Int. J. Appl. Math. Comput. Sci., 2002, Vol.12, No.1, 115–125

UNILATERAL CONTACT APPLICATIONS USING FEM SOFTWARE

M.E. STAVROULAKI* and G.E. STAVROULAKIS**

* Laboratory of Applied Mechanics, Department of Engineering Sciences,
Technical University of Crete, GR–73100 Chania, Greece,
e-mail: mstavr@mred.tuc.gr

** Division of Applied Mathematics and Mechanics,
University of Ioannina, GR–45110 Ioannina, Greece,
and Institute of Applied Mechanics,
Carolo Wilhelmina Technical University, D–38106 Braunschweig, Germany,
e-mail: gestavr@cc.uoi.gr

Nonsmooth analysis, inequality constrained optimization and variational inequalities are involved in the modelling of unilateral contact problems. The corresponding theoretical and algorithmic tools, which are part of the area known as nonsmooth mechanics, are by no means classical. In general purpose software some of these tools (perhaps in a simplified way) are currently available. Two engineering applications, a rubber-coated roller contact problem and a masonry wall, solved with MARC, are briefly presented, together with elements of the underlying theory.

Keywords: nonsmooth analysis, constrained optimization, contact problems

1. Introduction

The theoretical treatment and the numerical solution of unilateral contact problems, with or without friction, for linearly elastic and for more complicated nonlinear elastic or nonelastic media require novel tools from nondifferentiable (nonsmooth) analysis and optimization. Thanks to the work of pioneers in mechanics and mathematics, see, among others, (Antes and Panagiotopoulos, 1992; Chaudhary and Bathe, 1986; Hlavacek *et al.*, 1988; Kikuchi and Oden, 1988; Klarbring, 1999; Moreau *et al.*, 1988; Panagiotopoulos, 1988) and the references given therein, we now understand that the corresponding mechanical problems have the weak form of a (elliptic or parabolic) variational (or quasivariational, or even hemivariational) inequality, that under certain assumptions they can be written as energy optimization problems (generally, critical points) for nonsmooth potentials and that, perhaps after using some reformulations, they can be expressed as linear or nonlinear variational inequality problems. The application of nonsmooth tools characterises the whole branch of nonsmooth mechanics (Gao *et al.*, 2001; Moreau *et al.*, 1988).

Besides the very important theoretical consequences of all these correct formulations of the nonsmooth mechanical problems, one recognizes common mathematical problems which have been studied and solved numerically in other branches of science and technology. In fact, mathematical programming and numerical optimization provide a number of stable and effective algorithms for the solution of complicated engineering problems (see, among others, the expositions on complementarity problems in engineering and economics (Ferris and Pang, 1997) and on nonconvex and nonsmooth mechanics (Mistakidis and Stavroulakis, 1998)). A number of these developments have already been included in recent versions of general purpose engineering software (e.g. the programs based on the finite element method). Nevertheless, one cannot say that these complicated tools can be used as black-boxes. Maybe we will never see this kind of user-friendly intelligent software which would allow the production of useful results without theoretical knowledge. A minimum of theoretical background allows the effective application of the available computational tools, which means that less time is spent on computer modelling, and the production of accurate results with fewer iterations (convergent procedures). In the worst case one recognizes. The limits of the available tools and modifies analogously the design tasks.

In this contribution the above thoughts are briefly discussed by means of two industrial applications of contact mechanics. First, the appropriate formulations of the frictional contact problems are outlined and the available classes of solution algorithms are mentioned. In the next sections the solutions of two applications, one from the modelling of a roller abrasion test machine in mechanical engineering and the other from the area of earthquake engineering of monuments, are briefly presented. All numerical results have been produced by appropriate use of the general purpose program MARC. Since the scope of this contribution does not allow us to discuss all theoretical, numerical and practical aspects in detail, a number of references where more information can be found is included in appropriate parts of this text.

2. Elements of Contact Mechanics

The basic unilateral contact law is completely described by the following set of inequalities and the nonlinear complementarity relation:

$$[u]_N - g \leq 0, \quad -S_N \geq 0, \quad -S_N\left([u]_N - g\right) = 0. \quad (1)$$

Here $[u]_N$ is the distance between two parts coming in contact along the direction of their common normal, g is the initial opening and S_N is the corresponding contact stress. The first relation is the nonpenetration relation, the second represents the fact that only compressive contact stresses are allowed, and the complementarity relation says that either contact is realized, with possibly nonvanishing stress, or a separation, with zero contact stress, occurs. Relation (1) holds for every couple of nodes lying on opposite surfaces which come in potential contact. The node-to-node representation of the contact effect and the requirement of a common normal direction are consistent with a geometrically linear theory (small displacements and deformations). More general considerations are captured by node-to-surface techniques, which will not be discussed here (see, for instance, (MARC, 1996)). Equivalently, (1) is expressed by the multivalued, monotone contact law (see Fig. 1(a)):

$$-S_N = \begin{cases} 0, & \text{for} \quad [u]_N \leq g, \\ [0, +\infty], & \text{for} \quad [u]_N = g. \end{cases} \quad (2)$$

In turn, the last law is produced by subdifferentiating the nonsmooth, convex superpotential (see Fig. 1(b)):

$$\Phi_N\left([u]_N\right) = I_{U_{\mathrm{ad}}}\left([u]_N\right) = \begin{cases} 0, & \text{for } [u]_N \leq g, \\ +\infty, & \text{for } [u]_N = g, \end{cases} \quad (3)$$

where $U_{\mathrm{ad}} = \left\{[u]_N \in \mathbb{R}^1, [u]_N - g \leq 0\right\}$ stands for the set of kinematically admissible displacements and $I_{U_{\mathrm{ad}}}$ signifies the indicator function.

For the previously introduced unilateral contact joint the law reads as follows:

$$-S_N \in \partial I_{U_{\mathrm{ad}}}\left([u]\right),$$

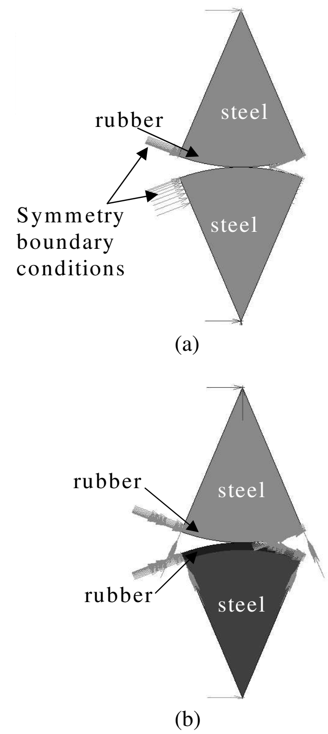

Fig. 1. Examined cases of contact for a quarter of the rollers.

while the variational inequality takes the form

$$-S_N \delta[u]_N \leq I_{U_{\mathrm{ad}}}\left([u]_N + \delta[u]_N\right) - I_{U_{\mathrm{ad}}}\left([u]_N\right), \quad \forall \delta[u]_N \in \mathbb{R}^1, \quad (4)$$

or, equivalently,

$$-S_N \delta[u]_N \leq 0, \quad \forall \delta[u]_N \in U_{\mathrm{ad}}. \quad (5)$$

Furthermore, let us consider a simplified static friction element, with limit frictional stress constant equal to T_0, which may be produced by the max-type nonsmooth potential:

$$\Phi\left([u]_T\right) = \max_{[u]_T}\left\{T_0[u]_T, -T_0[u]_T\right\}. \quad (6)$$

Let us recall here that the limit frictional stress of Coulomb's friction law is $T_0 = \mu\,|S_N|$ (cf. relation (18)). The coupling between normal and tangential relations introduced by this law makes the formulation of the problem more complicated. In order to be able to give an outline of the theory without being obliged to change the previously introduced framework for the contact effects, a given frictional limit model is assumed here. Comments on the coupled model, which has been used in the numerical examples, are given in Section 3.5. Therefore, the following subdifferential formulation of the stick-slip friction law can be written:

$$-S_T \in \partial \Phi\left([u]_T\right). \quad (7)$$

Relation (7) is, by definition, equivalent to the (local) variational inequality:

$$-S_T\delta[u]_T \leq \Phi\big([u]_T + \delta[u]_T\big) - \Phi\big([u]_T\big), \quad \forall [u]_T, \delta[u]_T \in \mathbb{R}^1. \quad (8)$$

A structural analysis problem is usually considered in the weak or variational form, i.e. the principle of virtual work (for compatible stresses and strains):

$$\mathbf{s}^T(\mathbf{e}^* - \mathbf{e}) = \mathbf{p}^T(\mathbf{u}^* - \mathbf{u}) + \mathbf{S}_N^T\big([\mathbf{u}]_N^* - [\mathbf{u}]_N\big) + \mathbf{S}_T^T\big([\mathbf{u}]_T^* - [\mathbf{u}]_T\big), \quad \forall \mathbf{e}^*, \mathbf{u}^*, [\mathbf{u}]_N^*, [\mathbf{u}]_T^*. \quad (9)$$

This relation is coupled with the local variational inequalities in the normal and tangential directions of the unilateral interface (cf. the contact law (4), (5) and the friction law (8)), which can be compactly written as:

$$-S_a\big([\mathbf{u}]_a^* - [\mathbf{u}]_a\big) \leq \Phi_a\big([\mathbf{u}]^*\big) - \Phi_a\big([\mathbf{u}]\big), \quad a = N, T, \quad (10)$$

for $\Phi_a([\mathbf{u}]^*) < \infty$. The arising problem is a variational inequality of the type

$$\mathbf{s}^T(\mathbf{e}^* - \mathbf{e}) - \mathbf{p}^T(\mathbf{u}^* - \mathbf{u}) + \Phi_N\big([\mathbf{u}]_N^*\big) - \Phi_N\big([\mathbf{u}]_N\big) + \Phi_T\big([\mathbf{u}]_T^*\big) - \Phi_T\big([\mathbf{u}]_T\big) \geq 0, \quad \forall \mathbf{e}^*, \mathbf{u}^*, [\mathbf{u}]_N^*, [\mathbf{u}]_T^* \quad (11)$$

and it replaces the classical variational equations or nonlinear mechanics. Here, Φ_N acts as a nonsmooth penalty function that enforces the nonpenetration requirement of the unilateral contact law. Alternativelly, one includes the corresponding inequalities (cf. (1)) in the set of admissible displacements $V_{\rm ad}$ (cf. the local forms (4) and (5)).

One should emphasize that the set of kinematically admissible displacements $U_{\rm ad}$ is defined, in general, by nonlinear inequalities, due to the nonlinear kinematic relations involved in the definition of the relative normal displacement $[u]_N$. This point has both theoretical consequences and practical implications. The potential energy optimization problem, like in all nonlinear elasticity problems, becomes nonconvex. In this case the admissible displacement set is, in general, nonconvex as well. Thus, a potential multiplicity of the solution is possible. Furthermore, the variational inequality (11) is in fact a nonconvex one and describes the critical points of the potential energy function $\Pi(u)$ within the set $U_{\rm ad}$. In other words, one may write the mechanical problem in the compact critical point form

$$\mathbf{0} \in \partial_{\rm Cl-R}\big(\Pi(u) + I_{U_{\rm ad}}\big), \quad (12)$$

where $\partial_{\rm Cl-R}$ is the Clarke-Rockafellar set-valued subdifferential of nonconvex and nonsmooth analysis. Further exploitation of this form gives rise to hemivariational inequality problems ((Panagiotopoulos, 1988; 1993) or to implicit variational inequality problems based on the quasidifferential theory (Demyanov *et al.*, 1996; Mistakidis and Stavroulakis, 1998)).

From the practical point of view, one first accepts that most numerical algorithms are able to find one of the many possible solutions. The numerical approximation, which is typically modelled on a load-incrementation technique, is usually based on some local linearization of the inherently nonlinear problem (large displacements, large deformations). The contact mechanism must be linearized analogously (approximation of the boundaries in the set $U_{\rm ad}$ with linear approximations). Moreover, the well-known point-to-point (or node-to-node) contact technique must be replaced by some more general point-to-surface method. Consequently, a search algorithm must be used for the determination of the points of contact and for the solution of the nonlinear problem. Usually this is accomplished by means of some iterative linearization technique.

In the case of small displacements and deformations in linear elastostatics, and for a frictionless unilateral contact problem, one has a quadratic minimization problem with linear inequality constraints: Find

$$\arg \min_{\mathbf{u} \in V_{\rm ad} = \mathbb{R}^n \cap \{\mathbf{Nu} - \mathbf{g} \leq \mathbf{0}\}} \left\{ \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} - \mathbf{p}^T\mathbf{u} \right\}. \quad (13)$$

The corresponding variational inequality is as follows: Find $\mathbf{u} \in V_{\rm ad} = \mathbb{R}^n \cap \{\mathbf{Nu} - \mathbf{g} \leq \mathbf{0}\}$ such that

$$\mathbf{u}^T\mathbf{K}(\mathbf{u}^* - \mathbf{u}) - \mathbf{p}^T(\mathbf{u}^* - \mathbf{u}) \geq 0, \quad \forall \mathbf{u}^* \in V_{\rm ad}. \quad (14)$$

Finally, the exploitation of the KKT optimality conditions of (13) or of the general form (12) leads to the well-known linear complementarity problem formulation of the static problem (Antes and Panagiotopoulos, 1992; Christensen *et al.*, 1998; Ferris and Pang, 1997; Klarbring, 1999; Mistakidis and Stavroulakis, 1998).

In the previous relations, the total number of effective displacement degrees of freedom in the finite element formulation, i.e. when classical support conditions are taken into account by neglecting the corresponding d.o.f.s, is assumed to be equal to n. In the definition of the admissible displacements set $V_{\rm ad}$ the nonpenetration requirement for each couple of unilateral nodes (the first relation in (1)) is expressed in a matrix form by means of matrix $\mathbf{N}$ and vectors $\mathbf{u}$ and $\mathbf{g}$. Furthermore, let us consider the mechanical problem in the form of a potential energy minimization, with subsidiary unilateral contact inequality constraints. From the theory of optimization one understands that these inequality constraints can be enforced by appropriate (inequality restricted) Lagrange multipliers, which have the physical meaning of point contact forces (the second relation in (1)). Moreover, instead of an inequality restricted potential energy optimization problem, where the only unknowns are the displacement degrees of freedom, one may formulate saddle point problems of the min-max type for a suitably defined Lagrangian function, which is a mixed formulation of the mechanical problem with displacements and contact forces as unknowns, as well as dual problems in terms of contact forces. From the mixed formulation one reconstructs, for instance, all the three relations of (1) and gets the mechanical problem in the form of a linear or nonlinear complementarity problem. Details on the theoretical analysis of the above mentioned problems can be found, e.g., in (Hlavacek *et al.*, 1988; Kikuchi and Oden, 1988; Klarbring, 1999; Mijar and Arora, 2000; Mistakidis and Stavroulakis, 1998; Panagiotopoulos, 1988). It is interesting to note here that the incompressibility condition, used in the first application of this paper, involves a restriction of equality type. If one introduces this restriction using the method of Lagrange multipliers, the corresponding multipliers (with the physical meaning of pressure) are not sign-restricted.

It is clear that specialized algorithms are required for the treatment of contact problems. They solve either the variational inequality (cf. (11) or (14)), or one of the other previously mentioned formulations of the problem. Algorithms which are based on smoothing techniques transform, roughtly speaking, the variational inequalities or the complementarity relation into approximate nonlinear equations, which are then solved by some general-purpose solver. Although the obtained accuracy, for example, the determination of stick and slip areas in friction, is lower, this formulation allows easier integration in general-purpose finite-element codes. In numerical examples an augmented Lagrangian multiplier method is used for the unilateral contact conditions and a kind of smoothing technique is used for the frictional effects. Other possible formulations as well as a theoretical justification of this method are discussed in (Christensen *et al.*, 1998; Leung *et al.*, 1998; Stavroulakis and Antes, 2000).

3. Contact Problem for Rubber Coated Rollers

3.1. Description of the Application

The modelling of two rubber-covered rollers (metallic tubes) which are supported at their ends and are coming in contact across their length is examined based on the modelling of a Roller Abrasion Test Machine, and a parametric investigation on the behaviour of rubber under different contact conditions is the main part of this work. The main aim was to investigate how the strain energy density and the resulting strains and stresses are influenced by the applied static loads (forces), considering different rubber elasticity models. The finite-element method was used in order to solve the static problem of two rubber-covered rollers in contact, which is the necessary first step before rolling contact is analysed. The main factors that influence the reliability and the accuracy of the finite-element solution are the selection of an appropriate material model for the rubber coating, the selection of friction properties at the contacting surfaces and the determination of an appropriate discretization of the finite-element model. For modelling and analysis, a general-purpose finite element code, MARC, was used (MARC, 1996; 1997). Several finite-element meshes were also used in order to achieve better accuracy of the results. The data for rubber modelling were extracted by experimental tests. The friction coefficient was estimated from a verification study based on a compression analysis of a rubber-coated roller between flat platens on a static test machine (Stavroulaki *et al.*, 2000).

The following cases of contact were studied, see Fig. 1:

- rubber—steel: A steel roller is coming in contact with a rubber-coated roller.
- rubber—rubber: Two rubber-coated rollers are coming in contact.

3.2. Finite-Element Modelling

The Roller Abrasion Test Machine includes two rollers with an outer diameter of steel tube equal to 105 mm; the thickness of steel is 9 mm and the thickness of rubber of the covered roller is 2.5 mm. One rubber layer was assumed to be 2.5 mm thick. Since the length of the cylindrical rollers, which is equal to 320 mm, is significantly more extensive than their diameters, a plane strain problem was assumed. In the present work, the interest is focused on the contact region, so a quarter of each roller was used for the finite-element model. This portion of the rollers is divided into quadrilateral (four-node isoparameric) elements, as shown in Fig. 2. A finer mesh was concentrated on the contact region, leading to a more efficient numerical discretization. For the modelling of the rubber incompressible material, the Herrmann incompressible elastic formulation was used (Herrmann, 1978). The main core of the cylinder was considered as rigid, and triangular isoparametric finite elements with appropriate high stiffness values were used for approximate modelling. Special care must be taken with modelling, since the changes in element types and sizes usually influence the final results (a change in the stiffness). Usually, a less accurate mesh is used for the region out of contact, in order to reduce the computational cost, but an analogy to the dimensions must always be kept.

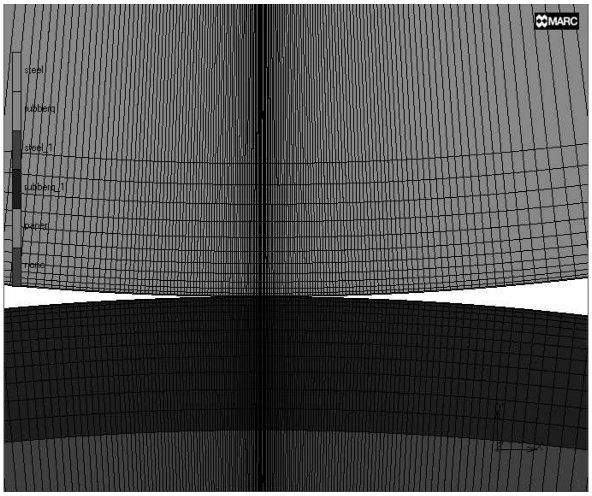

Fig. 2. Details of the finite element model (the rubber-rubber contact problem).

3.3. Material Modelling

The cover layer was a rubber material with hyper-elastic mechanical behaviour, characterised by its elastic strain energy function. Once the strain energy function is defined, the stresses and the material tangent modulus are evaluated for the total and updated Lagrange formulations.

Appealing to the notion of statistical mechanics and thermodynamics principles, the simplest model of rubber elasticity is the Neo-Hookean model represented as

$$W = C_{10}(I_1 - 3), \tag{15}$$

where $C_{10} = E/6$, I_i denotes the i-th strain invariant and E is the elasticity modulus.

This model exhibits a constant shear modulus, and gives a good correlation with the experimental data up to 40% strain in uniaxial tension and up to 90% strains in simple shear. In this application, a model with homogeneous, isotropic and incompressible material was used, with only one variable (Hinge and Maniatty, 1998). This model correlates well with experimental results according to (MARC, 1996). For a nonlinear elastic model (hyper-elastic) the third-order strain energy function of the generalised Rivlin type has been found to adequately characterise rubber in all the three modes of deformation (simple tension, equibiaxial tension and simple shear). The general form of the third-order deformation model for an incompressible rubber material ($I_3 = 1$) is given by

$$W = C_{01}(I_1 - 3) + C_{01}(I_2 - 3) + C_{11}(I_1 - 3)(I_2 - 3) + C_{20}(I_1 - 3)^2 + C_{30}(I_1 - 3)^3. \tag{16}$$

The Yeoh model differs from the above higher- order models in that it involves only the first strain invariant:

$$W = C_{01}(I_1 - 3) + C_{20}(I_1 - 3)^2 + C_{30}(I_1 - 3)^3. \tag{17}$$

This model is more versatile than the previous ones, because it fits various modes of deformation using the data obtained from a uniaxial tension test only. This leads to reduced requirements for material testing. However, caution needs to be exercised when applying this model to deformations involving low strains (Yeoh, 1995; 1997). On the other hand, the higher-order terms in the generalized Mooney-Rivlin polynomial strain energy function require test data from more that one test.

In this application, for the mild steel roller a linear elastic material behaviour was assumed. The characteristics of steel are: $E = 207\,$GN/m^2 (modulus of elasticity), $\nu = 0.30$ (Poisson ratio), density $\rho = 7850\,$Kg/m^3. For rubber material with the following characteristics of rubber: $E = 2.5\,$Mpa (modulus of elasticity), $\nu = 0.449$ (Poisson ratio) and density $\rho = 1150\,$Kg/m^3, an elastic behaviour was considered. For the Neo-Hookean model we have $C_{10} = E/6 = 0.4166\,$N/mm^2.

A nonlinear material model was also used (the Yeoh model of (17)) with $C_{10} = 0.2628$, $C_{20} = -0,0329$, $C_{30} = 0,0101$ and Bulk modulus $K = 2000\,$Mpa. These values were extracted from equivalent shear stress strain data that were converted from a lubricated compression data (Stavroulaki *et al.*, 2000).

3.4. Loads and Boundary Conditions

A perfect bonding between the steel roller and the rubber layer was considered. Symmetry boundary conditions were applied at the ends of the rubber-steel segment since only a local segment of the roller was modelled. The centre of the upper roller was set to have zero displacements and the centre node of the lower steel roller was constrained to have zero displacement in the horizontal (X) direction. A vertical displacement (in the Y direction) was applied at the centre of the lower roller in order to realize contact between the two rollers and nip forces to be developed. A static analysis was considered. Since the analysis is based on the nonlinear theory, the problem is solved incrementally. Two cases of vertical displacement of the lower roller, 0.25 mm and 0.10 m, were used in a parametric investigation.

More general approaches to applying forces on initially free parts (e.g. the upper roller) and considering the corresponding rigid body motions together with the unilateral effects are also possible (these are the so-called semi-coercive problems, see, e.g., (Adly and Goeleven, 2000; Panagiotopoulos, 1988, Ch. 4; Stavroulakis *et al.*, 1991)).

3.5. Analysis and Numerical Solution

In the contact analysis process it was necessary to define the contact bodies, the contact tolerance, the area in which the contact would occur, the contact procedure, the separation procedure and the type of friction. These particular definitions characterise the contact between the two bodies without the unnecessary detection of contact between them.

A Lagrange multiplier approach was used in this work, in order to enforce the contact region, the contacting surface displacements to be compatible with the target surface displacements. The developed contact forces are calculated directly from the external load and the nodal point forces, equivalent to the current element stresses. The resultant force transmitted from one surface to the other through a point of contact is resolved into a normal force P, acting along the common normal, which must be compressive due to the contact condition, and a tangential force Q in the tangent plane sustained by friction. Sliding is defined as the relative linear velocity between the two surfaces at the point of initial contact. It consists of the relative peripheral velocity of the surfaces at their point of contact. The tangential traction is due to friction, which is assumed to have a bilinear dependence on the slip velocity. In the static analysis problem, which is considered in this section, velocities are approximated by the incremental relative tangential displacements, so that for the incremental problem the formulation parallels the simplified static friction model of (6).

The region of contact is often unknown prior to the analysis and large changes in the contact area are possible including relative sliding with the Coulomb friction or possible separation after contact. Coulomb's law of friction is enforced by first evaluating the distributed surface traction from an estimate of the contact forces and then updating the traction corresponding to the conditions of sticking and sliding contact (cf. the algorithm proposed by Panagiotopoulos (1988)). In the modelling of friction, usually it is assumed that Coulomb's law of friction is applicable with a constant coefficient of friction. This simple friction model is frequently adequate in practice since a detailed characterisation of friction behaviour through laboratory experiments is a very difficult task (Chaudhary and Bathe, 1986). The type of contact that was assumed corresponds to a law of friction of a Coulomb type. A special type of a friction model, the one with the glue option according to the terminology used in MARC, which imposes no relative tangential motion, was also applied in order to solve the problem without slip. The friction stress is based upon the coefficient of friction μ and the normal stress at the surface. The coefficient of friction was considered to be equal to 1.0 for these applications.
The Coulomb friction model is given in the form

$$S_T \leq \mu |S_N| t, \tag{18}$$

where S_N is the normal stress, S_T is the tangential (friction) stress, μ is the friction coefficient, t is the tangential vector in the direction of the relative velocity (it is approximated by the tangential relative displacement $[u]_T$ in (6)) $t = v_r/|v_r|$, and v_r is the relative sliding velocity. The modified (smoothed) Coulomb friction model is given in the form

$$S_T \leq \mu |S_N| \frac{2}{n} \arctan\left(\frac{v_r}{v_{rc}}\right) t, \tag{19}$$

where v_{rc} is the value of the relative velocity when sliding occurs. A very large value of v_{rc} results in a reduced value of the effective friction. A very small value results in poor convergence. It is recommended that the value of v_{rc} be 1% or 10% of a typical relative sliding velocity (MARC, 1997). The procedure is actually a smoothing approach to the treatment of the frictional stick-slip nonsmoothness. Other possible smoothing techniques are discussed in (Kikuchi and Oden, 1988, p. 276; Leung *et al.*, 1998) and (Stavroulakis and Antes, 2000). For theoretical and analytical results related to quasistatic problems, for elastodynamics or more general mechanical models, the reader is referred to (Hlavacek *et al.*, 1988; Ionescu and Sofonea, 1993; Leung *et al.*, 1998; Rochdi *et al.*, 1998).

The algorithm implemented in the finite-element code which has been used in this investigation solves the unilateral contact problem exactly, by using an augmented Lagrangian technique. This class of techniques is well known in the mathematical optimization community (see, e.g., (Bertsekas, 1996)) and has been used in several contact mechanics investigations (see, e.g., (Mistakidis and Stavroulakis, 1998) and the recent review article (Mijar and Arora, 2000)). The nonsmooth relation of the frictional law (cf. (7) and (18)) was regularized, see (19). Given the higher sensitivity of the frictional mechanisms, which was demonstrated, e.g., in (Stavroulakis and Antes, 2000), this technique seems to be a robust one for the solution of frictional contact problems. This argument justifies its adoption by a commercial finite-element program.

3.6. Rubber-Steel and Rubber-Rubber Contact Problems

The changing mechanical behaviour of rubber is obvious rubber it is coming in contact with a stiff or a flexible material. The normal and shear stresses refer to the deformed plane at each node where they are calculated. From the results it is clear that the strain and stress on the contact plane and the strain energy density are useful in the estimation of the fatigue of the rubber. The strain energy

(W) helps us to localize the critical areas for possible future failure of the material. The total strain energy density $(\mathrm{d}W/\mathrm{d}V)$ is the sum of the distortional, TD, and dilatational components of strain energy density, TV. The strain energy density expresses the energy stored per unit volume of the material and is a useful design value, since it governs, e.g., crack and damage initiation and propagation.

In Figs. 3 and 4 the total strain energy density and the equivalent total strain are shown for the rubber-steel contact problem and for displacement of the lower steel roller equal to 0.10 and 0.25 mm, respectively. The developed nip forces are shown in Fig. 5 for the rubber-steel contact problem.

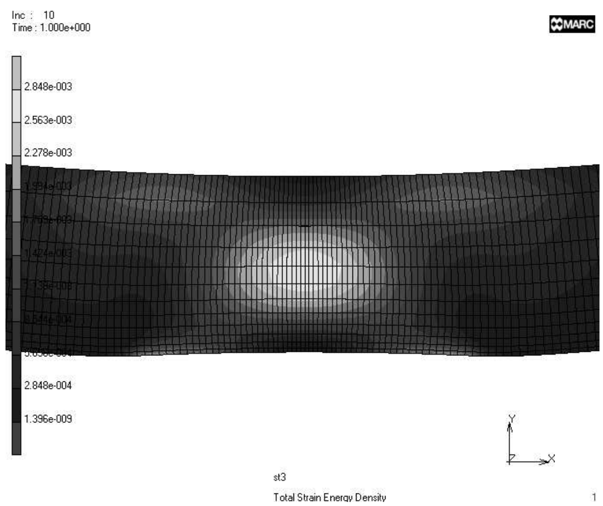

Fig. 3. Total strain energy density the for rubber-steel contact (0.10 mm).

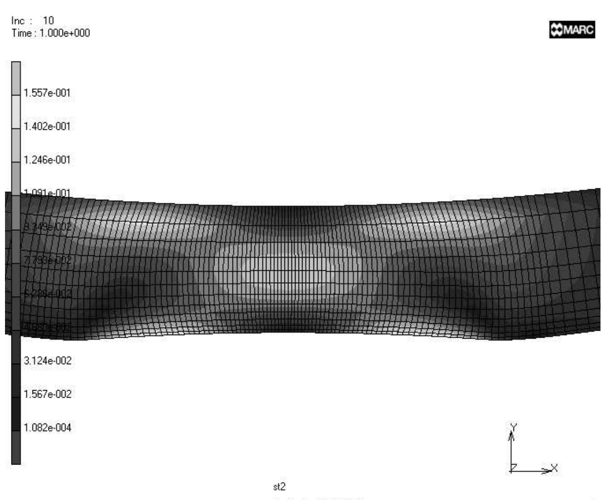

Fig. 4. Equivalent total strain for the rubber-steel contact problem (0.25 mm).

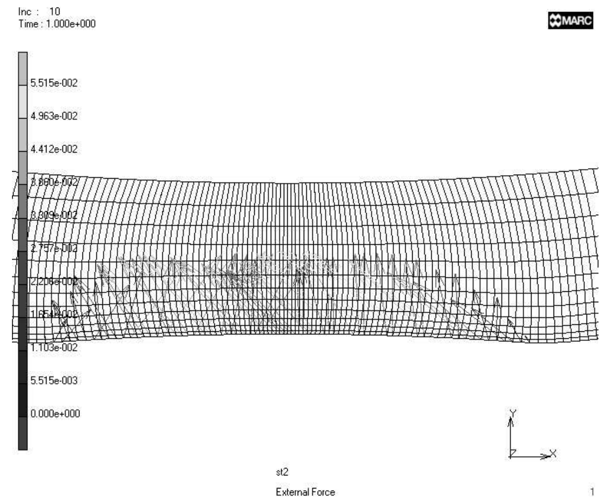

Fig. 5. Nip forces for the rubber-steel contact problem (0.25 mm).

Figures 6 and 7 show comparative results regarding the developed normal and shear stresses. The values are higher in the case of contact with rubber than those referring to steel, even though the assumptions about the friction coefficient are the same. Indicative results about the distribution along the length of the contact area for the rubber-steel contact problem are given in Figs. 8 and 9.

In Tabs. 1 and 2, results concerning the width of the contact area and the nip forces for various cases of the contact and the final finite-element model are given. The displacements and the equivalent total strain for an external initial displacement of 0.25 mm and the rubber-rubber contact problem are shown in Figs. 10 and 11, respectively. Figures 12 and 13 show comparative results regarding the developed normal and shear stresses for the rubber-rubber contact problem and for the final finite-element models.

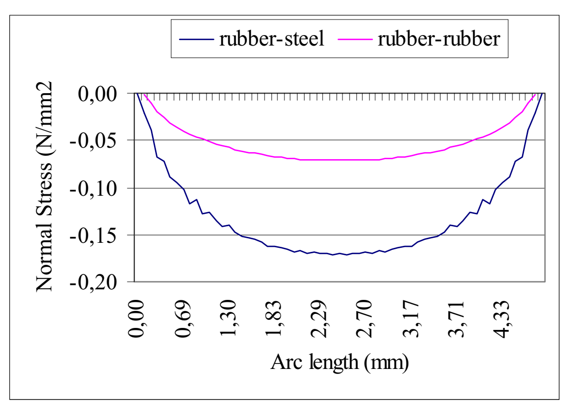

Fig. 6. Normal stresses for the rubber-steel and rubber-rubber contact problems (initial displacement 0.1 mm).

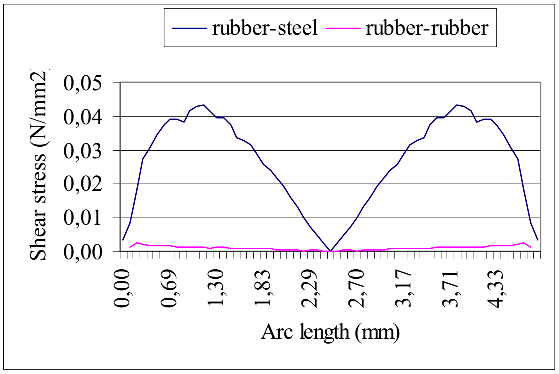

Fig. 7. Shear stresses for the rubber-steel and rubber-rubber contact problems (initial displacement 0.1 mm).

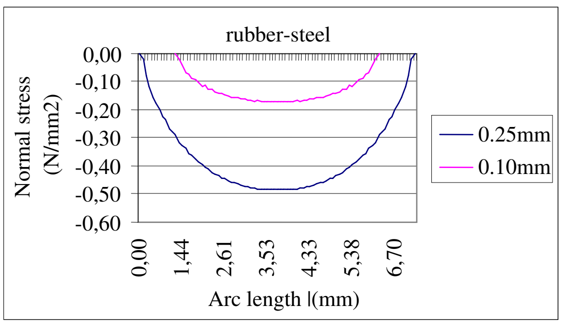

Fig. 8. Normal stresses for the rubber-steel contact problem and different loads.

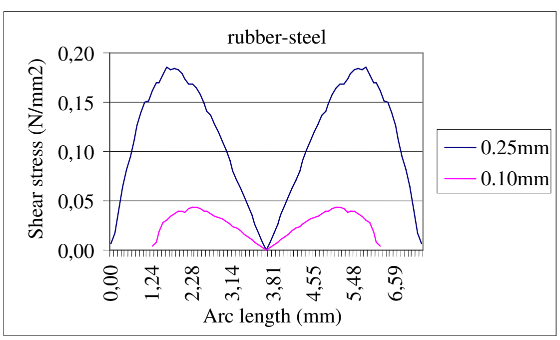

Fig. 9. Shear stresses for the rubber-steel contact problem and different loads.

3.7. Conclusions

In general, many factors are involved in the solution of contact problems by the finite-element method. These pertain to the way of contact modelling and the method of numerical analysis. In commercial finite-element codes attention must be given to the selection of proper factors. Special care must be taken with modelling using the finite-element method since the type of elements, their size and, in general, the loads and boundary assumptions influence the final results. A comparison with experimental data can lead to a verification of the finite-element model.

With respect to the material modelling, an acceptable accuracy was achieved since the loads were low. The linear and nonlinear models gave almost the same results. For applications with higher loads, another material model may be necessary (Stavroulaki *et al.*, 2000). For this model more experimental data are required. Especially for the model with higher-order functions in the strain energy function more tests must be used in order to calculate the constants of the function.

Table 1. Nip width (mm)* for the rubber-steel and the rubber-rubber contact problems (final finite-element model, linear and nonlinear material model for rubber).

Load case vs. material	rubber-steel R (LINEAR)	rubber-R (LINEAR)	rubber R (nonlinear model)
0.25 mm	7.69	8.31	8.31
0.10 mm	4.82	5.70	5.70

Table 2. Nip force (N/mm roller length).

Load case vs. material	rubber-steel R (LINEAR)	rubber-R (LINEAR)	rubber R (nonlinear model)
0.25 mm	2.43	1.35	1.70
0.10 mm	0.61	0.41	0.52

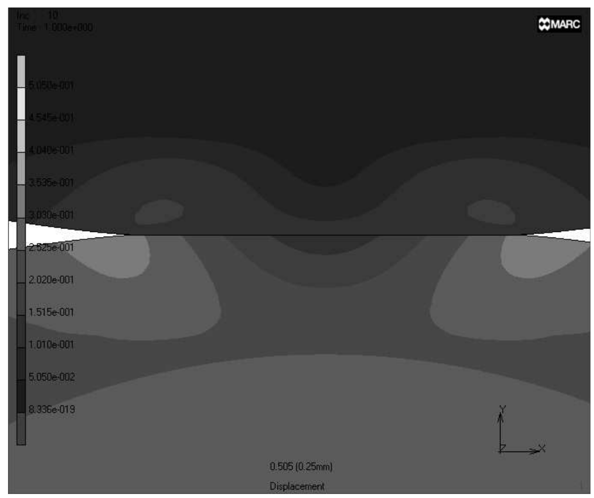

Fig. 10. Displacement distribution for an external initial displacement of 0.25 mm in the rubber-rubber contact problem.

* The length of the contact area refers to the deformed rubber condition. In the adjacent figures of XY, the arc length refers to the undeformed position of the nodes.

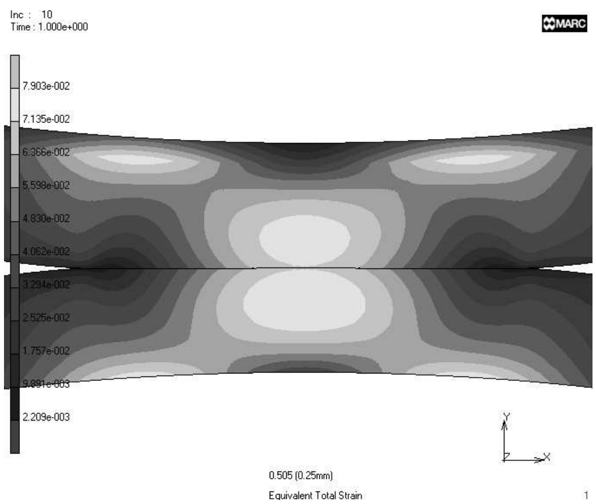

Fig. 11. Equivalent total strain for an external displacement of 0.25 mm in the rubber-rubber contact problem.

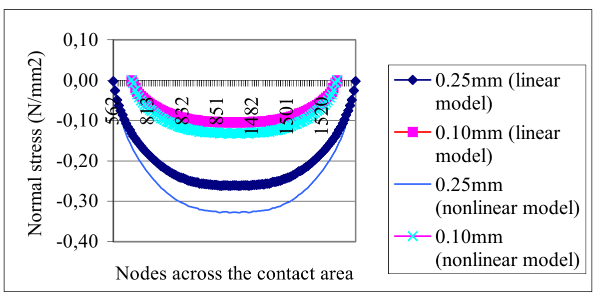

Fig. 12. Normal stresses for the rubber-rubber contact problem.

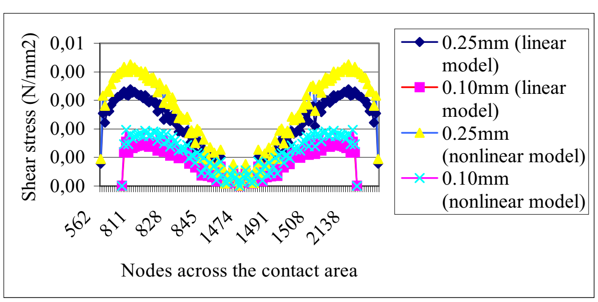

Fig. 13. Shear stresses for the rubber-rubber contact problem.

The distribution of stresses and strains across the length of the contact area indicate the conditions under which the rubber is deformed.

4. Contact Interaction Between Concrete Reinforcement and Masonry Walls

A masonry wall including reinforcing elements from reinforced concrete material over the door or window openings is shown in Fig. 14. The nonlinear behaviour of the masonry is modelled by means of appropriately modified elastoplastic laws. The unilateral contact interaction, including friction, follows the lines of the previous application. Finally, a dynamic analysis is performed by introducing a suitable, earthquake-like base excitation of the wall. An appropriately choosen time increment, which clearly demonstrates the unilateral effect (i.e. partial separation) between the wall and the reinforcing elements, is graphically depicted in Fig. 14. On the same plot the areas of plastification/cracking around the reinforcement are shown with isolines of the plastic work. A detailed parametric investigation shows that the reinforcement drastically reduces the areas of plastification and cracking. The results and application of this methodology to strengthening design of masonry monuments will be presented elsewhere.

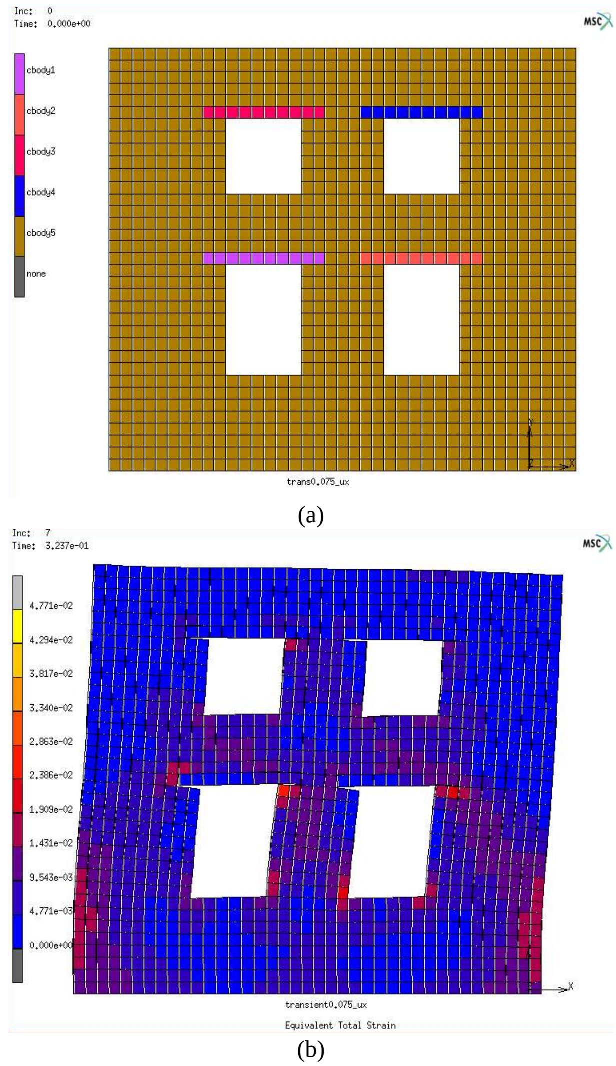

Fig. 14. Deformed instance of the masonry wall demonstrating the unilateral interactions between the wall and the reinforcing elements.

In conclusion, let us mention that contact analysis is now a well-established part of nonlinear computational mechanics. This ability is present in an increasing number of general-purpose programs. Extensions to frictional problems or, more general, problems studied within the area of nonsmooth mechanics are partially covered by the currently available programs. Since the required theoretical and algorithmic background does not belong to the material provided for an average student of engineering (or even of mathematics), special care must be taken with the effective use of these tools.

Acknowledgements

Part of the work reported here was initiated within an ESPRIT project of the European Union. The financial support, as well as the interesting discussion with our industrial partners, are gratefully acknowledged.

References

Adly S. and Goeleven D. (2000): *A discretization theory for a class of semi-coercive unilateral problems*. — Numer. Math., Vol. 87, No. 1, pp. 1–34.

Antes H. and Panagiotopoulos P.D. (1992): *The Boundary Integral Approach to Static and Dynamic Contact Problems. Equality and Inequality Methods*. — Basel: Birkhäuser.

Bathe K.J. (1996): *Finite Element Procedures*. — New Jersey: Prentice–Hall.

Batra R.C. (1980): *Rubber covered rolls. The nonlinear elastic problem*. — J. Appl. Mech., Vol. 47, pp. 82–86.

Bertsekas D.P. (1996): *Constrained optimization and Lagrange multiplier methods*. — Athena Scientific Press, MA.

Chaudhary A.B. and Bathe K.-J. (1986): *A solution method for static and dynamic analysis of three-dimensional contact problems with friction*. — Comput. Struct., Vol. 24, No. 6, pp. 855–873.

Christensen P.W., Klarbring A., Pang J.S., and N. Strömberg (1998): *Formulation and comparison of algorithms for frictional contact problems*. — Int. J. Num. Meth. Eng. Vol. 42, pp. 145–173.

Demyanov V.F., Stavroulakis G.E., Polyakova L.N. and Panagiotopoulos P.D. (1996): *Quasidifferentiability and Nonsmooth Modeling in Mechanics, Engineering and Economics*. — Dordrecht: Kluwer.

Ferris M.C. and Pang J.S. (1997): *Engineering and economic applications of complementarity problems*. — SIAM Rev., Vol. 39, pp. 669–713.

Gao D.Y., Ogden R.W. and Stavroulakis G.E. (Eds.)(2001): *Nonsmooth/Nonconvex Mechanics: Modeling, Analysis and Numerical Methods*. — Dordrecht: Kluwer.

Herrmann L.R. (1978): *Finite element analysis of contact problems*. — ASCE J. Eng. Mech. Div., Vol. 104, pp. 1043–1057.

Hinge K.C. and Maniatty A.M. (1998): *Model of steady rolling contact between layered rolls with thin media in the nip*. — Eng. Comput., Vol. 15, No. 7, pp. 956–976.

Hlavacek I., Haslinger J., Necas J. and Lovisek J. (1988): *Solution of Variational Inequalities in Mechanics*. — Berlin: Springer.

Ionescu I.R. and Sofonea M. (1993): *Functional and Numerical Methods in Viscoplasticity*. — Oxford: Oxford University Press.

Kikuchi N. and Oden J.T. (1988): *Contact Problems in Elasticity: A Study of Variational Ineualities and Finite Element Methods*. — Philadelphia: SIAM.

Klarbring A. (1999): *Contact, friction, discrete mechanical structures and mathematical programming*, In: New Developments in Contact Problems (P. Wriggers and P. Panagiotopoulos, Eds.). — CISM Courses and Lectures No. 384, Wien: Springer, Ch. 2, pp. 55–100.

Leung A.Y.T., Guoqing Ch. and Wanji Ch. (1998): *Smoothing Newton method for solving two- and three-dimensional frictional contact problems*. — Int. J. Num. Meth. Eng., Vol. 41, pp. 1001–1027.

MARC (1996): *Nonlinear Finite Element Analysis of Elastomer*. — MARC Analysis Research Corp.

MARC (1997): *Theory and User Information, Vol. A*. — MARC Analysis Research Corporation.

Mijar A.R. and Arora J.S. (2000): *Review of formulations for elastostatic frictional contact problems*. — Struct. Multidiscipl. Optim., Vol. 20, pp. 167–189.

Mistakidis E.S. and Stavroulakis G.E. (1998): *Nonconvex Optimization in Mechanics. Algorithms, Heuristics and Engineering Applications by the F.E.M.*. — Dordrecht: Kluwer.

Moreau J.J., Panagiotopoulos P.D. and Strang G. (1988): *Topics in Nonsmooth Mechanics*. — Basel: Birkhäuser.

Panagiotopoulos P.D. (1988): *Inequality Problems in Mechanics and Applications. Convex and Nonconvex Energy Functions*. — Basel: Birkhäuser.

Panagiotopoulos P.D. (1993): *Hemivariational Inequalities*. — Berlin: Springer.

Rochdi M., Shillor M. and Sofonea M. (1998): *Quasistatic viscoelastic contact with normal compliance and friction*. — J. Elasticity, Vol. 51, No. 2, pp. 105–126.

Stavroulaki M.E., Stevenson A. and Bowron St. (2000): *Finite element analysis of rubber coated rollers contact problem and the phenomena of friction*. — Proc. 4-th Int. Coll. Computational Methods for Shell and Spatial Structures IASS-IACM 2000, Chania, Athens, Greece (on CD-ROM).

Stavroulakis G.E., Panagiotopoulos P.D. and Al-Fahed A.M. (1991): *On the rigid body displacements and rotations in unilateral contact problems and applications*. — Comp. Struct., Vol. 40, pp. 599–614.

Stavroulakis G.E. and Antes H. (2000): *Nonlinear equation approach for inequality elastostatics. A 2-D BEM implementation*. — Comp. Struct., Vol. 75, No. 6, pp. 631–646.

Yeoh O.H. (1995): *Phenomenological theory of rubber elasticity*, In: Comprehensive Polymer Science (S.L. Aggarwal, Ed.). — 2nd Suppl., Pergamon Press.

Yeoh O.H. (1997): *Hyperelastic material models for finite element analysis of rubber*. — J. Nat. Rubber Res., Vol. 12, No. 3, pp. 142–153.