed stability margins for the afore-mentioned TPMRC-based LQ optimal regulator. To this end, observe first that here the matrix $\widehat{\Gamma}_N$ is nonsingular, since

$$\widehat{\Gamma}_N = \begin{bmatrix} 0.0301 & 0.0046 \\ 0.0046 & 0.0765 \end{bmatrix}$$

Note also that (56) and (63) hold and that (64) holds for $q = 0.9494$. As can be easily checked, (60) and (61) also hold. The matrices $\widehat{\Phi}$ and $\widehat{\Psi}$ are not asymptotically stable. In particular, the eigenvalues of $\widehat{\Phi}$ are $\lambda_1(\widehat{\Phi}) = 1.7992$ and $\lambda_2(\widehat{\Phi}) = 0.8226$. That is, the assumptions related to Case III of Theorem 12 are satisfied. Finally, in the present case (78) does not hold. That is, in our case, in order to compute guaranteed stability margins for the designed TPMRC-based LQ optimal regulators, the bounds provided by Theorems 6–8 and 12, and Propositions 2, and 3 can be used. Applying these results to the case of static TPMRC-based LQ optimal regulators, we obtain the bounds $\alpha = 0.2929$, $b = 0.2838$, $\gamma = 0.278$, $h = 0.0729$, $g = 0.0706$ and $m = 0.0692$ for $\sigma_{\min}(\mathbf{I} + \Omega(z))$ as $|z| = 1$. In the case of a dynamic TPMRC-based LQ optimal regulator, since $\sigma_{\max}(\mathbf{L}_u) = 0.6457$, we obtain the bounds $\alpha = 0.178$, $b = 0.1724$, $\gamma = 0.169$, $h = 0.0443$, $g = 0.0429$ and $m = 0.0421$ for $\sigma_{\min}(\mathbf{I} + \mathbf{T}(z))$ as $|z| = 1$. Note that, in the present case, $\phi = 133.21$, while $\vartheta = 2346.0352$. Hence, according to Remark 3, the margins provided by Theorems 6–8 are sharper than the margins provided by Theorem 12 and Propositions 2 and 3, respectively. \blacklozenge

Example 5. Consider the controllable and observable system with

$$A = \begin{bmatrix} 7 & -2.5 & -8 \\ -6.5 & 9 & -10 \\ -2 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1.5 & -1 \\ 3 & 3 \\ 5 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

Let the basic sampling period be $T_0 = 0.1$ sec and the uniform input multiplicity of the sampling be $N = 3$. Let $Q = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ and $\Gamma = \text{diag} \{20, 20\}$. In this case, since an observability index vector for the pair (A, B) is given by $(n_1, n_2) = (2, 1)$, we can chose the output multiplicities of the sampling M_1 and M_2 as $(M_1, M_2) = (4, 3)$. With this choice we have

$$W_N(T_0, 0) = \begin{bmatrix} 1.1842 & -1.4144 & -1.0141 \\ & 2.7437 & 1.3665 \\ \text{symmetry} & & 3.4309 \end{bmatrix}$$

$$B_N = \begin{bmatrix} 1.0882 & 0 & 0 \\ -1.2997 & 1.0269 & 0 \\ -0.9319 & 0.1513 & 1.5036 \end{bmatrix}, \quad \tilde{Q}_N = \begin{bmatrix} 1.0156 & -0.2421 & -0.6783 \\ 0.1461 & 1.2794 & \\ \text{symmetry} & & 19.2158 \end{bmatrix}$$

$$\tilde{\Gamma}_N = \begin{bmatrix} 27.0685 & 3.3055 & -8.9593 \\ 22.4114 & -3.4383 & \\ \text{symmetry} & 32.3490 & \end{bmatrix}, \quad \tilde{G}_N = \begin{bmatrix} 0.6966 & -0.0486 & -0.7312 \\ -0.8851 & -0.2835 & 1.1625 \\ -10.9007 & -5.6678 & 13.3090 \end{bmatrix}$$

The optimal fictitious state feedback is given by

$$F_N = \begin{bmatrix} 1.3456 & -0.8798 & -0.2897 \\ -0.3908 & 0.6431 & -0.5640 \\ 0.1995 & -0.5937 & 0.9746 \end{bmatrix}$$

As can be easily checked, in the case where we try to design a TPMRC-based LQ regulator with free L_u , the controller is unstable, since the matrix L_u is unstable. In order to obtain a stable controller, we set $L_u = \text{diag}\{0.3, -0.5, 0.6\}$. Then the matrix $[H \ D]$ has full column rank. Evaluating (42), we obtain

$$K = \begin{bmatrix} 19.3527 & -22.8180 & -25.4347 & 27.6246 & -9.1738 & 22.0474 & -11.7828 \\ -20.8644 & 20.9507 & 24.9884 & -23.8637 & 9.5287 & -19.3460 & 8.8867 \\ 12.2918 & -14.5028 & -16.2751 & 16.9141 & -5.6022 & 13.06 & -6.2055 \end{bmatrix}$$

Moreover, for the case of a static TPMRC-based LQ optimal regulator, evaluating (42), we obtain

$$K = \begin{bmatrix} 23.2916 & -25.5751 & -29.3034 & 30.4755 & -11.0346 & 24.6757 & -12.6858 \\ -16.7838 & 18.0522 & 20.9708 & -20.8678 & 7.7702 & -16.8911 & 8.0581 \\ 15.4688 & -16.5099 & -19.2614 & 18.8887 & -7.0682 & 15.1606 & -6.9673 \end{bmatrix}$$

The guaranteed stability margins of the afore-mentioned TPMRC-based LQ optimal regulator can be computed as follows: Observe first that here-matrix $\tilde{\Gamma}_N$ is nonsingular, since

$$\tilde{\Gamma}_N = \begin{bmatrix} 0.0487 & -0.0630 & -0.0235 \\ 0.1294 & 0.0453 & \\ \text{symmetry} & & 0.0945 \end{bmatrix}$$

Relations (56) and (63) hold and relation (64) holds for $q = 0.6515$. On the other hand, inequalities (60) and (61) do not hold. The matrices Φ and $\tilde{\Phi}$ are not asymptotically stable. In particular, the eigenvalues of $\tilde{\Phi}$ are $\lambda_1(\tilde{\Phi}) = 3.2762$, $\lambda_2(\tilde{\Phi}) = 1.6415$ and $\lambda_3(\tilde{\Phi}) = 0.4395$. That is, the assumptions related to Case III

of Theorem 12 are satisfied. Finally, in the present case relations (77) and (78) are satisfied. That is, in our case, in order to compute guaranteed stability margins for the designed TPMRC-based LQ optimal regulators, the bounds provided by Theorems 6, 8, 12 and 13, and Propositions 3 and 8 can be used. Applying these results to the case of static TPMRC-based LQ optimal regulators, we obtain the bounds $\alpha = 0.0320$, $\gamma = 0.0208$, $h = 9.1239 \times 10^{-4}$, $m = 5.9445 \times 10^{-4}$, $\hat{\alpha} = 0.551$ and $\hat{\gamma} = 0.359$ for $\sigma_{\min}(\mathbf{I} + \mathbf{\Omega}(z))$ as $|x| = 1$. In the case of the dynamic TPMRC-based LQ optimal regulator, since $\sigma_{\max}(\mathbf{L}_u) = 0.6$, we obtain the bounds $\alpha = 0.02$, $\gamma = 0.013$, $h = 5.7025 \times 10^{-4}$, $m = 3.7153 \times 10^{-4}$, $\hat{\alpha} = 0.3444$ and $\hat{\gamma} = 0.2244$ for $\sigma_{\min}(\mathbf{I} + \mathbf{T}(z))$ as $|z| = 1$. Note that, in the present case, $\phi = 3761.142$ while $\vartheta = 4634684.9424$ and $\mu = 4.9611$. Hence, according to Remark 3, the margins provided by Theorem 13 and Proposition 8 are sharper than the margins provided by Theorems 6, 8, 12 and Proposition 3.

From the above illustrative examples, it can be easily seen that the stability robustness analysis of a TPMRC-based LQ regulator is a hard task, and that the search for its guaranteed stability margins is quite complicated. It can also be seen that the guaranteed stability margins of a dynamic TPMRC-based LQ optimal regulator change according to the transition matrix of the controller itself. Finally, it becomes clear that, for a particular basic sampling period, the suggested guaranteed stability margins change with the values of the input and output multiplicities of the sampling, which become for this reason factors of crucial importance for the overall design procedure of a TPMRC-based LQ regulator.

7. Conclusions

In the present paper, the LQ optimal regulation problem for continuous-time systems has been solved using a novel multirate control scheme based on Two-Point-Multirate Controllers. On the basis of the proposed approach, the original LQ regulation problem has been transformed to a simplified single-rate-like discrete-time LQ regulation problem for the performance index with cross-product terms, for a relatively low-dimensional equivalent discrete time-invariant state-space model of the multirate digital control system. For this simplified problem, a fictitious static state feedback controller is needed to be computed. The proposed method is more practical and more effective as a design tool than currently used multirate schemes, since it provides the ability of the reconstruction of the action of static state feedback controllers, without resorting to the design of state estimators, and without introducing high-order exogenous dynamics in the control loop. Furthermore, LQ regulators designed on the basis of the proposed technique can possess any prescribed degree of stability, since there is a possibility to choose the transition matrices of the controllers arbitrarily.

Moreover, in this paper, the stability robustness properties of the TPMRC-based LQ regulator have been investigated. In particular, it has been clarified what the stability margins of such a type of LQ regulator are and how sensitive its stability is to small variations in the parameters of the plant. Guaranteed stability margins of the TPMRC-based LQ regulator have been derived in terms of the singular values

of the elementary cost and system matrices associated with the state-space description of the equivalent discrete-time single-rate-like LQ optimal design. The suggested guaranteed stability margins have been obtained on the basis of lower bounds on the minimum singular value the regulator's return difference matrix which obeys the so-called Modified Return Difference Equality. Sufficient conditions to guarantee the suggested stability margins are established and the connection between the suggested stability margins and the selection of cost weighting matrices is presented. From the theoretical analysis, as well as from the illustrative examples reported in the paper, it can be easily recognized that the stability margins of a dynamic TPMRC-based LQ optimal regulator change according to the transition matrix of the controller itself and can be set at satisfactory values by simply adjusting the TPMRC dynamics. Moreover, it has been shown that, for a particular basic sampling period, the suggested guaranteed stability margins change with the values of the input and output multiplicities of the sampling which become for this reason important factors of the overall design procedure of a TPMRC-based LQ regulator. Notice also that, due to the difficulty in obtaining a simple universal lower bound for the minimum singular value of the regulator's return difference matrix, analogous to the continuous-time case, our investigation on guaranteed stability margins has been focused on a broad variety of particular cases for the matrices of the state-space description of the equivalent LQR-CPT design. A universal analysis of the robustness properties of the TPMRC-based LQ regulator (as well as of other types of discrete-time regulators) would probably be plausible if a universal upper bound for the maximum singular value of the solution of the discrete algebraic Riccati equation was available. However, up to now, such a universal upper bound has not been found.

The results of the present paper can easily be extended in order to solve other important problems of the area of multirate digital systems, such as the pole placement problem and problems of the areas of adaptive and decentralized control. On the other hand, our investigation on guaranteed stability margins of the TPMRC-based LQ regulator can be used as a guideline to choose cross-product, state and input weighting matrices of the performance index for loop shaping with guaranteed stability margins. Furthermore, these results can be readily used as an easy test of stability robustness of other types of LQ optimal regulators, as e.g. the regulator proposed in (Apostolakis and Jordan, 1991). Work along this line is currently in progress.

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Appendices

Appendix A. Proof of Theorem 5.

Observe that, on the basis of (54), $F(z)$ can also be written as

$$\begin{aligned} F(z) = & I + KH(zI - \Phi)^{-1}B_N + L_u(zI - L_u)^{-1}KH(zI - \Phi)^{-1}B_N \\ & + z^{-1}L_u + z^{-1}L_u(zI - L_u)^{-1}L_u \end{aligned} \quad (A1)$$

and that the following relation holds:

$$z^{-1}L_u + z^{-1}L_u(zI - L_u)^{-1}L_u = L_u(zI - L_u)^{-1} \quad (A2)$$

Introducing (A2) in (A1), and defining

$$\Omega(z) = I + KH(zI - \Phi)^{-1}B_N \quad (A3)$$

relation (A1) yields

$$F(z) = z(zI - L_u)^{-1}\Omega(z)$$

The above relation implies that

$$\Omega(z) = (I - z^{-1}L_u)F(z)$$

Hence, in order to prove (55), it suffices to prove that

$$\Omega^T(z^{-1})(\tilde{\Gamma}_N + B_N^T P B_N)\Omega(z) = \tilde{\Gamma}_N + S(z) + W(z) \quad (A4)$$

To prove (A4), we work as follows: On the basis of (A3) and after some easy manipulations, the left-hand side of (A4) takes the form

$$\begin{aligned} \Omega^T(z^{-1})(\tilde{\Gamma}_N + B_N^T P B_N)\Omega(z) &= (\tilde{\Gamma}_N + B_N^T P B_N) \left[I + KH(zI - \Phi)^{-1} B_N \right] \\ &+ B_N^T(z^{-1}I - \Phi^T)^{-1} (KH)^T (\tilde{\Gamma}_N + B_N^T P B_N) \left[I + KH(zI - \Phi)^{-1} B_N \right] \end{aligned} \quad (\text{A5})$$

From (33) (or, alternatively, from (34) and (36)), we obtain

$$KH = (\tilde{\Gamma}_N + B_N^T P B_N)^{-1} (\tilde{G}_N^T + B_N^T P \Phi) \quad (\text{A6})$$

Introducing (A6) into (A5) yields

$$\begin{aligned} \Omega^T(z^{-1})(\tilde{\Gamma}_N + B_N^T P B_N)\Omega(z) &= \tilde{\Gamma}_N + B_N^T P B_N + (\tilde{G}_N^T + B_N^T P \Phi)(zI - \Phi)^{-1} B_N \\ &+ B_N^T(z^{-1}I - \Phi^T)^{-1} (\Phi^T P B_N + \tilde{G}_N) + B_N^T(z^{-1}I - \Phi^T)^{-1} \\ &\times (\Phi^T P B_N + \tilde{G}_N) (\tilde{\Gamma}_N + B_N^T P B_N)^{-1} (\tilde{G}_N^T + B_N^T P \Phi) \\ &\times (zI - \Phi)^{-1} B_N \end{aligned} \quad (\text{A7})$$

From the Riccati equation (37), we can easily conclude that

$$(\Phi^T P B_N + \tilde{G}_N) (\tilde{\Gamma}_N + B_N^T P B_N)^{-1} (\tilde{G}_N^T + B_N^T P \Phi) = \Phi^T P \Phi + \tilde{Q}_N - P \quad (\text{A8})$$

Introducing (A8) into (A7), we readily obtain

$$\begin{aligned} \Omega^T(z^{-1})(\tilde{\Gamma}_N + B_N^T P B_N)\Omega(z) &= \tilde{\Gamma}_N + B_N^T P B_N + (\tilde{G}_N^T + B_N^T P \Phi)(zI - \Phi)^{-1} B_N \\ &+ B_N^T(z^{-1}I - \Phi^T)^{-1} (\Phi^T P B_N + \tilde{G}_N) + B_N^T(z^{-1}I - \Phi^T)^{-1} \\ &\times (\Phi^T P \Phi + \tilde{Q}_N - P)(zI - \Phi)^{-1} B_N = \tilde{\Gamma}_N + S(z) + \Delta(z) \end{aligned}$$

where

$$\begin{aligned} \Delta(z) &= B_N^T P B_N + (\tilde{G}_N^T + B_N^T P \Phi)(zI - \Phi)^{-1} B_N \\ &+ B_N^T(z^{-1}I - \Phi^T)^{-1} (\Phi^T P B_N + \tilde{G}_N) \\ &+ B_N^T(z^{-1}I - \Phi^T)^{-1} \Phi^T P \Phi (zI - \Phi)^{-1} B_N \\ &- B_N^T(z^{-1}I - \Phi^T)^{-1} P(zI - \Phi)^{-1} B_N = W(z) + B_N^T G(z) B_N \end{aligned}$$

and

$$\begin{aligned} G(z) &= P + P \Phi(zI - \Phi)^{-1} + (z^{-1}I - \Phi^T)^{-1} \Phi^T P \\ &+ (z^{-1}I - \Phi^T)^{-1} \Phi^T P \Phi (zI - \Phi)^{-1} - (z^{-1}I - \Phi^T)^{-1} P(zI - \Phi)^{-1} \end{aligned} \quad (\text{A9})$$

From the above analysis, it becomes clear that, in order to prove (A4), it only remains to prove that $\mathbf{G}(z) = 0$. To this end, multiplying (A9) from the left by $z^{-1}\mathbf{I} - \Phi^T$ and from the right by $z\mathbf{I} - \Phi$ yields

$$\begin{aligned} (z^{-1}\mathbf{I} - \Phi^T)\mathbf{G}(z)(z\mathbf{I} - \Phi) &= (z^{-1}\mathbf{I} - \Phi^T)\mathbf{P}(z\mathbf{I} - \Phi) + (z^{-1}\mathbf{I} - \Phi^T)\mathbf{P}\Phi \\ &\quad + \Phi^T\mathbf{P}(z\mathbf{I} - \Phi) + \Phi^T\mathbf{P}\Phi - \mathbf{P} \end{aligned} \quad (\text{A10})$$

It is not difficult to see that the right-hand side of (A10) equals zero. Therefore, $\mathbf{G}(z) = 0$. As a consequence, relation (A4) holds. This completes the proof.

Appendix B. Proof of Theorem 6.

If (56) is satisfied, then from (55), we obtain

$$(\mathbf{I} - z\mathbf{L}_u^T)(\tilde{\Gamma}_N + \mathbf{B}_N^T\mathbf{P}\mathbf{B}_N)(\mathbf{I} - z^{-1}\mathbf{L}_u) \geq \mathbf{F}^{-T}(z^{-1})\tilde{\Gamma}_N\mathbf{F}^{-1}(z)$$

which implies that

$$\sigma_{\max}\left((\mathbf{I} - z\mathbf{L}_u^T)(\tilde{\Gamma}_N + \mathbf{B}_N^T\mathbf{P}\mathbf{B}_N)(\mathbf{I} - z^{-1}\mathbf{L}_u)\right) \geq \sigma_{\max}\left(\mathbf{F}^{-T}(z^{-1})\tilde{\Gamma}_N\mathbf{F}^{-1}(z)\right)$$

Therefore, following an argument similar to that reported in (Shaked, 1986), we can easily conclude that

$$\begin{aligned} \sigma_{\max}\left((\mathbf{I} - z\mathbf{L}_u^T)(\tilde{\Gamma}_N + \mathbf{B}_N^T\mathbf{P}\mathbf{B}_N)(\mathbf{I} - z^{-1}\mathbf{L}_u)\right) \\ \geq \sigma_{\max}^2(\mathbf{F}^{-1}(z))\sigma_{\min}(\tilde{\Gamma}_N) = \sigma_{\min}^{-2}(\mathbf{F}(z))\sigma_{\min}(\tilde{\Gamma}_N) \end{aligned} \quad (\text{B1})$$

On the other hand

$$\begin{aligned} \sigma_{\max}\left((\mathbf{I} - z\mathbf{L}_u^T)(\tilde{\Gamma}_N + \mathbf{B}_N^T\mathbf{P}\mathbf{B}_N)(\mathbf{I} - z^{-1}\mathbf{L}_u)\right) \\ \leq \sigma_{\max}(\mathbf{I} - z\mathbf{L}_u^T)\sigma_{\max}(\tilde{\Gamma}_N + \mathbf{B}_N^T\mathbf{P}\mathbf{B}_N)\sigma_{\max}(\mathbf{I} - z^{-1}\mathbf{L}_u) \end{aligned} \quad (\text{B2})$$

Let also note that for $|z| = 1$ the following inequalities hold:

$$\sigma_{\max}(\mathbf{I} - z^{-1}\mathbf{L}_u) \leq 1 + \sigma_{\max}(\mathbf{L}_u) \equiv \lambda, \quad \sigma_{\max}(\mathbf{I} - z\mathbf{L}_u^T) \leq \lambda \quad (\text{B3})$$

Combining relations (B1)–(B3), we obtain

$$\sigma_{\max}(\tilde{\Gamma}_N + \mathbf{B}_N^T\mathbf{P}\mathbf{B}_N)\lambda^2 \geq \sigma_{\min}^{-2}(\mathbf{F}(z))\sigma_{\min}(\tilde{\Gamma}_N)$$

or, equivalently,

$$\begin{aligned} \sigma_{\min}(\mathbf{F}(z)) \equiv \sigma_{\min}(\mathbf{I} + \mathbf{T}(z)) &\geq \lambda^{-1} \left[\frac{\sigma_{\min}(\tilde{\Gamma}_N)}{\sigma_{\max}(\tilde{\Gamma}_N + \mathbf{B}_N^T\mathbf{P}\mathbf{B}_N)} \right]^{\frac{1}{2}} \\ &\geq \lambda^{-1} \left[\frac{\sigma_{\min}(\tilde{\Gamma}_N)}{\sigma_{\max}(\tilde{\Gamma}_N) + \sigma_{\max}^2(\mathbf{B}_N)\sigma_{\max}(\mathbf{P})} \right]^{\frac{1}{2}} \end{aligned} \quad (\text{B4})$$

Observe now that the solution to (37), is also the solution to the following Riccati equation (Al-Rahmani and Franklin, 1990):

$$P = \hat{\Phi}^T P \hat{\Phi} + \hat{Q}_N - \hat{\Phi}^T P B_N (\tilde{\Gamma}_N + B_N^T P B_N)^{-1} B_N^T P \hat{\Phi} \quad (B5)$$

Equation (B5) can also be written down as

$$P = \hat{\Phi}^T P \hat{\Phi} + \hat{Q}_N - \hat{\Phi}^T P B_{N,\Gamma} (\mathbf{I} + B_{N,\Gamma}^T P B_{N,R})^{-1} B_{N,\Gamma}^T P \hat{\Phi}$$

Since by assumption $\hat{\Gamma}_N \neq 0$, from the results of (Komaroff, 1994), we have

$$P \leq \hat{\Phi} \hat{\Gamma}_N^{-1} \hat{\Phi} + \hat{Q}_N$$

Therefore

$$\begin{aligned} \sigma_{\max}(P) &\leq \sigma_{\max}(\hat{\Phi}^T \hat{\Gamma}_N^{-1} \hat{\Phi} + \hat{Q}_N) \leq \sigma_{\max}^2(\hat{\Phi}) \sigma_{\max}(\hat{\Gamma}_N^{-1}) + \sigma_{\max}(\hat{Q}_N) \\ &= \sigma_{\max}^2(\Phi - B_N \tilde{\Gamma}_N^{-1} \tilde{G}_N^T) \sigma_{\max}(\hat{\Gamma}_N^{-1}) + \sigma_{\max}(\mathbf{Q}_N - \tilde{G}_N \tilde{\Gamma}_N^{-1} \tilde{G}_N^T) \\ &\leq \frac{[\sigma_{\max}(\Phi) + \sigma_{\max}(B_N \tilde{\Gamma}_N^{-1}) \tilde{G}_N^T]^2}{\sigma_{\min}(\tilde{\Gamma}_N)} + \sigma_{\max}(\tilde{Q}_N) + \sigma_{\max}(\tilde{G}_N \tilde{\Gamma}_N^{-1} \tilde{G}_N^T) \\ &\leq \frac{[\sigma_{\max}(\Phi) + \sigma_{\max}(B_N) \sigma_{\max}(\tilde{\Gamma}_N^{-1}) \sigma_{\max}(\tilde{G}_N)]^2}{\sigma_{\min}(\tilde{\Gamma}_N)} \\ &\quad + \sigma_{\max}(\tilde{Q}_N) + \sigma_{\max}^2(\tilde{G}_N) \sigma_{\max}(\tilde{\Gamma}_N^{-1}) \\ &= \frac{[\sigma_{\min}(\tilde{\Gamma}_N) \sigma_{\max}(\Phi) + \sigma_{\max}(B_N) \sigma_{\max}(\tilde{G}_N)]^2}{\sigma_{\min}^2(\tilde{\Gamma}_N) \sigma_{\min}(\tilde{\Gamma}_N)} \\ &\quad + \frac{\sigma_{\min}(\tilde{\Gamma}_N) \sigma_{\max}(\tilde{Q}_N) + \sigma_{\max}^2(\tilde{G}_N)}{\sigma_{\min}(\tilde{\Gamma}_N)} \\ &\leq \frac{[\sigma_{\min}(\tilde{\Gamma}_N) \sigma_{\max}(\Phi) + \sigma_{\max}(B_N) \sigma_{\max}(\tilde{G}_N)]^2}{\sigma_{\min}^2(\tilde{\Gamma}_N) \sigma_{\min}^2(B_N)} \\ &\quad + \frac{\sigma_{\min}(\tilde{\Gamma}_N) \sigma_{\max}(\tilde{Q}_N) + \sigma_{\max}^2(\tilde{G}_N)}{\sigma_{\min}(\tilde{\Gamma}_N)} \\ &= \frac{[\sigma_{\min}(\tilde{\Gamma}_N) \sigma_{\max}(\Phi) + \sigma_{\max}(B_N) \sigma_{\max}(\tilde{G}_N)]^2 \sigma_{\min}(\tilde{\Gamma}_N)}{\sigma_{\min}^2(\tilde{\Gamma}_N) \sigma_{\min}^2(B_N)} \\ &\quad + \frac{\sigma_{\min}(\tilde{\Gamma}_N) \sigma_{\max}(\tilde{Q}_N) + \sigma_{\max}^2(\tilde{G}_N)}{\sigma_{\min}(\tilde{\Gamma}_N)} = \frac{\psi}{\sigma_{\min}(\tilde{\Gamma}_N) \sigma_{\min}^2(B_N)} = \phi \quad (B6) \end{aligned}$$

Taking into account (B6), relation (B4) yields $\sigma_{\min}(\mathbf{I} + \mathbf{T}(z)) \geq \alpha$, which is the sought result. Observe that $\alpha \in [0, 1]$. Relation (58) then follows from (44). This completes the proof.

Appendix C. Proof of Theorem 7.

Observe that the modified return difference equality (55) can also be expressed as

$$\begin{aligned} & \mathbf{F}^T(z^{-1})(\mathbf{I} - z\mathbf{L}_u^T)(\tilde{\Gamma}_N + \mathbf{B}_N^T\mathbf{P}\mathbf{B}_N)(\mathbf{I} - z^{-1}\mathbf{L}_u)\mathbf{F}(z) \\ &= \tilde{\Gamma}_N - \tilde{\mathbf{G}}_N^T\tilde{\mathbf{G}}_N + \mathbf{B}_N^T(z^{-1}\mathbf{I} - \Phi^T)^{-1}(\tilde{\mathbf{Q}}_N - \mathbf{I})(z\mathbf{I} - \Phi)^{-1}\mathbf{B}_N \\ &+ [\mathbf{B}_N^T(z^{-1}\mathbf{I} - \Phi^T)^{-1} + \tilde{\mathbf{G}}_N^T][(z\mathbf{I} - \Phi)^{-1}\mathbf{B}_N + \tilde{\mathbf{G}}_N] \end{aligned} \tag{C1}$$

Since by assumption $\tilde{\mathbf{Q}}_N - \mathbf{I} \geq 0$, we conclude that

$$\mathbf{B}_N^T(z^{-1}\mathbf{I} - \Phi^T)^{-1}(\tilde{\mathbf{Q}}_N - \mathbf{I})(z\mathbf{I} - \Phi)^{-1}\hat{\mathbf{B}} \geq 0 \tag{C2}$$

On the other hand,

$$[\mathbf{B}_N^T(z^{-1}\mathbf{I} - \Phi^T)^{-1} + \tilde{\mathbf{G}}_N^T][(z\mathbf{I} - \Phi)^{-1}\mathbf{B}_N + \tilde{\mathbf{G}}_N] \geq 0 \tag{C3}$$

Therefore, on the basis of (C2) and (C3), relation (C1) yields

$$\mathbf{F}^T(z^{-1})(\mathbf{I} - z\mathbf{L}_u^T)(\tilde{\Gamma}_N + \mathbf{B}_N^T\mathbf{P}\mathbf{B}_N)(\mathbf{I} - z^{-1}\mathbf{L}_u)\mathbf{F}(z) \geq \tilde{\Gamma}_N - \tilde{\mathbf{G}}_N^T\tilde{\mathbf{G}}_N$$

Since from (61), we have $\tilde{\Gamma}_N - \tilde{\mathbf{G}}_N^T\tilde{\mathbf{G}}_N \geq 0$, following an argument similar to that used in (Shaked, 1986), we can easily conclude that

$$\sigma_{\min}[\mathbf{I} + \mathbf{T}(z)] \geq \lambda^{-1} \left[\frac{\sigma_{\min}(\tilde{\Gamma}_N - \tilde{\mathbf{G}}_N^T\tilde{\mathbf{G}}_N)}{\sigma_{\max}(\tilde{\Gamma}_N + \sigma_{\max}^2(\mathbf{B}_N)\sigma_{\max}(\mathbf{P}))} \right]^{\frac{1}{2}}$$

Furthermore, since by assumption $\det \hat{\Gamma}_N \neq 0$, it follows that inequality (B6) holds. As a consequence, $\sigma_{\min}(\mathbf{I} + \mathbf{T}(z)) \geq b$. Also note also that, from (49), we obtain

$$\sigma_{\min}(\tilde{\Gamma}_N - \tilde{\mathbf{G}}_N^T\tilde{\mathbf{G}}_N) \leq \sigma_{\min}(\tilde{\Gamma}_N) - \sigma_{\min}(\tilde{\mathbf{G}}_N^T\tilde{\mathbf{G}}_N)$$

Therefore

$$\begin{aligned} 0 \leq \frac{\sigma_{\min}(\tilde{\Gamma}_N - \tilde{\mathbf{G}}_N^T\tilde{\mathbf{G}}_N)}{\sigma_{\max}(\tilde{\Gamma}_N) + \sigma_{\max}^2(\mathbf{B}_N)\phi} &\leq \frac{\sigma_{\min}(\tilde{\Gamma}_N) - \sigma_{\min}(\tilde{\mathbf{G}}_N^T\tilde{\mathbf{G}}_N)}{\sigma_{\max}(\tilde{\Gamma}_N) + \sigma_{\max}^2(\mathbf{B}_N)\phi} \\ &\leq \frac{\sigma_{\min}(\tilde{\Gamma}_N)}{\sigma_{\max}(\tilde{\Gamma}_N) + \sigma_{\max}^2(\mathbf{B}_N)\phi} \leq 1 \end{aligned}$$

Hence $b \in [0, 1]$ and relation (62) follows from (44). This completes the proof.

Appendix D. Proof of Theorem 8.

Observe that the modified return difference equality can also be written as

$$\begin{aligned} & \mathbf{F}^T(z^{-1})(\mathbf{I} - z\mathbf{L}_u^T)(\tilde{\Gamma}_N + \mathbf{B}_N^T\mathbf{P}\mathbf{B}_N)(\mathbf{I} - z^{-1}\mathbf{L}_u)\mathbf{F}(z) \\ &= \mathbf{B}_N^T(z^{-1}\mathbf{I} - \Phi^T)^{-1}(\tilde{\mathbf{Q}}_N - \tilde{\mathbf{G}}_N\tilde{\Gamma}_N^{-1}\tilde{\mathbf{G}}_N^T)(z\mathbf{I} - \Phi)^{-1}\mathbf{B}_N \\ &+ [\mathbf{I} + \mathbf{B}_N^T(z^{-1}\mathbf{I} - \Phi^T)^{-1}\tilde{\mathbf{G}}_N\tilde{\Gamma}_N^{-1}]\tilde{\Gamma}_N[\mathbf{I} + \tilde{\Gamma}_N^{-1}\tilde{\mathbf{G}}_N^T(z\mathbf{I} - \Phi)^{-1}\mathbf{B}_N] \end{aligned}$$

Taking into account (63), we can easily conclude that

$$\begin{aligned} & \mathbf{F}^T(z^{-1})(\mathbf{I} - z\mathbf{L}_u^T)(\tilde{\Gamma}_N + \mathbf{B}_N^T\mathbf{P}\mathbf{B}_N)(\mathbf{I} - z^{-1}\mathbf{L}_u)\mathbf{F}(z) \\ & \geq [\mathbf{I} + \mathbf{B}_N^T(z^{-1}\mathbf{I} - \Phi^T)^{-1}\tilde{\mathbf{G}}_N\tilde{\Gamma}_N^{-1}]\tilde{\Gamma}_N[\mathbf{I} + \tilde{\Gamma}_N^{-1}\tilde{\mathbf{G}}_N^T(z\mathbf{I} - \Phi)^{-1}\mathbf{B}_N] \end{aligned}$$

Also note that by assumption $\tilde{\Gamma}_N > 0$. Therefore

$$[\mathbf{I} + \mathbf{B}_N^T(z^{-1}\mathbf{I} - \Phi^T)^{-1}\tilde{\mathbf{G}}_N\tilde{\Gamma}_N^{-1}]\tilde{\Gamma}_N[\mathbf{I} + \tilde{\Gamma}_N^{-1}\tilde{\mathbf{G}}_N^T(z\mathbf{I} - \Phi)^{-1}\mathbf{B}_N] \geq 0$$

Then, following an argument similar to that used in (Shaked, 1986) we can easily conclude that

$$\begin{aligned} & \sigma_{\min}^2(\mathbf{F}(z)) \\ & \geq \lambda^{-2} \frac{\sigma_{\min} \left([\mathbf{I} + \mathbf{B}_N^T(z^{-1}\mathbf{I} - \Phi^T)^{-1}\tilde{\mathbf{G}}_N\tilde{\Gamma}_N^{-1}]\tilde{\Gamma}_N[\mathbf{I} + \tilde{\Gamma}_N^{-1}\tilde{\mathbf{G}}_N^T(z\mathbf{I} - \Phi)^{-1}\mathbf{B}_N] \right)}{\sigma_{\max}(\tilde{\Gamma}_N + \mathbf{B}_N^T\mathbf{P}\mathbf{B}_N)} \\ & \geq \lambda^{-2} \frac{\sigma_{\min}^2(\mathbf{I} + \tilde{\Gamma}_N^{-1}\tilde{\mathbf{G}}_N^T(z\mathbf{I} - \Phi)^{-1}\mathbf{B}_N)\sigma_{\min}(\tilde{\Gamma}_N)}{\sigma_{\max}(\tilde{\Gamma}_N + \mathbf{B}_N^T\mathbf{P}\mathbf{B}_N)} \\ & \geq \lambda^{-2} \frac{q^2\sigma_{\min}(\tilde{\Gamma}_N)}{\sigma_{\max}(\tilde{\Gamma}_N + \mathbf{B}_N^T\mathbf{P}\mathbf{B}_N)} \geq \lambda^{-2} \frac{q^2\sigma_{\min}(\tilde{\Gamma}_N)}{\sigma_{\max}(\tilde{\Gamma}_N) + \sigma_{\max}^2(\mathbf{B}_N)\sigma_{\max}(\mathbf{P})} \end{aligned}$$

Moreover, since by assumption $\det \tilde{\Gamma}_N \neq 0$, inequality (B6) holds. Therefore $\sigma_{\min}(\mathbf{I} + \mathbf{T}(z)) \geq \gamma$. Also note that, since $q \in [0, 1]$ and

$$\frac{\sigma_{\min}(\tilde{\Gamma}_N)}{\sigma_{\max}(\tilde{\Gamma}_N) + \sigma_{\max}^2(\mathbf{B}_N)\phi} \leq 1$$

we have $\gamma \in [0, 1]$. Relation (65) then follows from (44). This completes the proof.

Appendix E. Proof of Theorem 9.

From the modified return difference equality (48) and for \tilde{G}_N given by (66), we can easily obtain

$$\begin{aligned} &F^T(z^{-1})(I - zL_u^T)(\tilde{\Gamma}_N + B_N^T P B_N)(I - z^{-1}L_u)F(z) \\ &= \tilde{\Gamma}_N + B_N^T(z^{-1}I - \Phi^T)^{-1}\tilde{Q}_N(zI - \Phi)^{-1}B_N \\ &\quad + B_N^T\left[(z^{-1}I - \Phi^T)^{-1}\Phi^T\Sigma + \Sigma\Phi(zI - \Phi)^{-1}\right]B_N \end{aligned} \tag{E1}$$

Observe now that

$$\begin{aligned} &\Sigma + \Sigma\Phi(zI - \Phi)^{-1} + (z^{-1}I - \Phi^T)^{-1}\Phi^T\Sigma \\ &\quad + (z^{-1}I - \Phi^T)^{-1}(\Phi^T\Sigma\Phi - \Sigma)(zI - \Phi)^{-1} = 0 \end{aligned} \tag{E2}$$

Introducing (E2) into (E1), we obtain

$$\begin{aligned} &F^T(z^{-1})(I - zL_u^T)(\tilde{\Gamma}_N + B_N^T P B_N)(I - z^{-1}L_u)F(z) \\ &= \tilde{\Gamma}_N + B_N^T(z^{-1}I - \Phi^T)^{-1}\tilde{Q}_N(zI - \Phi)^{-1}B_N \\ &\quad + B_N^T\left[(z^{-1}I - \Phi^T)^{-1}(\Sigma - \Phi^T\Sigma\Phi)(zI - \Phi)^{-1} - \Sigma\right]B_N \end{aligned} \tag{E3}$$

Taking into account (67), relation (E3) becomes

$$\begin{aligned} &F^T(z^{-1})(I - zL_u^T)(\tilde{\Gamma}_N + B_N^T P B_N)(I - z^{-1}L_u)F(z) \\ &= \tilde{\Gamma}_N - B_N^T\Sigma B_N + 2B_N^T(z^{-1}I - \Phi^T)^{-1}\tilde{Q}_N(zI - \Phi)^{-1}B_N \end{aligned}$$

Since $\tilde{Q}_N > 0$, the following inequality holds:

$$F^T(z^{-1})(I - zL_u^T)(\tilde{\Gamma}_N + B_N^T P B_N)(I - z^{-1}L_u)F(z) = \tilde{\Gamma}_N - B_N^T\Sigma B_N \tag{E4}$$

Following an argument similar to that used in (Shaked, 1986) and taking into account that, by assumption, $\tilde{\Gamma}_N - B_N^T\Sigma B_N \geq 0$, we can easily conclude that

$$\sigma_{\min}(I + T(z)) \geq \lambda^{-1} \left[\frac{\sigma_{\min}(\tilde{\Gamma}_N - B_N^T\Sigma B_N)}{\sigma_{\max}(\tilde{\Gamma}_N) + \sigma_{\max}^2(B_N)\sigma_{\max}(P)} \right]^{\frac{1}{2}} \tag{E5}$$

Since $\det \tilde{\Gamma}_N \neq 0$, taking into account (B6), we finally obtain $\sigma_{\min}(I + T(z)) \geq d$. Also note that

$$\sigma_{\min}(\tilde{\Gamma}_N - B_N^T\Sigma B_N) \leq \sigma_{\min}(\tilde{\Gamma}_N) - \sigma_{\min}(B_N^T\Sigma B_N) \leq \sigma_{\min}(\tilde{\Gamma}_N)$$

Therefore $d \in [0, 1]$ and (68) follows from (44). This completes the proof.

Appendix F. Proof of Theorem 12.

Let us first realize that, in order to prove the theorem, it suffices to prove that $\sigma_{\max}(\mathbf{P}) \geq \vartheta$. To this end, we first focus our attention on Case I of the theorem, and we are looking for an upper bound on $\sigma_{\max}(\mathbf{P})$. To this end, observe that, since the solution of (37) also satisfies (B5) of Appendix B, according to the results in (Yasuda and Hirai, 1979) we can readily obtain

$$\sigma_{\max}(\mathbf{P}) \leq \sigma_{\max}(\widehat{\mathbf{Q}}_N) \left[1 - \sigma_{\max}^2(\widehat{\Phi}) \right]^{-1}$$

It is not difficult to see that

$$\begin{aligned} \sigma_{\max}(\widehat{\mathbf{Q}}_N) \left[1 - \sigma_{\max}^2(\widehat{\Phi}) \right]^{-1} &\leq \left[\sigma_{\max}(\widetilde{\mathbf{Q}}_N) + \frac{\sigma_{\max}^2(\widetilde{\mathbf{G}}_N)}{\sigma_{\min}(\widetilde{\mathbf{T}}_N)} \right] \left[1 - \sigma_{\max}^2(\widehat{\Phi}) \right]^{-1} \\ &= \left[\sigma_{\max}(\widetilde{\mathbf{Q}}_N) + \frac{\sigma_{\max}^2(\widetilde{\mathbf{G}}_N)}{\sigma_{\min}(\widetilde{\mathbf{T}}_N)} \right] \left[1 - \sigma_{\max}^2(\Phi - B_N \widetilde{\mathbf{T}}_N^{-1} \widetilde{\mathbf{G}}_N^T) \right]^{-1} \end{aligned}$$

Consequently, in this case $\sigma_{\max}(\mathbf{P}) \leq \vartheta$. Hence $\sigma_{\min}(\mathbf{I} + \mathbf{T}(z)) \geq h$ and (76) follows from (44).

In Case II, an upper bound for $\sigma_{\max}(\mathbf{P})$ can also be obtained on the basis of the results in (Shaked, 1986; Yasuda and Hirai, 1979). More precisely, we have

$$\begin{aligned} \sigma_{\max}(\mathbf{P}) &\leq \sigma_{\max}(\widehat{\mathbf{Q}}_N) \left[1 - \nu_{\max}^2(\widehat{\Phi}) \right]^{-1} \\ &\leq \left[\sigma_{\max}(\widetilde{\mathbf{Q}}_N) + \frac{\sigma_{\max}^2(\widetilde{\mathbf{G}}_N)}{\sigma_{\min}(\widetilde{\mathbf{T}}_N)} \right] \left[1 - \nu_{\max}^2(\Phi - B_N \widetilde{\mathbf{T}}_N^{-1} \widetilde{\mathbf{G}}_N^T) \right]^{-1} \end{aligned}$$

where $\nu_{\max}(\cdot)$ is defined in Theorem 10. Hence $\sigma_{\min}(\mathbf{I} + \mathbf{T}(z)) \geq h$, and (76) follows from (44).

Finally, in Case III of the theorem, in order to obtain an upper bound for $\sigma_{\max}(\mathbf{P})$, we work as follows: Observe first that the controllability (stabilizability) of the pair (Φ, B_N) implies that of the pair $(\widehat{\Phi}, B_N)$ (see (Al-Rahmani and Franklin, 1990) for details). Next, using an argument similar to that reported in (Shaked, 1986) (see the Appendix therein), we can easily obtain

$$\sigma_{\max}(\mathbf{P}) \leq \frac{\sigma_{\max}(\widehat{\mathbf{Q}}_N) + \omega^2}{1 - \hat{\rho}^2}$$

Since

$$\sigma_{\max}(\widehat{\mathbf{Q}}_N) \leq \sigma_{\max}(\widetilde{\mathbf{Q}}_N) + \frac{\sigma_{\max}^2(\widetilde{\mathbf{G}}_N)}{\sigma_{\min}(\widetilde{\mathbf{T}}_N)}$$

we finally obtain

$$\sigma_{\max}(\mathbf{P}) \leq \frac{\sigma_{\max}(\tilde{\mathbf{Q}}_N) + \frac{\sigma_{\max}^2(\tilde{\mathbf{G}}_N)}{\sigma_{\min}(\tilde{\mathbf{T}}_N)} + \omega^2}{1 - \hat{\rho}^2}$$

Therefore once again $\sigma_{\min}(\mathbf{I} + \mathbf{T}(z)) \geq h$ and (76) follows from (44). This completes the proof.

Appendix G. Proof of Theorem 13.

As was shown in the proof of Theorem 6, the solution to (37) also satisfies (B5). Suppose now that (77) and (78) hold. Then, according to the results presented in (Lee, 1997), the positive solution \mathbf{P} to the Riccati equation (B5) satisfies the inequality

$$\mathbf{P} \leq \frac{\nu_{\max}(\hat{\mathbf{Q}}_N)}{1 + \sigma_{\min}^2(\mathbf{B}_{N,\Gamma})\eta - \sigma_{\max}^2(\hat{\Phi})} \hat{\Phi}^T \hat{\Phi} + \hat{\mathbf{Q}}_N \tag{G1}$$

Therefore

$$\begin{aligned} \sigma_{\max}(\mathbf{P}) &\leq \sigma_{\max} \left(\frac{\nu_{\max}(\hat{\mathbf{Q}}_N)}{1 + \sigma_{\min}^2(\mathbf{B}_{N,\Gamma})\eta - \sigma_{\max}^2(\hat{\Phi})} \hat{\Phi}^T \hat{\Phi} + \hat{\mathbf{Q}}_N \right) \\ &\leq \frac{\nu_{\max}(\hat{\mathbf{Q}}_N)}{1 + \sigma_{\min}^2(\mathbf{B}_{N,\Gamma})\eta - \sigma_{\max}^2(\hat{\Phi})} \sigma_{\max}^2(\hat{\Phi}) + \sigma_{\max}(\hat{\mathbf{Q}}_N) \leq \mu \end{aligned} \tag{G2}$$

Taking into account (G2), relation (B4) of Appendix B, yields

$$\sigma_{\min}[\mathbf{I} + \mathbf{T}(z)] \geq \hat{\alpha} \tag{G3}$$

which is the result sought. Let also observe that $\hat{\alpha} \leq 1$. Relation (82) then follows from (44). This completes the proof.

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