ANALYSIS OF A VISCOELASTIC ANTIPLANE CONTACT PROBLEM WITH SLIP-DEPENDENT FRICTION

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We study a mathematical problem modelling the antiplane shear deformation of a viscoelastic body in frictional contact with a rigid foundation. The contact is bilateral and is modelled with a slip-dependent friction law. We present the classical formulation for the antiplane problem and write the corresponding variational formulation. Then we establish the existence of a unique weak solution to the model, by using the Banach fixed-point theorem and classical results for elliptic variational inequalities. Finally, we prove that the solution converges to the solution of the corresponding elastic problem as the viscosity converges to zero.

Keywords: antiplane problem, slip-dependent friciotn law, viscoelastic law

1. Introduction

The contact between deformable bodies is a phenomenon frequently found in industry and in everyday life. The contact of the braking pads with the wheel, of the tire with the road and the piston with the skirt are just simple examples. Because of the importance of this process, a considerable effort has been made in its modelling and numerical simulations, and the engineering literature concerning this topic is extensive. In the applied mathematics literature, general models for dynamic or quasistatic contact processes are recent. The reason is that, owing to the inherent complicated nature, contact phenomena are modelled by difficult nonlinear problems, which explains the slow progress in their mathematical analysis.

When two surfaces come into contact, a rearrangement of the surface asperities takes place. The surface asperities become more compliant and the slip becomes smoother. Clearly, these changes affect the friction process and we expect that the friction bound will decrease. Therefore, when we describe the physical setting, there is a need to allow for the resulting change or evolution of the contact boundary conditions. An important step in modelling the friction phenomenon was made in (Rabinowicz, 1965), where it was pointed out that the coefficient of friction varies with the tangential displacement. In this way a part of the elastic-plastic deformation of the interface is captured in the model. Other authors (see, e.g., (Ohnaka, 1996) for a review) performed a series of experiments in which it was possible to measure directly the interdependence between the different local variables on the contact surface. These experiments show that the friction depends mostly on the slip. Following these experiments, slip-weakening type friction laws were considered by many authors, see, e.g., (Campillo and Ionescu, 1997; Ionescu and Paumier 1996; Kuttler and Shillor, 1999). The changes in the current friction with the slip reflect mainly the elastic deformation on the surface asperities and possible chemical bonding.

In this paper, as in (Matei *et al.*, 2001), we consider a quasistatic contact problem between a rigid foundation and a cylinder. This problem is considered to be antiplane, i.e. the displacement is parallel to the generators of the cylinder and is independent of the axial coordinate. In recent years considerable attention has been paid to the analysis of antiplane shear deformations within the context of elasticity theory (see, e.g., (Horgan, 1995; Horgan, 1995; Horgan and Miller, 1994) and the references therein). The variational analysis including the existence of entropy solutions for a class of antiplane frictional problems was obtained recently in (Andreu *et al.*, 2000). The novelties in the present paper are connected with the fact that the friction bound is assumed to be historical dependent and the process is quasistatic, leading to an interesting and nonstandard variational problem. Moreover, whereas in (Matei *et al.*, 2001) the case of linear elastic materials is considered, here we deal with linear viscoelastic materials. Also, notice that the corresponding three-dimensional problem with slip dependent-friction was studied in (Chau *et al.*, 2001), using a similar technique to that employed in the present paper.

The first objective of this paper is to provide the variational analysis of the viscoelastic quasistatic antiplane contact problem including the existence of a unique weak solution to the model. The second objective is to obtain a convergence result as the coefficient of viscosity converges to zero.

The paper is structured as follows. In Section 2 we present the mechanical model for the quasistatic antiplane contact problem. In Section 3 we list the assumptions on the data, derive the variational formulation of the problem and state our main existence and uniqueness result, i.e. Theorem 1. The proof of this result is carried out in several steps in Section 4 and is based on the argument of Banach's fixed point. In Section 5 we obtain a convergence result. More exactly, we consider an elastic quasistatic antiplane contact problem for which we justify the existence and the uniqueness of the weak solution using an abstract result recently obtained in (Motreanu and Sofonea, 1999). Then we prove that the solution of the viscoelastic problem converges to the solution of the elastic problem as the viscosity coefficient converges to zero.

2. The Model of the Antiplane Contact Problem

We consider a body $\mathcal{B}$ identified with a region in $\mathbb{R}^3$ it occupies in a fixed and undistorted reference configuration. We assume that $\mathcal{B}$ is a cylinder with generators parallel to the x_3-axes with a cross-section which is a regular region Ω in the x_1, x_2-plane, $Ox_1x_2x_3$ being a Cartesian coordinate system. The cylinder is assumed to be sufficiently long so that end effects in the axial direction are negligible. Thus $\mathcal{B} = \Omega \times (-\infty, +\infty)$. Let $\partial\Omega = \Gamma$. We assume that Γ is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that the one-dimensional measure of Γ_1, denoted by $\operatorname{meas}\Gamma_1$, is strictly positive. Let $T > 0$ and let $[0,T]$ denote the time interval of interest. The cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty)$ and is in contact with a rigid foundation on $\Gamma_3 \times (-\infty, +\infty)$ during the process. Moreover, the cylinder is subjected to time-dependent volume forces of density $\boldsymbol{f}_0$ on $\mathcal{B}$ and to time-dependent surface tractions of density $\boldsymbol{f}_2$ on $\Gamma_2 \times (-\infty, +\infty)$. We assume that

$$\boldsymbol{f}_0 = (0, 0, f_0) \tag{1}$$

with $f_0 = f_0(x_1, x_2, t)$: $\Omega \times [0,T] \to \mathbb{R}$, and

$$\boldsymbol{f}_2 = (0, 0, f_2) \tag{2}$$

with $f_2 = f_2(x_1, x_2, t)$: $\Gamma_2 \times [0,T] \to \mathbb{R}$. The body forces (1) and the surface tractions (2) would be expected to give rise to a deformation of the elastic cylinder whose displacement, denoted by $\boldsymbol{u}$, is of the form

$$\boldsymbol{u} = (0, 0, u) \tag{3}$$

with $u = u(x_1, x_2, t)$: $\Omega \times [0,T] \to \mathbb{R}$. Such a kind of deformation is called an *antiplane shear*.

The infinitesimal strain tensor, denoted by $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$, is defined by

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i,j = 1,2,3, \tag{4}$$

where the index that follows a comma indicates a partial derivative with respect to the corresponding component of the spatial variable. Moreover, in the sequel, the convention of summation upon a repeated index is used. From (3) and (4) it follows that, in the case of the antiplane problem, the infinitesimal strain tensor becomes

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \begin{pmatrix} 0 & 0 & \frac{1}{2}u_{,1} \\ 0 & 0 & \frac{1}{2}u_{,2} \\ \frac{1}{2}u_{,1} & \frac{1}{2}u_{,2} & 0 \end{pmatrix}. \tag{5}$$

Let $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_{ij})$ denote the stress field. We consider the linear constitutive law for a viscoelastic material

$$\boldsymbol{\sigma} = 2\theta\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \lambda(\operatorname{tr}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}), \tag{6}$$

where $\lambda > 0$ and $\mu > 0$ are the Lamé coefficients, $\theta > 0$ is the coefficient of viscosity, $\operatorname{tr}\boldsymbol{\varepsilon}(\boldsymbol{u}) = \varepsilon_{ii}(\boldsymbol{u})$ and $\boldsymbol{I}$ is the unit tensor in $\mathbb{R}^3$. Here and below the dot above represents the derivative with respect to the time variable. In the case of the antiplane problem, from (5) and (6), the stress field becomes

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & \theta\dot{u}_{,1} + \mu u_{,1} \\ 0 & 0 & \theta\dot{u}_{,2} + \mu u_{,2} \\ \theta\dot{u}_{,1} + \mu u_{,1} & \theta\dot{u}_{,2} + \mu u_{,2} & 0 \end{pmatrix}. \tag{7}$$

We neglect the inertial term in the equation of motion and obtain the quasistatic approximation for the process. Thus, keeping in mind (1), (3) and (7), we deduce that the equation of equilibrium reduces to the following scalar equation:

$$\theta\Delta\dot{u} + \mu\Delta u + f_0 = 0 \quad \text{in} \quad \Omega \times (0,T). \tag{8}$$

Recall that, since the cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty) \times (0,T)$, the displacement field vanishes there. Thus (3) implies

$$u = 0 \quad \text{on} \quad \Gamma_1 \times (0,T). \tag{9}$$

Let $\boldsymbol{\nu}$ denote the unit normal on $\Gamma \times (-\infty, +\infty)$. We have

$$\boldsymbol{\nu} = (\nu_1, \nu_2, 0) \tag{10}$$

with $\nu_i = \nu_i(x_1, x_2)$: $\Gamma \to \mathbb{R}$, $i = 1, 2$. For a vector $\boldsymbol{v}$ we denote by v_ν and $\boldsymbol{v}_\tau$ its *normal* and *tangential* components on the boundary, given by

$$v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}, \tag{11}$$

respectively. In (11) and everywhere in this paper '$\cdot$' represents the inner product on the space $\mathbb{R}^d$ ($d = 2$ or 3). Moreover, for a given stress field $\boldsymbol{\sigma}$, we denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the *normal* and the *tangential* components on the boundary, respectively, i.e.

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}. \tag{12}$$

From (7) and (10) we deduce that the Cauchy stress vector is given by

$$\boldsymbol{\sigma}\boldsymbol{\nu} = (0, 0, \theta\, \partial_\nu \dot{u} + \mu \partial_\nu u). \tag{13}$$

Here and subsequently we use the notation $\partial_\nu u = u_{,1}\nu_1 + u_{,2}\nu_2$. Keeping in mind the traction boundary condition $\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2$ on $\Gamma_2 \times (0, T)$, it follows from (2) and (13) that

$$\theta\, \partial_\nu \dot{u} + \mu\, \partial_\nu u = f_2 \quad \text{on} \quad \Gamma_2 \times (0, T). \tag{14}$$

We now describe the frictional contact condition on $\Gamma_3 \times (-\infty, +\infty)$. Everywhere in this paper the notation $|\cdot|$ is used to denote the Euclidean norm on $\mathbb{R}^d$ ($d = 1$ or 3). First, we remark that from (3), (10) and (11) we obtain $u_\nu = 0$, which shows that the contact is *bilateral*, i.e. there is no loss of contact during the process. Using again (3), (10) and (11), we find

$$\boldsymbol{u}_\tau = (0, 0, u). \tag{15}$$

Similarly, from (7), (10) and (12) we obtain

$$\boldsymbol{\sigma}_\tau = (0, 0, \sigma_\tau), \tag{16}$$

where

$$\sigma_\tau = \theta\, \partial_\nu \dot{u} + \mu\, \partial_\nu u. \tag{17}$$

We assume that the friction is invariant with respect to the x_3 axis and for all $t \in [0, T]$ it is modelled by the following conditions on Γ_3:

$$\begin{cases} |\boldsymbol{\sigma}_\tau(t)| \le g\Big(\displaystyle\int_0^t |\dot{\boldsymbol{u}}_\tau(s)|\, \mathrm{d}s\Big), \\ |\boldsymbol{\sigma}_\tau(t)| < g\Big(\displaystyle\int_0^t |\dot{\boldsymbol{u}}_\tau(s)|\, \mathrm{d}s\Big) \Rightarrow \dot{\boldsymbol{u}}_\tau(t) = 0, \\ |\boldsymbol{\sigma}_\tau(t)| = g\Big(\displaystyle\int_0^t |\dot{\boldsymbol{u}}_\tau(s)|\, \mathrm{d}s\Big) \Rightarrow \exists\, \beta \ge 0 \\ \qquad\qquad \text{such that } \boldsymbol{\sigma}_\tau = -\beta \dot{\boldsymbol{u}}_\tau. \end{cases} \tag{18}$$

Here $g : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ is a given function and $\dot{\boldsymbol{u}}_\tau$ denotes the tangential velocity on the contact boundary. This is a version of Tresca's friction law where the friction bound g is assumed to depend on the accumulated slip of the surface. From the mechanical point of view, this dependence models the changes in the contact surface structure that resulted from sliding. In the previous formula, when the friction bound $g : \Gamma_3 \to \mathbb{R}_+$ was a given function, we obtained the classical Tresca friction law. In (18) the strict inequality holds in the *stick* zone and the equality in the *slip* zone.

Using now (15)–(18), it is straightforward to see that

$$\begin{cases} |\theta\, \partial_\nu \dot{u}(t) + \mu\, \partial_\nu u(t)| \le g\Big(\displaystyle\int_0^t |\dot{u}(s)|\, \mathrm{d}s\Big), \\ |\theta\, \partial_\nu \dot{u}(t) + \mu\, \partial_\nu u(t)| \\ \qquad < g\Big(\displaystyle\int_0^t |\dot{u}(s)|\, \mathrm{d}s\Big) \Rightarrow \dot{u}(t) = 0, \\ |\theta\, \partial_\nu \dot{u}(t) + \mu\, \partial_\nu u(t)| \\ \qquad = g\Big(\displaystyle\int_0^t |\dot{u}(s)|\, \mathrm{d}s\Big) \Rightarrow \exists \beta \ge 0 \\ \qquad \text{such that } \theta\, \partial_\nu \dot{u}(t) + \mu\, \partial_\nu u(t) = -\beta \dot{u}(t), \end{cases} \tag{19}$$

on Γ_3 for all $t \in [0, T]$.

Finally, we prescribe the initial displacement

$$u(0) = u_0 \text{ in } \Omega, \tag{20}$$

where u_0 is a given function in Ω.

Now, the mechanical model of the antiplane contact problem is complete and can be stated as follows.

Problem $\mathcal{P}$: *Find a displacement field* $u : \Omega \times [0, T] \to \mathbb{R}$ *such that* (8), (9), (14), (19) *and* (20) *hold.*

Note that when the displacement field u which solves Problem $\mathcal{P}$ is known, the stress tensor can be calculated using (7).

3. Variational Formulation and Main Result

In this section we derive the variational formulation of Problem $\mathcal{P}$ and state our main existence and uniqueness result, Theorem 1. To this end, we introduce the closed subspace of $H^1(\Omega)$ defined by

$$V = \{v \in H^1(\Omega) \,|\, v = 0 \text{ on } \Gamma_1\}.$$

Since $\operatorname{meas} \Gamma_1 > 0$, the Friedrichs-Poincaré inequality holds, i.e. there exists $C_P > 0$ which depends only on Ω and Γ_1 such that

$$\|u\|_{H^1(\Omega)} \le C_P \|\nabla u\|_{L^2(\Omega)^2} \quad \forall u \in V. \tag{21}$$

Using (21), we can introduce on V the inner product given by

$$(u, v)_V = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \quad \forall u, v \in V, \tag{22}$$

and let $\|\cdot\|_V$ be the associated norm, i.e.

$$\|v\|_V = \|\nabla v\|_{L^2(\Omega)^2}, \quad \forall v \in V. \tag{23}$$

From (21) and (23) it follows that $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_V$ are equivalent norms on V and therefore $(V, \|\cdot\|_V)$ is a real Hilbert space. By the Sobolev trace theorem and (21), there exists $C_0 > 0$ depending only on Ω and Γ such that

$$\|v\|_{L^2(\Gamma)} \le C_0 \|v\|_V, \quad \forall v \in V. \tag{24}$$

Here and subsequently, we still write v for the trace γv of v on Γ, for all $v \in V$.

In the study of the mechanical Problem $\mathcal{P}$, we assume that the forces and tractions have the regularity

$$f_0 \in L^\infty(0, T; L^2(\Omega)), \quad f_2 \in L^\infty(0, T; L^2(\Gamma_2)). \tag{25}$$

We suppose that the *friction bound* function g satisfies the following properties:

$$\begin{cases} (a)\ g : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+; \\ (b)\ \exists\, L_g > 0 \text{ such that } |g(x, r_1) - g(x, r_2)| \\ \qquad \le L_g |r_1 - r_2|, \forall\, r_1, r_2 \in \mathbb{R} \text{ a.e. } x \in \Gamma_3; \\ (c)\ \forall\, r \in \mathbb{R},\ g(\cdot, r) \text{ is Lebesgue measurable on } \Gamma_3; \\ (d)\ g(\cdot, 0) \in L^2(\Gamma_3). \end{cases} \tag{26}$$

The initial data are chosen such that

$$u_0 \in V. \tag{27}$$

For every $t \in [0, T]$ we need to consider the operator S_t defined by

$$\begin{aligned} &S_t\colon\ L^\infty(0, T; V) \to L^2(\Gamma), \\ &S_t(v) = \int_0^t |v(s)| \, \mathrm{d}s \ \text{ a.e. on } \Gamma. \end{aligned} \tag{28}$$

From (28) and (24) it follows that for all $v_1, v_2 \in L^\infty(0, T; V)$ the following inequality holds:

$$\|S_t(v_1) - S_t(v_2)\|_{L^2(\Gamma)} \le C \int_0^t \|v_1(s) - v_2(s)\|_V \, \mathrm{d}s. \tag{29}$$

Here and below C represents a positive constant whose value may change from line to line.

We define now the functional $j : L^2(\Gamma) \times V \to \mathbb{R}_+$ given by

$$j(\xi, v) = \int_{\Gamma_3} g(\xi)\, |v| \, \mathrm{d}a, \quad \forall\, \xi \in L^2(\Gamma), \quad \forall\, v \in V. \tag{30}$$

Using the conditions (26), we deduce that the integral in (30) is well defined.

Let $f : [0, T] \to V$ be given by

$$(f(t), v)_V = \int_\Omega f_0(t) v \, \mathrm{d}x + \int_{\Gamma_2} f_2(t) v \, \mathrm{d}a, \quad \forall v \in V \ \text{ a.e. } t \in (0, T). \tag{31}$$

The definition of f in (31) is based on the Riesz representation theorem, and we note that conditions (25) imply

$$f \in L^\infty(0, T; V). \tag{32}$$

The variational formulation of Problem $\mathcal{P}$ is based on the following result.

Lemma 1. *If u is a smooth solution to Problem $\mathcal{P}$, then $u(t) \in V$ and*

$$\begin{aligned} &\theta(\dot{u}(t), v - \dot{u}(t))_V + \mu\, (u(t), v - \dot{u}(t))_V \\ &\quad + j(S_t(\dot{u}), v) - j(S_t(\dot{u}), \dot{u}(t)) \\ &\quad \ge (f(t), v - \dot{u}(t))_V, \quad \forall v \in V \ \textit{a.e.}\ t \in (0, T). \end{aligned}$$

Proof. Let u denote a smooth solution to Problem $\mathcal{P}$. We have $u(t),\, \dot{u}(t) \in V$ a.e. $t \in (0, T)$ and, from (8), (9) and (14), we deduce that

$$\begin{aligned} &\theta \int_\Omega \nabla \dot{u}(t) \cdot \nabla(v - \dot{u}(t)) \, \mathrm{d}x \\ &\quad + \mu \int_\Omega \nabla u(t) \cdot \nabla(v - \dot{u}(t)) \, \mathrm{d}x \\ &= \int_\Omega f_0(t)(v - \dot{u}(t)) \, \mathrm{d}x + \int_{\Gamma_2} f_2(t)(v - \dot{u}(t)) \, \mathrm{d}a \\ &\quad + \int_{\Gamma_3} (\theta\, \partial_\nu \dot{u}(t) + \mu\, \partial_\nu u(t))(v - \dot{u}(t)) \, \mathrm{d}a, \\ &\qquad \forall\, v \in V \text{ a.e. } t \in (0, T). \end{aligned}$$

Thus, from (22) and (31) we obtain

$$\begin{aligned} &(\theta\, \dot{u}(t) + \mu\, u(t), v - \dot{u}(t))_V \\ &\quad - \int_{\Gamma_3} (\theta\, \partial_\nu \dot{u}(t) + \mu\, \partial_\nu u(t))(v - \dot{u}(t)) \, \mathrm{d}a \\ &= (f(t), v - \dot{u}(t))_V, \quad \forall v \in V \ \text{ a.e. } t \in (0, T). \end{aligned} \tag{33}$$

Using (19) and (30), we deduce that for all $t \in [0, T]$,

$$j(S_t(\dot{u}), \dot{u}(t)) = -\int_{\Gamma_3} (\theta\, \partial_\nu \dot{u}(t) + \mu\, \partial_\nu u(t)) \dot{u}(t) \, \mathrm{d}a, \tag{34}$$

and

$$j(S_t(\dot{u}), v) \geq -\int_{\Gamma_3} (\theta\, \partial_\nu \dot{u}(t) + \mu\, \partial_\nu u(t)) v \,\mathrm{d}a, \qquad \forall v \in V. \tag{35}$$

The inequality in Lemma 1 follows now from (33)–(35). ■

Lemma 1 and (20) lead us to consider the following variational problem:

Problem $\mathcal{PV}$: *Find a displacement field $u : [0,T] \to V$ such that*

$$\theta\, (\dot{u}(t), v - \dot{u}(t))_V + \mu\, (u(t), v - \dot{u}(t))_V + j(S_t(\dot{u}), v) - j(S_t(\dot{u}), \dot{u}(t)) \geq (f(t), v - \dot{u}(t))_V, \quad \forall v \in V \text{ a.e. } t \in (0,T), \tag{36}$$

$$u(0) = u_0. \tag{37}$$

Our main existence and uniqueness result, which we establish in Section 4, is the following:

Theorem 1. *Assume that (25)–(27) hold. Then the variational problem $\mathcal{PV}$ possesses a unique solution $u \in W^{1,\infty}(0,T;V)$.*

Here, to end this section, we present an interpretation of our result. We note that Problem $\mathcal{PV}$ represents the *variational formulation* of the viscoelastic quasistatic antiplane frictional contact problem $\mathcal{P}$ and therefore its solution may be interpreted as the *weak solution* to problem $\mathcal{P}$. Theorem 1 shows the unique solvability of Problem $\mathcal{PV}$ and therefore it shows that, under assumptions (25)–(27), Problem $\mathcal{P}$ has a unique weak solution.

4. Proof of Theorem 1

The proof of Theorem 1 will be carried out in several steps and is based on fixed-point arguments, similar to those used in (Chau *et al.*, 2001) with, however, a different choice of the operators since the settings in (Chau *et al.*, 2001) and here are different. Assume that (25)–(27) hold and let η and ξ be two elements of $L^\infty(0,T;V)$. We consider the following variational problem:

Problem $\mathcal{PV}_{\eta\xi}$: Find $v_{\eta\xi} : [0,T] \longrightarrow V$ such that

$$\theta(v_{\eta\xi}(t), v - v_{\eta\xi}(t))_V + \mu(\eta(t), v - v_{\eta\xi}(t))_V + j(S_t(\xi), v) - j(S_t(\xi), v_{\eta\xi}(t)) \geq (f(t), v - v_{\eta\xi}(t))_V, \quad \forall\, v \in V \text{ a.e. } t \in (0,T). \tag{38}$$

The unique solvability of the intermediate problem $\mathcal{PV}_{\eta\xi}$ follows from the following result:

Lemma 2. *There exists a unique solution $v_{\eta\xi}$ to Problem $\mathcal{PV}_{\eta\xi}$. Moreover, $v_{\eta\xi} \in L^\infty(0,T;V)$.*

Proof. It follows from classical results for elliptic variational inequalities that there exists a unique solution $v_{\eta\xi}(t) \in V$ that solves (38) a.e. $t \in (0,T)$.

Taking $v = 0_V$ in (38), we deduce that

$$\theta\|v_{\eta\xi}(t)\|_V \leq \|f(t)\|_V + \mu\|\eta(t)\|_V \text{ a.e. } t \in (0,T). \tag{39}$$

Keeping in mind (39), (32) and the regularity $\eta \in L^\infty(0,T;V)$, we obtain $v_{\eta\xi} \in L^\infty(0,T;V)$, which concludes the proof. ■

We consider now the operator Λ_η: $L^\infty(0,T;V) \longrightarrow L^\infty(0,T;V)$ defined for all $\eta \in L^\infty(0,T;V)$ by

$$\Lambda_\eta \xi = v_{\eta\xi}, \quad \forall\, \xi \in L^\infty(0,T;V). \tag{40}$$

Lemma 3. *For every $\eta \in L^\infty(0,T;V)$ the operator Λ_η has a unique fixed point $\xi_\eta \in L^\infty(0,T;V)$.*

Proof. Let $\eta \in L^\infty(0,T;V)$ and $\xi_i \in L^\infty(0,T;V)$, $i = 1, 2$. In order to simplify the notation, we denote by v_i the unique solution to Problem $\mathcal{PV}_{\eta\xi_i}$ for $i = 12$. Thus, from (38) we can write

$$\theta(v_i(t), v - v_i(t))_V + \mu(\eta(t), v - v_i(t))_V + j(S_t(\xi_i), v) - j(S_t(\xi_i), v_i(t)) \geq (f(t), v - v_i(t))_V, \quad \forall\, v \in V \text{ a.e. } t \in (0,T). \tag{41}$$

From (41), after some algebra, we find

$$\theta\|v_1(t) - v_2(t)\|_V^2 \leq j(S_t(\xi_1), v_2(t)) - j(S_t(\xi_2), v_2(t)) + j(S_t(\xi_2), v_1(t)) - j(S_t(\xi_1), v_1(t)) \text{ a.e. } t \in (0,T). \tag{42}$$

Using now (30), (26), (29) and (24), it follows that

$$j(S_t(\xi_1), v_2(t)) - j(S_t(\xi_2), v_2(t)) + j(S_t(\xi_2), v_1(t)) - j(S_t(\xi_1), v_1(t)) \leq C \int_0^t \|\xi_1(s) - \xi_2(s)\|_V \,\mathrm{d}s \times \|v_1(t) - v_2(t)\|_V \text{ a.e. } t \in (0,T). \tag{43}$$

From (42) and (43) we deduce that

$$\|v_1(t) - v_2(t)\|_V \le C \int_0^t \|\xi_1(s) - \xi_2(s)\|_V \,\mathrm{d}s \ \text{ a.e. } \ t \in (0,T).$$

Now, the definition (40) implies

$$\|\Lambda_\eta \xi_1(t) - \Lambda_\eta \xi_2(t)\|_V \le C\int_0^t \|\xi_1(s) - \xi_2(s)\|_V \,\mathrm{d}s \ \text{ a.e. } \ t\in(0,T). \quad (44)$$

Set

$$\|v\|_\zeta = \inf\big\{M > 0 \,|\, e^{-\zeta t}\|v(t)\|_V \le M \ \text{ a.e. } \ t \in (0,T)\big\}, \quad \forall\, v \in L^\infty(0,T;V),$$

with $\zeta > 0$ to be determined later. Clearly, $\|\cdot\|_\zeta$ defines a norm on the space $L^\infty(0,T;V)$ which is equivalent to the standard norm $\|\cdot\|_{L^\infty(0,T;V)}$. Using now (44) and the definition of $\|\cdot\|_\zeta$, we can write

$$\begin{aligned} e^{-\zeta t}\|\Lambda_\eta \xi_1(t) - \Lambda_\eta \xi_2(t)\|_V &\le C\, e^{-\zeta t} \int_0^t e^{\zeta s} e^{-\zeta s}\|\xi_1(s) - \xi_2(s)\|_V \,\mathrm{d}s \\ &\le C\, e^{-\zeta t}\|\xi_1 - \xi_2\|_\zeta \int_0^t e^{\zeta s}\,\mathrm{d}s \\ &\le \frac{C}{\zeta}\,\|\xi_1 - \xi_2\|_\zeta \ \text{ a.e. } \ t \in (0,T). \end{aligned}$$

Consequently, we deduce that

$$\|\Lambda_\eta \xi_1 - \Lambda_\eta \xi_2\|_\zeta \le \frac{C}{\zeta}\,\|\xi_1 - \xi_2\|_\zeta.$$

Taking ζ such that $\zeta > C$, we conclude that the operator Λ_η is a contraction on the space $(L^\infty(0,T;V), \|\cdot\|_\zeta)$. By the Banach fixed point theorem, the operator Λ_η has a unique fixed-point $\xi_\eta \in L^\infty(0,T;V)$. ■

In what follows, we continue to write

$$v_\eta = v_{\eta\xi_\eta}, \quad \forall\, \eta \in L^\infty(0,T;V). \quad (45)$$

Keeping in mind that ξ_η is the unique fixed point of the operator Λ_η, from (40) and (45) we have

$$v_\eta = \xi_\eta. \quad (46)$$

Using (46) and the fact that v_η is the unique solution to Problem $\mathcal{PV}_{\eta\xi_\eta}$, we can write

$$\begin{aligned} &\theta(v_\eta(t), v - v_\eta(t))_V + j(S_t(v_\eta), v) \\ &\quad - j(S_t(v_\eta), v_\eta(t)) \\ &\ge (f(t) - \mu\eta(t), v - v_\eta(t)), \\ &\qquad \forall\, v \in V \ \text{ a.e. } \ t \in (0,T). \quad (47) \end{aligned}$$

We define now the function $u_\eta\colon [0,T] \longrightarrow V$ by

$$u_\eta(t) = \int_0^t v_\eta(s)\,\mathrm{d}s + u_0, \quad \forall\, t \in [0,T]. \quad (48)$$

We also define the operator $\Lambda : L^\infty(0,T;V) \longrightarrow L^\infty(0,T;V)$ by the formula

$$\Lambda\eta = u_\eta, \quad \forall\, \eta \in L^\infty(0,T;V). \quad (49)$$

Lemma 4. *The operator Λ has a unique fixed point $\eta^* \in L^\infty(0,T;V)$.*

Proof. Let $\eta_1, \eta_2 \in L^\infty(0,T;V)$ and let $v_i = v_{\eta_i}$, $u_i = u_{\eta_i}$, for $i = 1, 2$. Using (47) and arguments similar to those used in the proof of Lemma 3, we obtain

$$\|v_1(s) - v_2(s)\|_V \le C\Big(\|\eta_1(s) - \eta_2(s)\|_V + \int_0^s \|v_1(r) - v_2(r)\|_V \,\mathrm{d}r\Big) \ \text{ a.e. } \ s \in (0,T).$$

Integrating the previous inequality on $[0,\, t]$ with a fixed t and using a Gronwall-type argument, we obtain

$$\int_0^t \|v_1(s) - v_2(s)\|_V \,\mathrm{d}s \le C \int_0^t \|\eta_1(s) - \eta_2(s)\|_V \,\mathrm{d}s, \quad \forall\, t \in [0,T]. \quad (50)$$

The relations (48)–(50) give

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V \le C \int_0^t \|\eta_1(s) - \eta_2(s)\|_V \,\mathrm{d}s, \quad \forall\, t \in [0,T].$$

Keeping in mind the definition of $\|\cdot\|_\zeta$, Lemma 4 follows from the previous inequality, after using a fixed-point argument similar to that presented in Lemma 3. ■

Now we have all the ingredients to prove Theorem 1.

Proof of Theorem 1. (Existence) Let $\eta^* \in L^\infty(0,T;V)$ be the unique fixed point of the operator Λ and let $u_{\eta^*} \in W^{1,\infty}(0,T;V)$ be the function defined by the relation (48) for $\eta = \eta^*$. We have $\dot{u}_{\eta^*} = v_{\eta^*}$ and, from (47), it follows that

$$\begin{aligned} &\theta(\dot{u}_{\eta^*}(t), v - \dot{u}_{\eta^*}(t))_V + j(S_t(\dot{u}_{\eta^*}), v) \\ &\quad - j(S_t(\dot{u}_{\eta^*}), \dot{u}_{\eta^*}(t)) \\ &\ge (f(t) - \mu\eta^*(t), v - \dot{u}_{\eta^*}(t)), \\ &\qquad \forall\, v \in V \ \text{ a.e. } \ t \in (0,T). \quad (51) \end{aligned}$$

The inequality (36) follows now from (51) and (49), using the fact that η^* is the fixed point of the operator Λ.

The definition (48) implies $u_{\eta^*}(0) = u_0$ so that (37) is fulfilled. We conclude now that u_{η^*} is a solution to Problem $\mathcal{PV}$.

(Uniqueness) Let $u_i \in W^{1,\infty}(0,T;V)$ be two solutions to Problem $\mathcal{PV}$, $i = 1,2$. Setting $v_i = \dot{u}_i$, $i = 1,2$, we have

$$u_i(t) = \int_0^t v_i(s)\,\mathrm{d}s + u_0, \quad \forall\, t \in [0,T]. \tag{52}$$

From (36) it follows that

$$\begin{aligned}\theta(v_i(t), v - v_i(t))_V &+ j(S_t(v_i), v) - j(S_t(v_i), v_i(t))\\ &\geq (f(t) - \mu u_i(t), v - v_i(t))_V \quad \text{a.e. } t \in (0,T).\end{aligned}$$

Taking $v = v_2(t)$ (resp. $v = v_1(t)$) when $i = 1$ (resp. $i = 2$) and adding the two resulting inequalities, we obtain

$$\begin{aligned}&\theta\|v_1(t) - v_2(t)\|_V^2\\ &\quad\leq j(S_t(v_1), v_2(t))\\ &\qquad -j(S_t(v_2), v_2(t)) + j(S_t(v_2), v_1(t))\\ &\qquad -j(S_t(v_1), v_1(t)) + \mu\|u_1(t) - u_2(t)\|_V\\ &\qquad \times\|v_1(t) - v_2(t)\|_V \quad \text{a.e. } t \in (0,T).\end{aligned} \tag{53}$$

Using (30), (26), (29) and (24), we deduce that

$$\begin{aligned}&j(S_t(v_1), v_2(t)) - j(S_t(v_2), v_2(t))\\ &\quad + j(S_t(v_2), v_1(t)) - j(S_t(v_1), v_1(t))\\ &\quad \leq L_g\|S_t(v_1) - S_t(v_2)\|_{L^2(\Gamma_3)}\,\|v_1(t) - v_2(t)\|_{L^2(\Gamma_3)}\\ &\quad \leq C\left(\int_0^t \|v_1(s) - v_2(s)\|_V\,ds\right)\|v_1(t) - v_2(t)\|_V\\ &\qquad\qquad \text{a.e. } t \in (0,T).\end{aligned}$$

Combining the last inequality and (53), for a.e. $t \in (0,T)$ we obtain

$$\begin{aligned}&\theta\|v_1(t) - v_2(t)\|_V^2\\ &\quad \leq \Big(C\int_0^t \|v_1(s) - v_2(s)\|_V\,\mathrm{d}s\\ &\qquad + \mu\|u_1(t) - u_2(t)\|_V\Big)\|v_1(t) - v_2(t)\|_V.\end{aligned}$$

From this inequality and (52), we deduce that

$$\begin{aligned}&\|v_1(t) - v_2(t)\|_V\\ &\quad \leq C\int_0^t \|v_1(s) - v_2(s)\|_V\,\mathrm{d}s \quad \text{a.e. } t \in (0,T).\end{aligned}$$

Using a Gronwall-type argument, we find that $v_1 = v_2$. Finally, from (52), we deduce that $u_1 = u_2$, which concludes the uniqueness part of Theorem 1. ■

5. A Convergence Result

In this section we assume that the friction bound does not depend on the slip and we investigate the behaviour of the solution as the coefficient of viscosity tends to zero. Therefore, everywhere in the sequel we consider a given function g which satisfies

$$g \in L^\infty(\Gamma_3), \quad g \geq 0 \ \text{ a.e. on } \Gamma_3, \tag{54}$$

and we denote by j the functional

$$j\colon\ V \longrightarrow \mathbb{R}_+, \quad j(v) = \int_{\Gamma_3} g|v|\,\mathrm{d}a, \quad \forall\, v \in V.$$

We assume that

$$\begin{aligned}&f_0 \in W^{1,\infty}(0,T;L^2(\Omega))\\ &f_2 \in W^{1,\infty}(0,T;L^2(\Gamma_2)),\end{aligned} \tag{55}$$

and the initial data u_0 satisfy

$$u_0 \in V, \quad \mu(u_0, v)_V + j(v) \geq (f(0), v)_V, \quad \forall\, v \in V, \tag{56}$$

where f is given by (31). We consider the following variational problems:

Problem $\mathcal{PV}_\theta$: *Find a displacement field $u_\theta : [0,T] \longrightarrow V$ such that*

$$\begin{aligned}&\theta(\dot{u}_\theta(t), v - \dot{u}_\theta(t))_V + \mu(u_\theta(t), v - \dot{u}_\theta(t))_V\\ &\quad + j(v) - j(\dot{u}_\theta(t))\\ &\quad \geq (f(t), v - \dot{u}_\theta(t))_V, \quad \forall v \in V \ \textit{a.e. } t \in (0,T),\end{aligned} \tag{57}$$

$$u_\theta(0) = u_0. \tag{58}$$

Problem $\mathcal{PV}_\mathcal{E}$: *Find a displacement field $u : [0,T] \rightarrow V$ such that*

$$\begin{aligned}&\mu\,(u(t), v - \dot{u}(t))_V + j(v) - j(\dot{u}(t))\\ &\quad \geq (f(t), v - \dot{u}(t))_V, \quad \forall v \in V \ \textit{a.e. } t \in (0,T),\end{aligned} \tag{59}$$

$$u(0) = u_0. \tag{60}$$

Clearly, Problem $\mathcal{PV}_\mathcal{E}$ represents the variational formulation of the antiplane frictional contact problem for linear elastic materials, i.e. the problem obtained when (6) is replaced by the elastic constitutive law

$$\boldsymbol{\sigma} = \lambda(\mathrm{tr}\,\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}).$$

It follows from Theorem 1 that Problem $\mathcal{PV}_\theta$ has a unique solution with regularity $u_\theta \in W^{1,\infty}(0,T;V)$. Moreover, since $j\colon\ V \longrightarrow \mathbb{R}_+$ is a continuous seminorm, keeping in mind (31), (55) and (56), it follows from Corollary 2.2 in (Motreanu and Sofonea, 1999) that Problem $\mathcal{PV}_\mathcal{E}$ has a unique solution $u \in W^{1,\infty}(0,T;V)$.

Our main result in this section is the following:

Theorem 2. *Assume that* (54)–(56) *hold. Then the solution* u_θ *to Problem* $\mathcal{PV}_\theta$ *converges to the solution* u *to Problem* $\mathcal{PV}_\mathcal{E}$*, i.e.*

$$u_\theta \to u \quad in\ L^\infty(0,T;V) \ \ as\ \ \theta \searrow 0. \tag{61}$$

From this result we conclude that the weak solution to the antiplane viscoelastic problem with Tresca's friction law may be approached by the weak solution to the antiplane elastic problem with Tresca's friction law when the coefficient of viscosity θ is small enough. In addition to the mathematical interest in the convergence result (61), this is of importance from a mechanical point of view, as it indicates that the case of elasticity with friction may be considered as a limit case of viscoelasticity with friction.

Proof. Let $\theta > 0$. We set $v = \dot u(t)$ in (57) and $v = \dot u_\theta(t)$ in (59). We add the corresponding inequalities to obtain

$$\mu(u(t) - u_\theta(t), \dot u(t) - \dot u_\theta(t))_V \le \theta(\dot u_\theta(t), \dot u(t) - \dot u_\theta(t))_V \ \text{ a.e. } \ t \in (0,T),$$

which implies

$$\frac{1}{2}\mu\frac{\mathrm{d}}{\mathrm{d}t}\|u(t) - u_\theta(t)\|_V^2 \le \theta(\dot u_\theta(t), \dot u(t) - \dot u_\theta(t))_V \ \text{ a.e. } \ t \in (0,T).$$

Let $s \in [0,T]$. We deduce from the previous inequality, (58) and (60) that

$$\begin{aligned}\mu\|u(s) - u_\theta(s)\|_V^2 &\le 2\theta\int_0^s \big(\dot u_\theta(t), \dot u(t) - \dot u_\theta(t)\big)_V\,\mathrm{d}t,\\ &\le 2\theta\int_0^s \big((\dot u(t), \dot u(t) - \dot u_\theta(t))_V - \|\dot u(t) - \dot u_\theta(t)\|_V^2\big)\,\mathrm{d}t,\\ &\le 2\theta\int_0^s \big(\|\dot u(t)\|_V\|\dot u(t) - \dot u_\theta(t)\|_V \\ &\quad - \|\dot u(t) - \dot u_\theta(t)\|_V^2\big)\,\mathrm{d}t.\end{aligned}$$

We use now the inequality

$$a\,b \le \frac{a^2}{4} + b^2, \quad \forall\, a, b > 0$$

to obtain

$$\mu\|u(s) - u_\theta(s)\|_V^2 \le 2\theta\int_0^s \frac{\|\dot u(t)\|_V^2}{4}\,\mathrm{d}t.$$

Consequently, we have

$$\mu\,\|u_\theta(s) - u(s)\|_V^2 \le \frac{\theta}{2}\int_0^T \|\dot u(t)\|_V^2\,\mathrm{d}t,$$

which implies (61). ■

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