ENLARGED EXACT COMPENSATION IN DISTRIBUTED SYSTEMS

LARBI AFIFI*, ABDELHAKIM CHAFIAI*
ABDELAZIZ BEL FEKIH**

* Faculty of Sciences, University Hassan II Ain Chock
B.P.5366, Maârif, Casablanca, Morocco
e-mail: {afifi,chafiai}@facsc-achok.ac.ma

** Faculty of Sciences, University Abdelmalek Saadi
B.P. 2121, Tetouan, Morocco
e-mail: belfekih@hotmail.com

In this work, we examine, through the observation of a class of linear distributed systems, the possibility of reducing the effect of disturbances (pollution, etc.), by making observations within a given margin of tolerance using a control term. This problem is called enlarged exact remediability. We show that with a convenient choice of input and output operators (actuators and sensors, respectively), the considered control problem has a unique optimal solution, which will be given. We also study the relationship between the notion of remediability, introduced in previous works, and that of enlarged exact remediability.

Keywords: distributed systems, remediability, control, actuators, sensors

1. Introduction

In this work, motivated by environmental problems, we study, through the observation of a class of disturbed linear systems the possibility of reducing in finite time T the effect of a disturbance (pollution, etc.) by taking an observation in a given region of tolerance using a control term. This constitutes an extension of previous works on remediability (Afifi *et al.*, 1998; 1999; 2000) or disturbance rejection and decoupling (Otsuka, 1991; Rabah and Malabare, 1997).

With no loss of generality, we consider a class of disturbed linear systems described by the following equation:

$$\begin{cases} \dot{z}(t) = Az(t) + f(t) + Bu(t), & 0 < t < T, \\ z(0) = z_0, \end{cases} \tag{1}$$

where A generates a strongly continuous semigroup (s.c.s.g.) $(S(t))_{t\geq 0}$ on the space X, $B \in \mathcal{L}(\mathcal{U}; X)$, $u \in L^2(0, T; \mathcal{U})$ is the control, X and $\mathcal{U}$ are two real Hilbert spaces. Moreover, $z_0 \in \mathcal{D}(A)$, a dense subspace of X (Curtain and Zwart, 1995). The term f (representing pollution, infection, etc.) is supposed to be unknown and the system (1) is augmented by the output equation

$$y(t) = Cz(t), \tag{2}$$

where $C \in \mathcal{L}(X, Y)$, Y being the observation space (a Hilbert space). The solution to (1) is given by

$$z_{u,f}(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)\,\mathrm{d}s + \int_0^t S(t-s)f(s)\,\mathrm{d}s.$$

If we denote by $y_{u,f}$ the corresponding observation, then in the case where $f = 0$ and $u = 0$ (normal case), the observation is given by

$$y_{0,0}(t) = CS(t)z_0.$$

But if the system is disturbed by a term f, the observation becomes

$$y_{0,f}(t) = CS(t)z_0 + \int_0^t CS(t-s)f(s)\,\mathrm{d}s \neq CS(t)z_0.$$

Then we introduce a control term Bu in order to reduce the effect of this disturbance by taking the observation at final time T within a given region of tolerance $\mathcal{C}$, i.e.

$$y_{u,f}(T) = CS(T)z_0 + \int_0^T CS(T-s)f(s)\,\mathrm{d}s + \int_0^T CS(T-s)Bu(s)\,\mathrm{d}s \in \mathcal{C}, \tag{3}$$

where $\mathcal{C}$ is a nonempty, convex and closed subset of Y such that $CS(T)z_0 \in \mathcal{C}$. This will be called enlarged exact remediability. The actuators and sensors are respectively the input and output parameters of the system. For a disturbance $f \in L^2(0,T;X)$, actuators ensuring the existence of a control u satisfying (3) are termed $\mathcal{C}$-efficient. In the particular case where $\mathcal{C}$ is the closed ball $\mathcal{B}(CS(T)z_0, \varepsilon) \subset Y$, centred at $CS(T)z_0$ and with a radius $\varepsilon > 0$, these actuators are said to be ε-efficient.

Exact remediability is a strong notion and its realization is difficult. We then introduce the notion of enlarged exact remediability, which is weaker and more practical. We give its characterization, particularly in the case of a ball $\mathcal{B}(CS(T)z_0, \varepsilon) \subset Y$. We study the optimal control problem (Lee and Marcus, 1967), and we show, under the weak remediability hypothesis, that the cost is reduced compared with the exact remediabilty case. Finally, we study some particular situations.

This paper is organized as follows. In Section 2, we briefly recall the notions of remediability and efficient actuators, and we give the principal results, which will be used later in this work. In Section 3, we define and characterize the notion of enlarged exact remediability, define $\mathcal{C}$-efficient actuators and give their characterization particularly in the case where $\mathcal{C} = \mathcal{B}(CS(T)z_0, \varepsilon)$. We also study the relationship between the notions of remediability and enlarged exact remediability, and hence between efficient actuators and $\mathcal{C}$-efficient actuators. In Section 4, we study the problem of enlarged exact remediability with minimal energy, using an extension of the Hilbert Uniqueness Method (H.U.M.) (Lions, 1988). Finally, in Section 5, we examine particular situations related to the choice of $\mathcal{C}$.

2. The Notion of Remediability

In this part, we recall the notions of exact and weak remediabilities and efficient actuators, as well as the principal characterization results (Afifi *et al.*, 1998; 1999; 2000). We consider the system described by (1), augmented by the output equation (2). Let H and R be the linear operators defined by

$$H: \begin{array}{rcl} L^2(0,T;\mathcal{U}) & \longrightarrow & X, \\ u & \longmapsto & Hu = \displaystyle\int_0^T S(T-s)Bu(s)\,\mathrm{d}s \end{array} \tag{4}$$

and

$$R: \begin{array}{rcl} L^2(0,T;X) & \longrightarrow & Y, \\ f & \longmapsto & Rf = \displaystyle\int_0^T CS(T-s)f(s)\,\mathrm{d}s. \end{array} \tag{5}$$

We have

$$y_{u,f}(T) = CS(T)z_0 + CHu + Rf. \tag{6}$$

2.1. Definitions and Characterizations

Let us recall the following definitions.

Definition 1. (i) We say that a disturbance f is exactly remediable on $[0,T]$ if there exists $u \in L^2(0,T;\mathcal{U})$ such that

$$y_{u,f}(T) - CS(T)z_0 = 0. \tag{7}$$

(ii) We say that (1), augmented by (2), is exactly remediable on $[0,T]$ if any disturbance $f \in L^2(0,T;X)$ is exactly remediable on $[0,T]$.

Definition 2. (i) A disturbance f is called weakly remediable on $[0,T]$ if for any $\varepsilon > 0$ there exists $u \in L^2(0,T;\mathcal{U})$ such that

$$\|y_{u,f}(T) - CS(T)z_0\| < \varepsilon. \tag{8}$$

(ii) The system (1), augmented by (2), is said to be weakly remediable on $[0,T]$ if any disturbance $f \in L^2(0,T;X)$ is weakly remediable on $[0,T]$.

Let $B^\star$, $R^\star$, $S^\star(\cdot)$ and $C^\star$ be the adjoint operators of B, R, S and C, respectively. Furthermore, let X', $\mathcal{U}'$ and Y' be the dual spaces of X, $\mathcal{U}$ and Y, respectively. The operator $R^\star$ is given by

$$R^\star: \begin{array}{rcl} Y' & \longrightarrow & L^2(0,T;X'), \\ \theta & \longmapsto & R^\star\theta = S^\star(T-\cdot)C^\star\theta. \end{array} \tag{9}$$

We have the following characterization.

Proposition 1. *(i) A disturbance f is exactly remediable on $[0,T]$ if and only if*

$$Rf \in \mathrm{Im}(CH). \tag{10}$$

(ii) The system (1), augmented by (2), is exactly remediable on $[0,T]$ if and only if there exists $\gamma > 0$ such that

$$\|R^\star\theta\|_{L^2(0,T;X')} \le \gamma \|B^\star R^\star\theta\|_{L^2(0,T;\,\mathcal{U}')}, \quad \forall\,\theta \in Y'. \tag{11}$$

For weak remediability, we have the following result.

Proposition 2. *(i) A disturbance f is weakly remediable on $[0,T]$ if and only if*

$$Rf \in \overline{\mathrm{Im}(CH)}. \tag{12}$$

(ii) The system (1), augmented by (2), is weakly remediable on $[0,T]$ if and only if

$$\ker(B^\star R^\star) = \ker(R^\star). \tag{13}$$

2.2. Efficient Actuators

Let Ω be an open and bounded subset of $\mathbb{R}^n$, with a sufficiently regular boundary $\Gamma = \partial\Omega$, Ω being the geometrical support of the analysed system (1). We assume that the space $X = L^2(\Omega)$ has an orthonormal basis of eigenfunctions $\{\varphi_{nj}\}$ for $n \geq 1$ and $j = 1, r_n$ of A such that

$$\varphi_{nj} = \lambda_n \varphi_{nj} \quad \text{for} \quad j = 1, \dots, r_n, \quad n \geq 1$$

with $\lambda_n \searrow -\infty$. The semigroup $(S(t))_{t\geq 0}$ generated by A is defined by

$$S(t)z = \sum_{n\geq 1} e^{\lambda_n t} \sum_{j=1}^{r_n} \langle z, \varphi_{nj}\rangle \varphi_{nj}. \tag{14}$$

This is the case if A is a selfadjoint operator on X with compact resolvent $(sI - A)^{-1}$.

In the case of p actuators $(\Omega_i, g_i)_{i=1,p}$ we have $\mathcal{U} = \mathbb{R}^p$, and the operator B is defined by (Curtain and Pritchard, 1978; El Jai and Pritchard, 1988)

$$B: \begin{array}{rcl} \mathbb{R}^p & \longrightarrow & L^2(\Omega), \\ u(t) & \longmapsto & Bu(t) = \sum_{i=1}^{p} g_i u_i(t), \end{array}$$

where $u = (u_1, \dots, u_p)^{\mathrm{tr}} \in L^2(0, T; \mathbb{R}^p)$ and $g_i \in L^2(\Omega_i)$ with $\Omega_i = \mathrm{supp}(g_i) \subset \Omega$ for $i = 1, p$ and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. We have

$$B^\star z = \big(\langle g_1, z\rangle, \dots, \langle g_p, z\rangle\big)^{\mathrm{tr}} \quad \text{for} \quad z \in X',$$

where in the general case M^{tr} is the transpose of M and $\langle\cdot,\cdot\rangle_X \equiv \langle\cdot,\cdot\rangle$ is the inner product on X. If the support of $k \in X$ is $D = \mathrm{supp}(k)$, we have $\langle k, \cdot\rangle_X = \langle k, \cdot\rangle_{L^2(D)}$.

Let us remark that in the case of pointwise actuators, the operator B is unbounded, but the results are analogous by replacing the state space X by a space V such that $V' \subset X \subset V$, with continuous injections (X is identified with its dual).

Definition 3. Actuators $(\Omega_i, g_i)_{i=1,p}$ ensuring the weak remediability of the system (1), augmented by (2), are said to be efficient.

If the output of the system is given by q sensors $(D_i, h_i)_{i=1,q}$ with $h_i \in X$, $D_i = \mathrm{supp}(h_i) \subset \Omega$ for $i = 1, q$ and $D_i \cap D_j = \emptyset$ for $i \neq j$ (Curtain and Pritchard, 1978; El Jai and Pritchard, 1988), the operator C is defined by

$$C: \begin{array}{rcl} L^2(\Omega) & \longrightarrow & \mathbb{R}^q, \\ z & \longmapsto & Cz = \big(\langle h_1, z\rangle, \dots, \langle h_q, z\rangle\big)^{\mathrm{tr}}, \end{array}$$

and

$$C^\star\theta = \sum_{i=1}^{q} \theta_i h_i \quad \text{for} \quad \theta = (\theta_1, \dots, \theta_q)^{\mathrm{tr}} \in \mathbb{R}^q.$$

In the case of actuators $(\Omega_i, g_i)_{i=1,p}$, sensors $(D_i, h_i)_{i=1,q}$ and the s.c.s.g. given by (14), the characterization of efficient actuators is given by the following proposition.

Proposition 3. *Actuators $(\Omega_i, g_i)_{i=1,p}$, are efficient if and only if*

$$\bigcap_{n\geq 1} \ker(M_n G_n^{\mathrm{tr}}) = \{0\} \tag{15}$$

with

$$M_n = \big(\langle g_i, \varphi_{nj}\rangle\big)_{\substack{i=1,p \\ j=1,r_n}}, \quad G_n = \big(\langle h_i, \varphi_{nj}\rangle\big)_{\substack{i=1,q \\ j=1,r_n}}.$$

Let us note that if there exists n_0 such that

$$\mathrm{rank}(M_{n_0} G_{n_0}^{\mathrm{tr}}) = q, \tag{16}$$

then the actuators $(\Omega_i, g_i)_{i=1,p}$ are efficient.

Example 1. Consider the following system:

$$\begin{cases} \dfrac{\partial z}{\partial t}(x,t) = \Delta z(x,t) + g_1(x)u_1(t) + f(x,t) \\ \qquad\qquad \text{in} \quad]0,1[\times]0,T[, \\ z(x,0) = z_0 \quad \text{in} \quad]0,1[, \\ z(x,t) = 0 \quad \text{on} \quad \{0,1\}\times]0,T[, \end{cases}$$

augmented by the output equation

$$y = Cz = \langle h_1, z\rangle$$

with $u_1 \in L^2(0,T;\mathbb{R})$, $g_1 \in L^2(\Omega_1)$, $h_1 \in L^2(D_1)$, $\Omega_1 = \mathrm{supp}(g_1)$, $D_1 = \mathrm{supp}(h_1) \subset \Omega$ and $f \in L^2(0,T;L^2(\Omega))$. The eigenvectors of the Laplacian Δ are defined by

$$\varphi_n(\xi) = \sqrt{2}\sin(n\pi\xi), \quad \forall\, n \geq 1,$$

and the associated eigenvalues are simple and given by

$$\lambda_n = -n^2\pi^2, \quad \forall\, n \geq 1.$$

In the case of one sensor (D, h), with $D = \mathrm{supp}(h) \subset]0,1[$, let n_0 be such that $\langle h, \varphi_{n_0}\rangle \neq 0$. An actuator (Ω_1, g_1) is efficient if $\langle g_1, \varphi_{n_0}\rangle \neq 0$, or

$$\int_{\Omega_1} g_1(\xi)\sin(n_0\pi\xi)\,\mathrm{d}\xi \neq 0.$$

Hence, e.g., if $g_1 = \varphi_{n_0}$, (Ω_1, g_1) is efficient.

In the case of pointwise sensors, the operator C is unbounded. Then the results are the same if the domain $\mathcal{D}(C)$ of the output operator C is contained in X and invariant by the semigroup $(S(t))_{t\geq 0}$.

2.3. Exact Remediability with Minimal Energy

For z_0 in X and $f \in L^2(0,T;X)$, is there an optimal control $u \in L^2(0,T;\,\mathcal{U})$ such that $y_{u,f}(T) = CS(T)z_0$, i.e. a control which minimizes the function $J(v) = \|v\|^2$ on the set $\{v \in L^2(0,T;\,\mathcal{U}) \mid y_{v,f}(T) = CS(T)z_0\}$? This problem can be solved using an extension of the approach H.U.M.

For $\theta \in Y' \equiv Y$, let

$$\|\theta\|_{\mathcal{F}} = \left[\int_0^T \|B^\star S^\star(T-s)C^\star\theta\|^2_{\mathcal{U}'}\,\mathrm{d}s\right]^{\frac{1}{2}}, \tag{17}$$

where $\mathcal{F}$ is a space which will be precised later, $\|\cdot\|_{\mathcal{F}}$ is a semi-norm.

If $\ker(C^\star) = \{0\}$, then the system (1), augmented by (2), is weakly remediable on $[0,T]$ if and only if $\|\cdot\|_{\mathcal{F}}$ is a norm on Y. We suppose that $\|\cdot\|_{\mathcal{F}}$ is a norm. Let $\mathcal{F}$ be the completion of the space Y with respect to the norm $\|\cdot\|_{\mathcal{F}}$. $\mathcal{F}$ is denoted by

$$\mathcal{F} = \overline{Y}^{\|\cdot\|_{\mathcal{F}}}. \tag{18}$$

$(\mathcal{F}, \langle\cdot,\cdot\rangle_{\mathcal{F}})$ is a Hilbert space with the inner product

$$\langle\theta,\sigma\rangle_{\mathcal{F}} = \int_0^T \Big\langle B^\star S^\star(T-s)C^\star\theta, B^\star S^\star(T-s)C^\star\sigma\Big\rangle\,\mathrm{d}s, \quad \forall\,\theta,\sigma \in \mathcal{F}. \tag{19}$$

Y is contained in $\mathcal{F}$ with continuous injection. Let Λ be the operator defined by

$$\Lambda = CHH^\star C^\star : \begin{array}{rcl} Y & \longrightarrow & Y, \\ \theta & \longmapsto & \Lambda\theta = \displaystyle\int_0^T CS(T-s) \\ & & \times BB^\star S^\star(T-s)C^\star\theta\,\mathrm{d}s, \\ & & \text{for } \theta \in Y' \equiv Y. \end{array} \tag{20}$$

We have

$$\langle\Lambda\theta,\sigma\rangle_Y = \langle\theta,\sigma\rangle_{\mathcal{F}}, \quad \forall\,\theta,\sigma \in Y. \tag{21}$$

Let $\mathcal{F}'$ be the dual space of $\mathcal{F}$. Λ has a unique extension as an isomorphism Λ: $\mathcal{F} \longrightarrow \mathcal{F}'$ such that

$$\langle\Lambda\theta,\sigma\rangle_Y = \langle\theta,\sigma\rangle_{\mathcal{F}}, \quad \forall\,\theta,\sigma \in \mathcal{F} \tag{22}$$

and

$$\|\Lambda\theta\|_{\mathcal{F}'} = \|\theta\|_{\mathcal{F}}, \quad \forall\,\theta \in \mathcal{F}. \tag{23}$$

We have the following result.

Proposition 4. *If the observation $Rf \in \mathcal{F}'$, then there exists a unique $\theta_f \in \mathcal{F}$ such that $\Lambda\theta_f = -Rf$, and the control*

$$u_{\theta_f}(t) = B^\star S^\star(T-t)C^\star\theta_f \tag{24}$$

satisfies

$$y_{u_{\theta_f},f}(T) - CS(T)z_0 = 0. \tag{25}$$

Moreover, u_{θ_f} is optimal and

$$\|u_{\theta_f}\|_{L^2(0,T;\,\mathcal{U})} = \|\theta_f\|_{\mathcal{F}}. \tag{26}$$

3. Notion of Enlarged Exact Remediability

3.1. Enlarged Exact Remediability

In this part, we introduce enlarged exact remediability, which is a more general notion than exact remediability, and consists in studying the possibility of taking an observation at the final time, in a region of tolerance $\mathcal{C}$, where $\mathcal{C}$ is a given closed and convex subset of Y. We examine the case where $\mathcal{C}$ is a closed ball $\mathcal{B}(CS(T)z_0,\varepsilon)$, and then we introduce and characterize the notion of $\mathcal{C}$-efficient and ε-efficient actuators.

Definition 4. A disturbance $f \in L^2(0,T;X)$ is called $\mathcal{C}$-remediable on $[0,T]$ if there exists a control $u \in L^2(0,T;\mathcal{U})$ such that

$$y_{u,f}(T) \in \mathcal{C}. \tag{27}$$

It is easy to show the following characterization result.

Proposition 5. *The statments below are equivalent:*
(i) f is $\mathcal{C}$-remediable on $[0,T]$,
(ii)

$$\mathrm{Im}(CH) \cap \mathcal{C}_1 \neq \emptyset, \tag{28}$$

where $\mathcal{C}_1 = \mathcal{C} - CS(T)z_0 - Rf$.

Let us note that if $\mathcal{C} = \{CS(T)z_0\}$, we have a problem of exact remediability, and if $CS(T)z_0$ is an interior point of $\mathcal{C}$, then weak remediability implies $\mathcal{C}$-remediability, but the converse is not true. The following section is focused on the case where $\mathcal{C}$ is a closed ball.

3.2. Case of $\mathcal{C}=\mathcal{B}(CS(T)z_0,\varepsilon)$

Definition 5. A disturbance f is $\mathcal{B}(CS(T)z_0,\varepsilon)$-remediable or ε-remediable on $[0,T]$ if there exists a control $u \in L^2(0,T;\mathcal{U})$ such that

$$\|CHu + Rf\| \leq \varepsilon. \tag{29}$$

If F is a closed subspace of Y and P_F is the orthogonal projection on F, we have the following result.

Proposition 6. *The statments below are equivalent:*
(i) f *is* ε*-remediable on* $[0,T]$,
(ii)

$$\operatorname{Im}(CH) \cap \mathcal{B}(Rf,\varepsilon) \neq \emptyset, \tag{30}$$

(iii)

$$\|P_{\ker(B^\star R^\star)}\ (Rf)\| < \varepsilon. \tag{31}$$

Proof. The equivalence between (i) and (ii) follows from Definition 5. In turn, the equivalence between (ii) and (iii) results from the fact that

$$(ii) \Longleftrightarrow d\big(Rf, \overline{\operatorname{Im}(CH)}\big) < \varepsilon$$

as

$$\overline{\operatorname{Im}(CH)} \oplus \ker(H^\star C^\star) = Y.$$

Since

$$Rf = Rf_1 + Rf_2 \ \text{ with } \ Rf_1 \in \overline{\operatorname{Im}(CH)}$$

and

$$Rf_2 = P_{\ker(H^\star C^\star)}\ (Rf) \in \ker(H^\star C^\star),$$

for $w \in \overline{\operatorname{Im}(CH)}$ we have

$$d(Rf,w)^2 = \|w - Rf\|^2 = \|w - Rf_1\|^2 + \|Rf_2\|^2.$$

As $\langle w - Rf_1, Rf_2\rangle = 0$, we get

$$\begin{aligned} & d\big(Rf, \overline{\operatorname{Im}(CH)}\big)^2 \\ &= \inf_{w\in\overline{\operatorname{Im}(CH)}} d(Rf,w)^2 \\ &= \|Rf_2\|^2 + \inf_{w\in\overline{\operatorname{Im}(CH)}} \|w - Rf_1\|^2 \\ &= \|Rf_2\|^2 = \|P_{\ker(H^\star C^\star)}\ (Rf)\|^2 \end{aligned}$$

because $Rf_1 \in \overline{\operatorname{Im}(CH)}$. Using $H^\star C^\star = B^\star R^\star$, we have the desired conclusion. ■

Let us remark that if f is ε-remediable, then f is ε'-remediable for any $\varepsilon' \geq \varepsilon$. The converse is not true. Indeed, if $\Omega_1 = D_1 =]0,1[$, $h = \varphi_{n_1}$ and $g = \varphi_{n_2}$ with $n_1 \neq n_2$, then $\ker(B^\star R^\star) = \mathbb{R}$, and the result is true for any f such that $\varepsilon < |Rf| < \varepsilon'$.

3.3. ε-Efficient Actuators

In this section, we introduce and characterize $\mathcal{C}$-efficient actuators, essentially in the case where $\mathcal{C} = \mathcal{B}(CS(T)z_0,\varepsilon)$.

Definition 6. For a fixed disturbance $f \in L^2(0,T;X)$, actuators $(\Omega_i, g_i)_{i=1,p}$ ensuring the $\mathcal{C}$-remediability of f are said to be $\mathcal{C}$-efficient. If $\mathcal{C} = \mathcal{B}(CS(T)z_0,\varepsilon)$, these actuators are called ε-efficient.

In the case of p actuators $(\Omega_i, g_i)_{i=1,p}$ and an output given by q sensors $(D_i, h_i)_{i=1,q}$, the characterization of ε-efficient actuators is given by the following result.

Proposition 7. *Actuators* $(\Omega_i, g_i)_{i=1,p}$ *are* ε*-efficient for a fixed* $f \in L^2(0,T;X)$ *if and only if*

$$\|P_F\ (Rf)\| < \varepsilon, \tag{32}$$

where $F = \bigcap_{n\geq 1} \ker(M_n G_n^{\mathrm{tr}})$.

Proof. The result follows from Proposition 6 and the fact that $\ker(B^\star R^\star) = F$. ■

If the system (1), augmented by (2), is weakly remediable on $[0,T]$, then any disturbance $f \in L^2(0,T;X)$ is ε-remediable on $[0,T]$ for any $\varepsilon > 0$. Then efficient actuators are ε-efficient for every $\varepsilon > 0$ and $f \in L^2(0,T;X)$. But actuators can be ε-efficient for a given $f \in L^2(0,T;X)$ without being efficient. This is illustrated by the following example.

Example 2. As in Example 1, we consider the system

$$\begin{cases} \dfrac{\partial z}{\partial t}(x,t) &= \Delta z(x,t) + g_1(x)u_1(t) + f(x,t) \\ & \qquad \text{in } \ ]0,1[\times]0,T[, \\ z(x,0) &= z_0 \quad \text{in } \ ]0,1[, \\ z(x,t) &= 0 \quad\ \text{on } \ \{0,1\}\times]0,T[, \end{cases}$$

augmented by the output equation

$$y = Cz = \langle h_1, z\rangle$$

with $u_1 \in L^2(0,T;\mathbb{R})$, $g_1 \in L^2(\Omega_1)$, $h_1 \in L^2(D_1)$, $\Omega_1 = \operatorname{supp}(g_1)$, $D_1 = \operatorname{supp}(h_1) \subset \Omega$ and $f \in L^2(0,T;L^2(\Omega))$. Then, with the same notation and for $\Omega_1 = D_1 =]0,1[$, $h = \varphi_{n_1}$ and $g = \varphi_{n_2}$ with $n_1 \neq n_2$, the actuator (Ω_1, g_1) is not efficient, but for f defined by $f(\cdot,s) = e^{(-1-\alpha)\lambda_{n_1}(T-s)}\varphi_{n_1}$, (Ω_1, g_1) is ε-efficient for a convenient α.

In the following proposition, we show that there is equivalence between weak remediablity and ε-remediability for any $f \in L^2(0,T;X)$.

Proposition 8. *The system (1), augmented by (2), is weakly remediable on* $[0,T]$ *if and only if there exists* $\varepsilon > 0$ *such that any* $f \in L^2(0,T;X)$ *is* ε*-remediable on* $[0,T]$.

Proof. Let $\varepsilon > 0$ be such that (1), augmented by (2), is ε-remediable on $[0,T]$ for any $f \in L^2(0,T;X)$. Then

$$\|P_{\ker(B^\star R^\star)}\ (Rf)\| < \varepsilon, \quad \forall\, f \in L^2(0,T;X).$$

Consequently, $P_{\ker(B^\star R^\star)}(\mathrm{Im}\, R) = \{0\}$, since $\overline{\mathrm{Im}(R)} \oplus \ker(R^\star) = Y$. Then, necessarily, $\ker(B^\star R^\star) \subset \ker(R^\star)$, and hence we obtain weak remediability.

The converse follows immediatly from the definition of weak remediability. ■

4. Enlarged Exact Remediability with Minimal Energy

Let $\mathcal{C}$ be a nonempty closed convex subset of Y, $z_0 \in X$ and $f \in L^2(0,T;X)$. We consider the following problem of enlarged exact remediability with minimal energy:

$$(P) \begin{cases} \min\ J(u) \quad \text{with} \quad J(u) = \|u\|^2 \\ \text{subject to} \ \ y_{u,f}(T) \in \mathcal{C}. \end{cases} \tag{33}$$

If the disturbance f is $\mathcal{C}$-remediable, the problem (P) is well defined and has a unique solution in the set of admissible controls defined by

$$\mathcal{U}_{ad} = \big\{u \in L^2(0,T;\mathcal{U}) \ \mid \ y_{u,f}(T) \in \mathcal{C}\big\}.$$

The solution to (P), denoted by $v^\star$, is characterized by

$$J'(v^\star)(v - v^\star) \geq 0, \quad \forall\, v \in \mathcal{U}_{ad}, \tag{34}$$

i.e.

$$\langle v^\star, v - v^\star\rangle \geq 0, \quad \forall\, v \in \mathcal{U}_{ad}.$$

Let us note that (P) is a generalization of the following exact remediability problem:

$$(P_1) \begin{cases} \min\ \|u\|^2 \\ \text{subject to} \ \ y_{u,f}(T) = CS(T)z_0, \end{cases}$$

since in this case we have $\mathcal{C} = \{CS(T)z_0\}$. If $u^\star$ is the solution to (P_1), we have

$$\|v^\star\| \leq \|u^\star\|.$$

Hence the optimal cost of (P) is reduced with respect to Problem (P_1).

Problem (P) is also a generalization of ε-remediability one, since it is sufficient to consider $\mathcal{C} = \mathcal{B}(CS(T)z_0, \varepsilon)$. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two nonempty, closed and convex subsets of Y such that $\mathcal{C}_1 \subset \mathcal{C}_2$, then $\mathcal{C}_1$-remediability implies $\mathcal{C}_2$-remediability, and the cost is decreasing when $\mathcal{C}$ is increasing.

Next, we will solve Problem (P) using an extension of the H.U.M. approach and a penalization method (Bel Fekih, 1990; Bergounioux, 1994). First, let us show preliminary results, which will be used to demonstrate the main result of this section.

We consider the following criterion:

$$J_\alpha(y,v) = \frac{1}{\alpha}\|y_{v,f}(T) - y\|^2 + \|v\|^2 \tag{35}$$

with $y \in Y$, $v \in L^2(0,T;\mathcal{U})$ and $\alpha > 0$, and the minimization problem

$$(P_\alpha) \begin{cases} \min\ J_\alpha(y,v), \\ (y,v) \in \mathcal{C} \times L^2(0,T;\mathcal{U}). \end{cases} \tag{36}$$

We have the following existence result.

Lemma 1. *(P_α) admits a solution $(y_\alpha, v_\alpha) \in \mathcal{C} \times L^2(0,T;\mathcal{U})$ characterized by*

$$\langle d_\alpha, y - y_\alpha\rangle \geq 0, \quad \forall\, y \in \mathcal{C}, \tag{37}$$

$$v_\alpha(t) = B^\star p_\alpha(t), \quad 0 < t < T, \tag{38}$$

where d_α is given by

$$d_\alpha = \frac{1}{\alpha}\big[y_\alpha - y_{v_\alpha,f}(T)\big] \in Y \tag{39}$$

and p_α is the solution of the adjoint equation

$$\begin{cases} -p'_\alpha(t) \ = \ A^\star p_\alpha(t), \quad 0 < t < T, \\ \ \ p_\alpha(T) \ = \ C^\star d_\alpha. \end{cases} \tag{40}$$

Proof. Let $(y_\alpha^{(k)}, v_\alpha^{(k)})_{k\geq 0}$ be a minimizing sequence, so that

$$J_\alpha(y_\alpha^{(k)}, v_\alpha^{(k)}) \searrow \inf_{(y,v)\in\mathcal{C}\times L^2(0,T;\mathcal{U})} J_\alpha(y,v) \text{ as } k \nearrow +\infty.$$

The sequence $(J_\alpha(y_\alpha^{(k)}, v_\alpha^{(k)}))_{k\geq 0}$ is bounded because it is convergent and

$$\|v_\alpha^{(k)}\|^2 \leq J_\alpha(y_\alpha^{(k)}, v_\alpha^{(k)}) \leq C_1, \quad \forall\, k \geq 0.$$

Then $(v_\alpha^{(k)})_{k\geq 0}$ is bounded in $L^2(0,T;\mathcal{U})$. Since the mapping $v \in L^2(0,T;\mathcal{U}) \longrightarrow y_{v,f}(T) - y_{0,f}(T)$ is linear and continuous, there exists a constant $C_2 > 0$ such that

$$\|y_{v,f}(T) - y_{0,f}(T)\| \leq C_2\|v\|, \quad \forall\, v \in L^2(0,T;\mathcal{U}).$$

Hence, $(y_{v_\alpha^{(k)},f}(T))_{k\geq 0}$ is bounded. Since

$$\|y_{v_\alpha^{(k)},f}(T) - y_\alpha^k\|^2 \leq \alpha J_\alpha\big(y_\alpha^{(k)}, v_\alpha^{(k)}\big) \leq \alpha C_1, \quad \forall\, k \geq 0,$$

$(y_\alpha^k)_{k\geq 0}$ is bounded in Y. Hence there exists a subsequence which converges to an element (y_α, v_α). Using the continuity of J_α on $Y \times L^2(0,T;\mathcal{U})$, we have

$$\begin{aligned} J_\alpha(y_\alpha, v_\alpha) &\leq \liminf_{k\to+\infty} J_\alpha\big(y_\alpha^{(k)}, v_\alpha^{(k)}\big) \\ &= \inf_{(y,v)\in\mathcal{C}\times L^2(0,T;\mathcal{U})} J_\alpha(y,v). \end{aligned}$$

$\mathcal{C}$ is closed, $(y_\alpha, v_\alpha) \in \mathcal{C} \times L^2(0,T;\mathcal{U})$ and hence

$$J_\alpha(y_\alpha, v_\alpha) \geq \inf_{(y,v)\in\mathcal{C}\times L^2(0,T;\mathcal{U})} J_\alpha(y,v).$$

Consequently, (y_α, v_α) is a solution to (P_α).

On the other hand, (y_α, v_α) satisfies the following necessary condition:

$$\frac{1}{2}J'_\alpha(y_\alpha, v_\alpha)\cdot\big((y,v)-(y_\alpha,v_\alpha)\big)\geq 0,\qquad \forall\,(y,v)\in\mathcal{C}\times L^2(0,T;\mathcal{U}).$$

We have

$$\begin{aligned}J'_\alpha(y_\alpha,v_\alpha)\cdot(y,v)&=\frac{2}{\alpha}\Big[\big\langle y_{v_\alpha,f}(T)-y_\alpha, y_{v,f}(T)-CS(T)z_0-Rf\big\rangle\\&\quad-\big\langle y_{v_\alpha,f}(T)-y_\alpha, y\big\rangle\Big]+2\langle v_\alpha, v\rangle.\end{aligned}$$

Then

$$\begin{aligned}J'_\alpha(y_\alpha,v_\alpha)(y-y_\alpha,v-v_\alpha)&=\frac{2}{\alpha}\Big[\big\langle y_{v_\alpha,f}(T)-y_\alpha, y_{v,f}(T)-y_{v_\alpha,f}(T)\big\rangle\\&\quad-\big\langle y_{v_\alpha,f}(T)-y_\alpha, y-y_\alpha\big\rangle\Big]\\&\quad+2\langle v_\alpha, v-v_\alpha\rangle.\end{aligned}$$

The necessary condition can be written as

$$\begin{aligned}&\frac{1}{\alpha}\big\langle y_{v_\alpha,f}(T)-y_\alpha, y_{v,f}(T)-y_{v_\alpha,f}(T)\big\rangle\\&-\frac{1}{\alpha}\big\langle y_{v_\alpha,f}(T)-y_\alpha, y-y_\alpha\big\rangle+\langle v_\alpha, v-v_\alpha\rangle\geq 0,\\&\qquad\forall\,(y,v)\in\mathcal{C}\times L^2(0,T;\mathcal{U}).\end{aligned}\tag{41}$$

By replacing v and y in (41) by v_α and y_α, respectively, we have the following inequalities, where the element d_α given by (39) appears:

$$\begin{aligned}&\frac{1}{\alpha}\big\langle y_\alpha-y_{v_\alpha,f}(T), y-y_\alpha\big\rangle\geq 0,\quad\forall\,y\in\mathcal{C},\\&-\frac{1}{\alpha}\big\langle y_\alpha-y_{v_\alpha,f}(T), y_{v,f}(T)-y_{v_\alpha,f}(T)\big\rangle\\&\quad+\langle v_\alpha, v-v_\alpha\rangle\geq 0,\quad\forall\,v\in L^2(0,T;\mathcal{U}).\end{aligned}\tag{42}$$

Consider the adjoint equation (40). By integration by parts we have

$$\begin{aligned}&\big\langle z'_{v,f}(\cdot)-Az_{v,f}(\cdot), p_\alpha(\cdot)\big\rangle-\big\langle z_{v,f}(\cdot), -p'_\alpha(\cdot)-A^\star p_\alpha(\cdot)\big\rangle\\&\quad=\big\langle z_{v,f}(T), p_\alpha(T)\big\rangle-\big\langle z_{v,f}(0), p_\alpha(0)\big\rangle\end{aligned}$$

or, equivalently,

$$\begin{aligned}&\big\langle Bv(\cdot)+f(\cdot), p_\alpha(\cdot)\big\rangle\\&\quad=\big\langle z_{v,f}(T), p_\alpha(T)\big\rangle-\big\langle z_0, p_\alpha(0)\big\rangle.\end{aligned}\tag{43}$$

With $v=v_\alpha$ and by considering the difference with (43), we have

$$\big\langle B\big(v(\cdot)-v_\alpha(\cdot)\big), p_\alpha(\cdot)\big\rangle=\big\langle z_{v,f}(T)-z_{v_\alpha,f}(T), p_\alpha(T)\big\rangle,\qquad\forall\,v\in L^2(0,T;\mathcal{U}).$$

Then

$$\begin{aligned}&\big\langle B\big(v(\cdot)-v_\alpha(\cdot)\big), p_\alpha(\cdot)\big\rangle\\&\quad=\frac{1}{\alpha}\big\langle y_{v,f}(T)-y_{v_\alpha,f}(T), y_\alpha-y_{v_\alpha,f}(T)\big\rangle.\end{aligned}\tag{44}$$

Using this relation in (42), we obtain

$$\begin{cases}\langle d_\alpha, y-y_\alpha\rangle\geq 0,\quad\forall\,y\in\mathcal{C},\\[4pt] -\big\langle v-v_\alpha, B^\star p_\alpha(\cdot)\big\rangle+\langle v_\alpha, v-v_\alpha\rangle\geq 0,\\ \qquad\qquad\forall\,v\in L^2(0,T;\mathcal{U}).\end{cases}$$

The last inequality implies $v_\alpha(t)=B^\star p_\alpha(t)$. ■

Lemma 2. *From the sequence $(y_\alpha, v_\alpha, d_\alpha)_{\alpha>0}$, we can extract a subsequence which converges to an element $(y^\star, v^\star, d^\star)$ characterized as follows:*

(i) $y^\star=y_{v^\star,f}(T)$,

(ii) $v^\star$ *is the solution to* (P),

(iii) the sequence $(J_\alpha(y_\alpha,v_\alpha))_{\alpha>0}$ is bounded and decreasing with a limit $\|v^\star\|^2$ and $(v_\alpha)_{\alpha>0}$ converges strongly to $v^\star$ in $L^2(0,T;\mathcal{U})$,

(iv) the control $v^\star$ is given by

$$v^\star(t)=B^\star p(t),\quad 0<t<T,\tag{45}$$

where $p(t)$ is the solution to the following adjoint equation:

$$\begin{cases}-p'(t)&=&A^\star p(t),\quad 0<t<T,\\ p(T)&=&C^\star d^\star,\end{cases}\tag{46}$$

(v) the element $d^\star$ is characterized by

$$\big\langle d^\star, y-y_{v^\star,f}(T)\big\rangle\geq 0,\qquad\forall\,y\in\mathcal{C}\cap\big(CS(T)z_0+Rf+\mathcal{F}'\big).\tag{47}$$

Proof. First show that the sequence $(y_\alpha, v_\alpha, d_\alpha)_{\alpha>0}$ is bounded. Indeed, under the $\mathcal{C}$-remediability hypothesis, there exists $\hat{u}\in L^2(0,T;\mathcal{U})$ such that $y_{\hat{u},f}(T)\in\mathcal{C}$. By noting that $\hat{y}=y_{\hat{u},f}(T)$, we have $(\hat{y},\hat{u})\in\mathcal{C}\times L^2(0,T;\mathcal{U})$, and hence

$$J_\alpha(y_\alpha,v_\alpha)\leq J_\alpha(\hat{y},\hat{u})=\|\hat{u}\|^2,\quad\forall\,\alpha>0.$$

Then

$$\|y_{v_\alpha,f}(T)-y_\alpha\|^2 \le \alpha J_\alpha(y_\alpha,v_\alpha)$$
$$\le \alpha J_\alpha(\hat{y},\hat{u}) = \alpha\|\hat{u}\|^2, \quad \forall\,\alpha>0.$$

Consequently, $\lim_{\alpha\to 0}\|y_{v_\alpha,f}(T)-y_\alpha\| = 0$ and $(J_\alpha(y_\alpha,v_\alpha))_{\alpha>0}$ is bounded.

If $\beta\le\alpha$, then $J_\alpha(y,v)\le J_\beta(y,v),\quad \forall\,(y,v)\in\mathcal{C}\times L^2(0,T;\mathcal{U})$, and hence

$$J_\alpha(y_\alpha,v_\alpha)\le J_\alpha(y_\beta,v_\beta)\le J_\beta(y_\beta,v_\beta). \tag{48}$$

$(v_\alpha)_{\alpha>0}$ is then bounded, since

$$\|v_\alpha\|^2\le J_\alpha(y_\alpha,v_\alpha)\le J_\alpha(\hat{y},\hat{u})=\|\hat{u}\|^2.$$

The map $v\in L^2(0,T;\mathcal{U})\to y_{v,f}(T)\in Y$ is affine and continuous. We have

$$\|y_{v,f}(T)-CS(T)z_0-Rf\|\le K\|v\|,$$
$$\forall\,v\in L^2(0,T;\mathcal{U}),\quad K>0. \tag{49}$$

Then $(y_{v_\alpha,f}(T))_{\alpha>0}$ is bounded in Y. This shows that $y_\alpha=(y_\alpha-y_{v_\alpha,f}(T))+y_{v_\alpha,f}(T)$ is bounded in Y.

In the general case, $(d_\alpha)_{\alpha>0}$ is not bounded, but it is with respect to the norm $\|.\|_{\mathcal{F}}$ because $p_\alpha(t)=S^\star(T-t)C^\star d_\alpha$, and hence

$$\|v_\alpha\|^2=\|B^\star S^\star(T-\cdot)C^\star d_\alpha\|^2=\|d_\alpha\|^2_{\mathcal{F}}.$$

Accordingly, we get that d_α is bounded in $\mathcal{F}$. Finally, $(y_\alpha,v_\alpha,d_\alpha)_{\alpha>0}$ is bounded in $Y\times L^2(0,T;\mathcal{U})\times\mathcal{F}$.

Now, let us consider a subsequence of $(y_\alpha,v_\alpha,d_\alpha)_{\alpha>0}$, also denoted by $(y_\alpha,v_\alpha,d_\alpha)_{\alpha>0}$ and converging to an element $(y^\star,v^\star,d^\star)$, which will be characterized hereafter.

(i) The weak convergence of $(v_\alpha)_{\alpha>0}$ to $v^\star$ implies the convergence of $(y_{v_\alpha,f}(T))_{\alpha>0}$ to $y_{v^\star,f}(T)$ (using (49)). Then

$$y^\star=\lim_{\alpha\to0} y_\alpha=\lim_{\alpha\to0}(y_\alpha-y_{v_\alpha,f}(T))+\lim_{\alpha\to0} y_{v_\alpha,f}(T)$$
$$=y_{v^\star,f}(T).$$

(ii) Since $\mathcal{C}$ is closed, $y_{v^\star,f}(T)=\lim_{\alpha\to0} y_\alpha\in\mathcal{C}$. On the other hand, if $v\in L^2(0,T;\mathcal{U})$, then

$$\|v_\alpha\|^2\le J_\alpha(y_\alpha,v_\alpha)\le J_\alpha\big(y_{v,f}(T),v\big)=\|v\|^2. \tag{50}$$

As $\alpha\to0$, we have

$$\|v^\star\|^2\le\liminf_{\alpha\to0}\|v_\alpha\|^2\le\|v\|^2.$$

Then $v^\star$ is a solution to Problem (P).

(iii) Using (48), it is easy to see that the sequence $(J_\alpha(y_\alpha,v_\alpha))_{\alpha>0}$ is bounded and decreasing, so it converges. Using (50) with $v=v^\star$, we have

$$\|v_\alpha\|^2\le\liminf_{\alpha\to0}\|v_\alpha\|^2\le\lim_{\alpha\to0}J_\alpha(y_\alpha,v_\alpha)\le\|v^\star\|^2.$$

Then $\lim_{\alpha\to0}J_\alpha(y_\alpha,v_\alpha)=\|v^\star\|^2$. On the other hand, the same inequality implies

$$\|v_\alpha\|^2\le\liminf_{\alpha\to0}\|v_\alpha\|^2\le\limsup_{\alpha\to0}\|v_\alpha\|^2\le\|v^\star\|^2.$$

Then $\|v_\alpha\|^2$ converges to $\|v^\star\|^2$, which implies, using the weak convergence of $(v_\alpha)_{\alpha>0}$ to $v^\star$, that $(v_\alpha)_{\alpha>0}$ converges strongly in $L^2(0,T;\mathcal{U})$.

(iv) Using (38), we have

$$v_\alpha(t)=B^\star p_\alpha(t)=B^\star S^\star(T-\cdot)C^\star d_\alpha.$$

Then for $w\in L^2(0,T;\mathcal{U})$, we get

$$\langle v_\alpha,w\rangle=\Big\langle d_\alpha,\int_0^T CS(T-t)Bw(t)\,\mathrm{d}t\Big\rangle$$
$$=\big\langle d_\alpha,y_{w,f}(T)-CS(T)z_0-Rf\big\rangle.$$

Since $y_{w,f}(T)-CS(T)z_0-Rf\in\mathcal{F}$, the weak convergence of v_α to $v^\star$ in $L^2(0,T;\mathcal{U})$ and that of d_α to $d^\star$ in $\mathcal{F}$ imply

$$\langle v^\star,w\rangle=\big\langle d^\star,y_{w,f}(T)-CS(T)z_0-Rf\big\rangle$$
$$=\Big\langle d^\star,\int_0^T CS(T-t)Bw(t)\,\mathrm{d}t\Big\rangle$$
$$=\int_0^T\big\langle B^\star S^\star(T-t)C^\star d^\star,w(t)\big\rangle\,\mathrm{d}t,$$

and hence

$$v^\star(t)=B^\star S^\star(T-t)C^\star d^\star=B^\star p(t).$$

(v) Inequality (37) can be written as

$$\big\langle d_\alpha,y_\alpha-CS(T)z_0-Rf\big\rangle$$
$$\le\big\langle d_\alpha,y-CS(T)z_0-Rf\big\rangle,\quad\forall\,y\in\mathcal{C}, \tag{51}$$

and, using (39), we have

$$J_\alpha(y_\alpha,v_\alpha)=\big\langle d_\alpha,y_\alpha-y_{v_\alpha,f}(T)\big\rangle+\|v_\alpha\|^2.$$

Then

$$\langle d_\alpha,y_\alpha\rangle=\big\langle d_\alpha,y_{v_\alpha,f}(T)\big\rangle-\|v_\alpha\|^2+J_\alpha(y_\alpha,v_\alpha). \tag{52}$$

But

$$\left\langle d_\alpha, y_{v_\alpha,f}(T)\right\rangle = \left\langle d_\alpha, CS(T)z_0 + Rf\right\rangle + \left\langle d_\alpha, \int_0^T CS(T-t)Bv_\alpha(t)\,\mathrm{d}t\right\rangle$$
$$= \left\langle d_\alpha, CS(T)z_0 + Rf\right\rangle + \int_0^T \left\langle B^\star S^\star(T-t)C^\star d_\alpha, v_\alpha(t)\right\rangle \mathrm{d}t$$
$$= \left\langle d_\alpha, CS(T)z_0 + Rf\right\rangle + \|v_\alpha\|^2.$$

By using (52) this gives

$$\langle d_\alpha, y_\alpha\rangle = \left\langle d_\alpha, CS(T)z_0 + Rf\right\rangle + J_\alpha(y_\alpha, v_\alpha).$$

Then

$$\lim_{\alpha\to 0} \left\langle d_\alpha, y_\alpha - CS(T)z_0 - Rf\right\rangle = \lim_{\alpha\to 0} J_\alpha(y_\alpha, v_\alpha) = \|v^\star\|^2$$
$$= \int_0^T \left\langle B^\star S^\star(T-t)C^\star d^\star, v^\star(t)\right\rangle \mathrm{d}t$$
$$= \left\langle d^\star, \int_0^T CS(T-t)Bv^\star(t)\,\mathrm{d}t\right\rangle$$
$$= \left\langle d^\star, y_{v^\star,f}(T) - CS(T)z_0 - Rf\right\rangle.$$

For $y - CS(T)z_0 - Rf \in \mathcal{F}'$ and as $\alpha \to 0$ in (51), we obtain

$$\left\langle d^\star, y_{v^\star,f}(T) - CS(T)z_0 - Rf\right\rangle \le \left\langle d^\star, y - CS(T)z_0 - Rf\right\rangle, \quad \forall\, y \in \mathcal{C} \cap \left(CS(T)z_0 + Rf + \mathcal{F}'\right)$$

or, equivalently,

$$\left\langle d^\star, y - y_{v^\star,f}(T)\right\rangle \ge 0, \quad \forall\, y \in \mathcal{C} \cap \left(CS(T)z_0 + Rf + \mathcal{F}'\right).$$

■

In the following result, which is a generalization of Proposition 4, we give a solution to Problem (P).

Proposition 9. *If $\mathcal{C}$ is a nonempty, closed and convex set of Y, and*

$$\mathcal{C} \cap \left(CS(T)z_0 + Rf + \mathcal{F}'\right) \neq \emptyset, \tag{53}$$

then

(i) there exists a unique $\theta_f \in \mathcal{F}$ such that

$$\Lambda\theta_f + CS(T)z_0 + Rf \in \mathcal{C}$$

and

$$\left\langle \theta_f, y - \Lambda\theta_f - CS(T)z_0 - Rf\right\rangle \ge 0, \quad \forall\, y \in \mathcal{C} \cap \left(CS(T)z_0 + Rf + \mathcal{F}'\right), \tag{54}$$

(ii) the control

$$u_{\theta_f}(t) = B^\star S^\star(T-t)C^\star\theta_f, \quad 0 < t < T \tag{55}$$

is the unique solution to (P). Moreover, u_{θ_f} is optimal and

$$\|u_{\theta_f}\|^2 = \|\theta_f\|^2_{\mathcal{F}} = \left\langle y_{u_{\theta_f},f}(T) - CS(T)z_0 - Rf, \theta_f\right\rangle. \tag{56}$$

Proof. Let $\theta_f = d^\star$, $u_{\theta_f} = v^\star$ and $y_{v^\star,f}(T) = \Lambda\theta_f + CS(T)z_0 + Rf$. Then the existence of θ_f and u_{θ_f} follows from Lemma 2.

For the unicity, let θ_f and $\sigma_f \in \mathcal{F}$ be such that

$$\left\langle \theta_f, y - \Lambda\theta_f - CS(T)z_0 - Rf\right\rangle \ge 0,$$
$$\left\langle \sigma_f, y - \Lambda\sigma_f - CS(T)z_0 - Rf\right\rangle \ge 0, \quad \forall\, y \in \mathcal{C} \cap \left(CS(T)z_0 + Rf + \mathcal{F}'\right).$$

Then, for $y = \Lambda\theta_f + CS(T)z_0 + Rf \in \mathcal{C}$, from the second inequality we deduce that

$$\langle \sigma_f, \Lambda\theta_f - \Lambda\sigma_f\rangle \ge 0,$$

and using the first inequality with $y = \Lambda\sigma_f + CS(T)z_0 + Rf \in \mathcal{C}$, we have

$$\langle \theta_f, \Lambda\sigma_f - \Lambda\theta_f\rangle \ge 0.$$

Then

$$\langle \sigma_f, \Lambda\theta_f - \Lambda\sigma_f\rangle + \langle \theta_f, \Lambda\sigma_f - \Lambda\theta_f\rangle = -\left\langle \theta_f - \sigma_f, \Lambda(\theta_f - \sigma_f)\right\rangle = -\|\theta_f - \sigma_f\|^2_{\mathcal{F}} \ge 0,$$

and hence $\theta_f = \sigma_f$. ■

In the next section, we examine some particular situations concerning the choice of $\mathcal{C}$.

5. Particular Cases

5.1. Case of Classic Exact Remediability

For $\mathcal{C} = \{CS(T)z_0\}$ and $Rf \in \mathcal{F}'$, the inequality (54) is trivial. Therefore, in order to have θ_f, it is sufficient to solve the equation $\Lambda\theta_f = -Rf$, and hence the solution to the optimal control is given by $u_{\theta_f}(t) = B^\star S^\star(T-t)C^\star\theta_f$. We then obtain the solution given by Proposition 4 in the case of the exact remediability problem.

5.2. Case of Linear Constraints

In this part, we consider the case of linear constraints, which is frequent in optimization theory. More precisely, without loss of generality, we consider the case where $\mathcal{C}$ is a closed subspace of Y. Since $y_{u_{\theta_f},f}(T) = CS(T)z_0 + Rf + \Lambda\theta_f$, the inequality (54) becomes

$$\big\langle \theta_f, y - y_{u_{\theta_f},f}(T)\big\rangle \geq 0, \qquad \forall\, y \in \mathcal{C} \cap \big(CS(T)z_0 + Rf + \mathcal{F}'\big)$$

or, equivalently,

$$\big\langle \theta_f, y_{u_{\theta_f},f}(T)\big\rangle \leq \langle \theta_f, y\rangle, \qquad \forall y \in \mathcal{C} \cap \big(CS(T)z_0 + Rf + \mathcal{F}'\big).$$

If $y = y_{u,f}(T)$ with $u \in L^2(0,T;\mathcal{U})$, then y describes the whole set $CS(T)z_0 + Rf + \mathcal{F}'$ when u describes $L^2(0,T;\mathcal{U})$, and hence

$$\big\langle \theta_f, y_{u_{\theta_f},f}(T)\big\rangle \leq \big\langle \theta_f, y_{u,f}(T)\big\rangle, \tag{57}$$

for all u such that $y_{u,f}(T) \in \mathcal{C}$.

Let

$$\mathcal{U}_\mathcal{C} = \big\{u \in L^2(0,T;\mathcal{U}) \ \mid\ y_{u,f}(T) \in \mathcal{C}\big\}$$

and

$$\mathcal{L} : \begin{array}{rcl} L^2(0,T;\mathcal{U}) & \longrightarrow & Y, \\ u & \longmapsto & \mathcal{L}u = CHu. \end{array}$$

For $w_0 \in L^2(0,T;\mathcal{U})$ such that $\mathcal{L}w_0 = CS(T)z_0 + Rf$, we have

$$\mathcal{U}_\mathcal{C} = -w_0 + \mathcal{L}^{-1}(\mathcal{C}),$$

where $\mathcal{L}^{-1}(\mathcal{C}) = \{u \in L^2(0,T;\mathcal{U}) \mid \mathcal{L}(u) \in \mathcal{C}\}$. Indeed,

$$\begin{aligned} u \in \mathcal{U}_\mathcal{C} &\iff CS(T)z_0 + Rf + \mathcal{L}u \in \mathcal{C} \\ &\iff \mathcal{L}w_0 + \mathcal{L}u \in \mathcal{C} \iff \mathcal{L}(u + w_0) \in \mathcal{C} \\ &\iff u \in -w_0 + \mathcal{L}^{-1}(\mathcal{C}). \end{aligned}$$

Since $y_{u,f}(T) = \mathcal{L}w_0 + \mathcal{L}u$, the inequality (57) can be written as

$$\langle \theta_f, \mathcal{L}w_0 + \mathcal{L}u_{\theta_f}\rangle \leq \langle \theta_f, \mathcal{L}w_0 + \mathcal{L}u\rangle, \quad \forall\, u \in \mathcal{U}_\mathcal{C},$$

i.e.

$$\langle \theta_f, \mathcal{L}u_{\theta_f}\rangle \leq \langle \theta_f, \mathcal{L}u\rangle, \quad \forall\, u \in \mathcal{U}_\mathcal{C}$$

or, equivalently, with $u = -w_0 + v$, where $v \in \mathcal{L}^{-1}(\mathcal{C})$,

$$\langle \theta_f, \mathcal{L}u_{\theta_f}\rangle \leq \langle \theta_f, -\mathcal{L}w_0 + \mathcal{L}v\rangle, \quad \forall\, v \in \mathcal{U}_\mathcal{C}.$$

By setting $y = \mathcal{L}v$, (57) becomes

$$\langle \theta_f, \mathcal{L}u_{\theta_f} + \mathcal{L}w_0\rangle \leq \langle \theta_f, y\rangle, \quad \forall\, y \in \mathcal{C} \cap \operatorname{Im}\mathcal{L}.$$

For $y = 0$, we have $\langle \theta_f, \mathcal{L}u_{\theta_f} + \mathcal{L}w_0\rangle \leq 0$, and by replacing y by $-y$, we have

$$\langle \theta_f, \mathcal{L}u_{\theta_f} + \mathcal{L}w_0\rangle \leq -\langle \theta_f, y\rangle, \quad \forall\, y \in \mathcal{C} \cap \operatorname{Im}\mathcal{L}.$$

Then

$$\langle \theta_f, \mathcal{L}u_{\theta_f} + \mathcal{L}w_0\rangle \leq \langle \theta_f, y\rangle \leq -\langle \theta_f, \mathcal{L}u_{\theta_f} + \mathcal{L}w_0\rangle, \qquad \forall\, y \in \mathcal{C} \cap \operatorname{Im}\mathcal{L},$$

$$|\langle \theta_f, y\rangle| \leq -\langle \theta_f, \mathcal{L}u_{\theta_f} + \mathcal{L}w_0\rangle, \quad \forall\, y \in \mathcal{C} \cap \operatorname{Im}\mathcal{L}$$

and $\langle \theta_f, y\rangle = 0$, $\forall\, y \in \mathcal{C} \cap \operatorname{Im}\mathcal{L}$, because for $y \in \mathcal{C}$ we have $\langle \theta_f, y\rangle \neq 0$. Then $\alpha y \in \mathcal{C}, \forall\, \alpha$, and hence

$$|\alpha| \geq \frac{-\langle \theta_f, \mathcal{L}u_{\theta_f} + \mathcal{L}w_0\rangle}{|\langle \theta_f, y\rangle|}, \quad \forall\, \alpha,$$

which is impossible. Then $\theta_f \in (\mathcal{C} \cap \operatorname{Im}\mathcal{L})^\perp$. Since $\mathcal{L}u_{\theta_f} + \mathcal{L}w_0 = y_{u_{\theta_f},f} - Rf$, we get $\langle \theta_f, \mathcal{L}u_{\theta_f} + \mathcal{L}w_0\rangle = \langle \theta_f, y_{u_{\theta_f},f}\rangle = 0$ because $y_{u_{\theta_f},f} \in \mathcal{C}$.

Corollary 1. *If there exists a unique $\theta_f \in \mathcal{F}$ such that*

$$\theta_f \in (\mathcal{C} \cap \operatorname{Im}\mathcal{L})^\perp, \quad CS(T)z_0 + \Lambda\theta_f + Rf \in \mathcal{C}, \tag{58}$$

where $(\mathcal{C} \cap \operatorname{Im}\mathcal{L})^\perp = \{\phi \in \mathcal{F} \mid \langle \phi, y\rangle = 0, \ \forall\, y \in \mathcal{C} \cap \operatorname{Im}\mathcal{L}\}$, the corresponding control is the optimal one ensuring the enlarged exact remediability with respect to the subspace $\mathcal{C}$.

Note that, using (58), we have $\langle \theta_f, CS(T)z_0 + \Lambda\theta_f + Rf\rangle = 0$. Then

$$J(u_{\theta_f}) = \|\theta_f\|^2_\mathcal{F} = \langle \theta_f, \Lambda\theta_f\rangle = -\big\langle \theta_f, CS(T)z_0 + Rf\big\rangle.$$

The orthogonality is considered as a duality between $\mathcal{F}$ and $\mathcal{F}'$.

5.3. Case of the Constraint of "Bounded Observation"

Let $\mathcal{C} = \mathcal{B}(CS(T)z_0, \varepsilon)$. We have $y_{u,f}(T) \in \mathcal{C} \iff \|CHu + Rf\| \leq \varepsilon$, and if $\|Rf\| \leq \varepsilon$, it is sufficient to consider the zero control. Assume then that $\|Rf\| > \varepsilon$. For $a > 0$, consider the operator $\Gamma_a : y \in Y \longrightarrow (aI + \Lambda)y \in Y$. We have

$$\langle \Gamma_a y, y\rangle = \big\langle (aI + \Lambda)y, y\big\rangle = a\|y\|^2 + \|y\|^2_\mathcal{F}.$$

Γ_a is then an isomorphism $Y \longrightarrow Y$. By setting $y_a \in Y$, the unique solution to the equation

$$(aI + \Lambda)y_a = Rf, \tag{59}$$

we have the following result.

Corollary 2. *y_a is the solution to (54), if and only if $a\|y_a\| \geq \varepsilon$.*

Proof. Using (59), we have $y - CS(T)z_0 - \Lambda y_a - Rf = y - CS(T)z_0 + ay_a$. Then

$$\begin{aligned} &\big\langle y_a, y - CS(T)z_0 - \Lambda y_a - Rf \big\rangle \\ &\qquad = \big\langle y_a, y - CS(T)z_0 + ay_a \big\rangle \\ &\qquad = \big\langle y_a, y - CS(T)z_0 \big\rangle + a\|y_a\|^2, \end{aligned} \tag{60}$$

and for $y \in \mathcal{C} \cap (CS(T)z_0 + Rf + \mathcal{F}')$ we have

$$\langle y_a, y - CS(T)z_0 \rangle \geq -\varepsilon \|y_a\|.$$

Then

$$\big\langle y_a, y - CS(T)z_0 - \Lambda y_a - Rf \big\rangle \geq \|y_a\| \big(a\|y_a\| - \varepsilon\big). \tag{61}$$

The condition is sufficient because if $a\|y_a\| \geq \varepsilon$, then y_a satisfies (54).

This condition is also necessary, because using the hypothesis we have

$$\begin{aligned} &\big\langle y_a, y - CS(T)z_0 - \Lambda y_a - Rf \big\rangle \geq 0, \\ &\qquad \forall\, y \in \mathcal{C} \cap \big(CS(T)z_0 + Rf + \mathcal{F}'\big). \end{aligned} \tag{62}$$

Since $y_a \in Y$, the inner product is defined for $y \in Y$. On the other hand, since the affine space $CS(T)z_0 + Rf + \mathcal{F}'$ is dense in Y, the inequality (54) is true for $y \in \mathcal{C}$.

Let

$$y = CS(T)z_0 - \varepsilon \frac{y_a}{\|y_a\|} \in \mathcal{C}.$$

Using (60), we have

$$\begin{aligned} &\big\langle y_a, y - CS(T)z_0 - \Lambda y_a - Rf \big\rangle \\ &\qquad = \big\langle y_a, y - CS(T)z_0 \big\rangle + a\|y_a\|^2 \\ &\qquad = \|y_a\|(a\|y_a\| - \varepsilon) \geq 0, \end{aligned}$$

and hence $a\|y_a\| \geq \varepsilon$. ■

6. Conclusion

In this work, we defined and characterized the notion of enlarged exact remediability, which is a generalization of the notion of exact remediability, introduced in previous works, and also the notion of $\mathcal{C}$-efficient actuators. Then we studied the relationship between weak remediability and $\mathcal{C}$-remediability, and hence between efficient actuators and $\mathcal{C}$-efficient actuators in the case of a ball $\mathcal{C} = \mathcal{B}(CS(T)z_0, \varepsilon)$.

Using an extension of the H.U.M. approach as well as penalization and optimization techniques, we showed how to determine the optimal control ensuring enlarged exact remediability, and that the cost is reduced with respect to the problem of exact remediability. As an application, we examined particular cases related to the choice of the constraints or the region of tolerance $\mathcal{C}$.

Finally, let us note that this work can be extended to the problem of regional enlarged remediability and also to other systems.

References

Afifi L., Chafiai A. and El Jai A. (1998): *Compensation spatiale en temps fini dans les systèmes distribués.* — Int. Rep., No. 14, L.T.S. Université de Perpignan, France.

Afifi L., Chafiai A. and El Jai A. (1999): *Regionally efficient and strategic actuators.* — Int. J. Syst. Sci., (accepted).

Afifi L., Chafiai A. and El Jai A. (2000): *Sensors and actuators for compensation in hyperbolic systems.* — Proc. 14-th Int. Symp. *Mathematical Theory of Networks and Systems, MTNS 2000*, Perpignan, France, (CD–ROM).

Bel Fekih A. (1990): *Une nouvelle approche pour l'analyse et le contrôle des systèmes distribués.* — Thèse de doctorat d'état, Faculté des sciences, Kénitra, Morocco.

Bergounioux M. (1994): *Optimal control of parabolic problems with state constraints: A penalization method for optimality conditions.* — Appl. Math. Optim., Vol. 29, No. 3, pp. 285–307.

Curtain R.F. and Pritchard A.J. (1978): *Infinite Dimensional Linear Systems Theory.* — New York: Springer.

Curtain R.F. and Zwart H. (1995): *An Introduction to Infinite Dimensional Linear Systems Theory.* — New York: Springer.

El Jai A. and Pritchard A.J. (1988): *Sensors and Actuators in Distributed Systems Analysis.* — New York: Wiley.

Lee E.B. and Marcus L. (1967): *Foundation of Optimal Control Theory.* — New York: Wiley.

Lions J.L. (1988): *Contrôlabilité Exacte.* — Paris: Masson.

Otsuka N. (1991): *Simultaneous decoupling and disturbance rejection problems for infinite dimensional systems.* — IMA J. Math. Contr. Inf., Vol. 8, pp. 165–178.

Rabah R. and Malabare M. (1997): *A note on decoupling for linear infinite dimensional systems.* — Proc. 4-th IFAC Conf. *System Structure and Control*, Bucharest, Romania, pp. 87–83.

Received: 16 February 2001
Revised: 4 July 2001