CONTROL STRUCTURE IN OPTIMIZATION PROBLEMS OF BAR SYSTEMS

LESZEK MIKULSKI*

* Faculty of Civil Engineering, Cracow University of Technology
ul. Warszawska 24, 31–155 Cracow, Poland
e-mail: mikul@nethq.net

Optimal design problems in mechanics can be mathematically formulated as optimal control tasks. The minimum principle is employed in solving such problems. This principle allows us to write down optimal design problems as Multipoint Boundary Value Problems (MPBVPs). The dimension of MPBVPs is an essential restriction that decides on numerical difficulties. Optimal control theory does not give much information about the control structure, i.e., about the sequence of the forms of the right-hand sides of state equations appearing successively in time. The correctness of the assumed control structure can be checked after obtaining the solution of the boundary problem. For the numerical solution, we use hybrid procedures which are a connection of the multiple shooting method with that of collocation.

Keywords: optimization, minimum principle, elastic structures

1. Optimization Framework

An optimized bar system is represented by its model, which should be as close to reality as possible. As the variables connected with the model we consider: parameters describing physical properties of an object with its geometry, factors acting on the object (load, temperature, environment) and parameters describing a system response. According to the accessible knowledge about individual parameters, we distinguish the following classes of tasks: analysis, synthesis, identification and control. Optimization can appear in synthesis, identification and control. So far in constructional design the analysis has been a basic class, although it does not contain optimization. A proper formulation of an optimization problem consists in choosing control variables, a criterion and necessary constraints, and so it is a complicated task. An improper formulation may cancel any advantages which are due to optimization.

1.1. Formulation of the Optimization Task

1. State equations with boundary conditions and interior point conditions:

$$y'(x) = \begin{cases} f_1(x, y, u), & x \in (0, x_1), \\ \vdots & \\ f_m(x, y, u), & x \in (x_{m-1}, l), \end{cases} \tag{1}$$

$$\begin{aligned}
&f : \mathbb{R}^{n+k+1} \to \mathbb{R}^n, && y(0) = y_0, \\
&y : [0, l] \to \mathbb{R}^n, && u : [0, l] \to \mathbb{R}^k, \\
&\psi(y(l), l) = 0, && \psi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^q, \\
&N_j(y_j(x_i), x_i) = 0, && N_j : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}, \\
&\quad i = 1, 2, \dots, m-1,\ j \in E_1 \subset \{1, 2, \dots, n\} && \\
&\underline{\psi}_j(y_j(x_i^+), y_j(x_i^-)) = 0, && \underline{\psi}_j : \mathbb{R}^2 \to \mathbb{R}, \\
&\quad i = 1, \dots, m-1, \quad j \in E_2 \subset \{1, 2, \dots, n\}, && \\
&E_1 \cap E_2 = \emptyset. &&
\end{aligned} \tag{2}$$

2. Constraints:

$$\begin{aligned}
&C(y(x), u(x)) \le 0, \quad C : \mathbb{R}^{n+k} \to \mathbb{R}^t, \quad x \in (0, l), \\
&S(y(x)) \le 0, \quad S : \mathbb{R}^n \to \mathbb{R}^t, \quad u(x) \in V \subset \mathbb{R}^k.
\end{aligned} \tag{3}$$

   In the optimization process it is crucial if the constraints (3) depend explicitly on control.

3. Optimization criterion:

   (a) Bolza and Lagrange-type functionals (4):

$$J[u, l] = \phi(y(l), l) + \int_0^l L\big(y(x), u(x), x\big)\,\mathrm{d}x \tag{4}$$

   (b) Maximum-type functionals:
   Min-max problems can be transformed into

constrained Mayer-type problems by introducing an additional control parameter p (**min-max** task) (Bulirsch *et al.*, 1991; Pesch, 1994) :

$$\begin{aligned} J[u] &= \max_{x\in[0,l]} |M(y(x),u(x),x)|, \\ p = \max|M(y(x),u(x),x)|, &\quad J[u,p]=p, \\ p - |M(y(x),u(x),x)| &\geq 0. \end{aligned} \tag{5}$$

1.2. Necessary Conditions of Optimization

Necessary conditions for optimal control will be set in this section. Define the Hamilton function H and the function Φ as follows:

$$\begin{aligned} H(y,u,\lambda) &:= L(y,u) + \lambda^T f(y,u) + \mu C(y,u), \\ \Phi(y,x,\nu) &:= \phi(y,x) + \nu^T \psi(y,x), \\ \mu &= -C_u^{-1}(L_u + \lambda^T f_u) \quad \text{if} \quad C_u \neq 0. \end{aligned} \tag{6}$$

The optimal solution satisfies the necessary conditions

$$\lambda'^T = -H_y, \quad u = \arg\min_{u\in V} H, \quad \lambda^T(l) = \Phi_y|_{x=l}, \tag{7}$$

where $\lambda : [0,l] \rightarrow \mathbb{R}^n$ and $\nu \in \mathbb{R}^q$ are the so-called Lagrange multipliers.

1.2.1. State Variable Inequality Constraints

The cases of constraints dependent on state variables (Oberle and Grimm, 1989; Pesch, 1994) undergo a particular analysis:

$$\begin{aligned} S\big(y(x)\big) &\leq 0 : \mathbb{R}^n \rightarrow \mathbb{R}^t, \\ S\big(y(x)\big) &\equiv 0 \quad \text{for} \quad x \in [x_1, x_2] \subset [0,l]. \end{aligned} \tag{8}$$

It is impossible to determine the control directly from the equation $S(y(x)) = 0$. We can distinguish the following two cases:

1. A constraint is active in an interval: $S(y(x)) = 0$ for $x \in [x_1, x_2]$.
2. A constraint remains active at an interior point x_b: $S(y(x_b)) = 0$.

The second condition will be scrutinized. For the optimal solution of the problem with the conditions (8), there exist multipliers σ_i which enable us to express jumps of adjoint functions at the interior point x_b:

$$\lambda^T(x_b^+) = \lambda^T(x_b^-) - \sigma^T S_y\big(y(x_b)\big), \quad \sigma_i \geq 0. \tag{9}$$

Moreover, the Hamilton function is continuous at the point x_b, $H(x_b^+, y, \lambda, u) = H(x_b^-, y, \lambda, u)$.

1.3. Constraints on State Variables at a Fixed Point

The internal condition at a fixed point x_i can generally be written down as follows:

$$\begin{aligned} \underline{\psi}_j\big(y_j(x_i^+), y_j(x_i^-)\big) = 0, &\quad i = 1,\dots,m-1, \\ j \in E_2, &\quad \underline{\psi}_j : \mathbb{R}^2 \rightarrow \mathbb{R}. \end{aligned} \tag{10}$$

The adjoint variables $\lambda(x)$ are discontinuous at the point x_i:

$$\lambda_j({x_i}^+) = \lambda_j({x_i}^-) - \nu \frac{\partial \psi}{\partial y}. \tag{11}$$

In optimum design tasks, the conditions (10) have in most cases the form

$$y_j(x_i) = 0, \quad \lambda_j({x_i}^+) = \lambda_j({x_i}^-) - \nu, \tag{12}$$

or

$$y_j({x_i}^+) = y_j({x_i}^-) + D_j, \quad \lambda_j(x_i) = 0. \tag{13}$$

Optimal control theory makes it possible to formulate an optimization problem with constraints on state variables as a multipoint boundary value problem (MPBVP).

2. Numerical Solution of Optimal Control Problems

Indirect or direct methods are used for a numerical solution of optimal control problems. By definition, indirect methods involve adjoint variables, the Hamilton function, the minimum principle, and necessary conditions are employed in formulating MPBVPs. Numerical methods producing solutions to MPBVPs are based on the Indirect Multiple Shooting method. The crucial points of an indirect method algorithm with suitable computer programs are shown in Fig. 1.

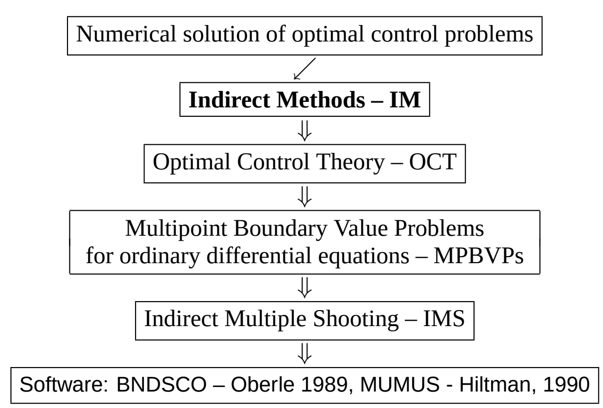


Fig. 1. Characterization of indirect methods for solving optimal control problems.

An essential feature of the Indirect Multiple Shooting method is the verification of necessary conditions of optimization, which enables us to reach a solution that satisfies the conditions and is a good candidate for the optimal solution. In this method a certain starting solution with a structure suitable for the problem is iteratively corrected. The structure of the solution can be complicated and the choice of initial values for which an IMS method is convergent can be difficult. However, an indirect method requires the knowledge of optimal control theory, and setting down MPBVPs is not effortless. Using an indirect method gives a wealth of information that provides a deep insight into the optimal solution structure. Taking into account adjoint equations is burdensome but it makes it possible to check many necessary conditions in order to eliminate solutions which are not optimal.

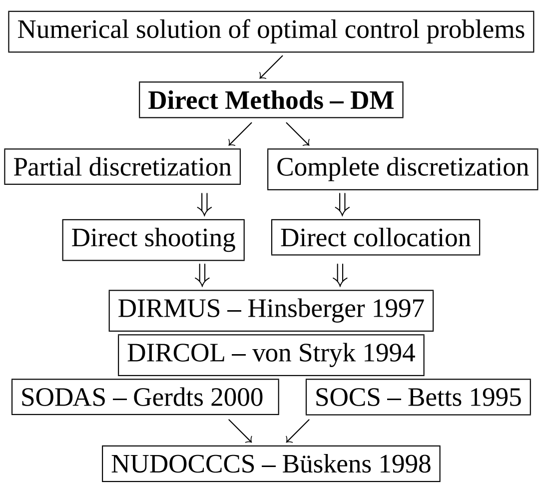


Fig. 2. Characterization of direct methods for solving optimal control problems. DIRCOL – A Direct Collocation Method for the Numerical Solution of Optimal Control Problems (Von Stryk, 2002), DIRMUS – DIRect MUltiple Shooting method (Hinsberger, 1996), NUDOCCCS – NUmerical Discretization method for Optimal Control problems with Constraints in Control and State, Version 9.0 (Pesch, 2002).

3. Optimization of Multispan Steel Girders

Now we present one of many subjects related to the optimal design of continuous beams which are elements of an actual object. The beams are made out of St3SX steel and they have an I-section with a constant height of the web and the changeable width of the flange. This width is the control variable in the optimization problem. The continuous beams are subjected to a dead weight q_1 and a useful load q_2. For the three-span beam of a floor we set down loads and on this basis the analytical useful load $q_2 = 65\,$KN/m was accepted. The three combinations of loads which exhausted the most disadvantageous possibilities were analyzed. The right deflection, angle of deflection, bending moment and shearing force were involved in the aforementioned combinations.

3.1. Formulation of Optimization Tasks

The three-span beam is optimized in the presence of constraints for four different criteria. The program of the optimization tasks is formulated as follows (Mikulski, 1999): For the beam acted upon by manifold loads it is necessary to determine an optimal distribution of the width of the cross-section $B(x)$, which corresponds to the minimum of an accepted cost function with the assumed constraints. As the optimization criteria, in the successive tasks we assume the following:

1. Volume of the beam with limited stresses:

$$\begin{aligned} J &= \int_0^l A(x)\,\mathrm{d}x, \quad \sigma_x \le f_d, \\ A(x) &= 0.036u(x) + 0.01757. \end{aligned} \tag{14}$$

2. Volume of the beam with limited deflections:

$$J = \int_0^l A(x)\,\mathrm{d}x, \quad |y_k| \le y_{\mathrm{dop}}, \quad k = 1, 5, 9. \tag{15}$$

3. Maximum value of deflection with a fixed volume:

$$J = \sum \alpha_k \max_x |y_k|, \quad k = 1, 5, 9, \quad \int_0^l A(x)\,\mathrm{d}x = V_0. \tag{16}$$

4. Maximum value of a normal stress with a constant volume

$$J = \sum \alpha_k \max_x \sigma_x^k, \quad k = 1, 2, 3, \quad \int_0^l A(x) dx = V_0. \tag{17}$$

State equations, loads in characteristic intervals, boundary conditions and internal point conditions are successively tabulated ($I = 0.01977u(x) + 0.00314$).

The unknown values $Q_1, \ldots, Q_6$ will be determined from the following equations:

$$\begin{gathered} Q_i' = 0, \quad i = 1, 2, 3, 4, 5, 6, \\ y_1(16) = y_1(32) = 0, \quad y_5(16) = y_5(32) = 0, \\ y_9(16) = y_9(32) = 0. \end{gathered} \tag{18}$$

An extended version of the minimum principle is employed in formulating necessary conditions of optimality. It takes into account the constraints on state variables and

Table 1. State equations.

Variant	1	2	3	4
I	$y_1' = y_2$	$y_2' = y_3/EI$	$y_3' = y_4$	$y_4' = -(q_I(x) + \gamma A(x))$
II	$y_5' = y_6$	$y_6' = y_7/EI$	$y_7' = y_8$	$y_8' = -(q_{II}(x) + \gamma A(x))$
III	$y_9' = y_{10}$	$y_{11}' = y_3/EI$	$y_{11}' = y_{12}$	$y_{12}' = -(q_{III}(x) + \gamma A(x))$

Table 2. Load $(l_1 = 16,\ l_2 = 32,\ l_3 = 48\,\mathrm{m})$.

Variant	$(0, l_1)$	(l_1, l_2)	(l_2, l_3)
I	$q_1 + q_2$	q_1	$q_1 + q_2$
II	q_1	$q_1 + q_2$	q_1
III	$q_1 + q_2$	$q_1 + q_2$	q_1

their discontinuities at a finite number of points. The optimal control can be determined by solving the system of equations

$$y' = \frac{\partial H}{\partial \lambda}, \quad \lambda' = -\frac{\partial H}{\partial y}, \quad u = \arg\min_{u \in V} H, \tag{19}$$

with boundary conditions for state variables and transversality conditions for adjoint functions. A detailed discusion of the necessary conditions will be given in the next section.

3.2. Optimization of the Beam with Respect to the Maximum Deflection

The sum of the maximum deflections (16) is the cost function in the discussed problem. Introducing the control parameter p, we reduce the optimization problem to the Mayer form. In the formulated task we search for control u which minimizes the value of the parameter p:

$$\begin{aligned} &\min_u p \\ &p(u) = \alpha_1 \max_x |y_1| + \alpha_2 \max_x |y_5| + \alpha_3 \max_x |y_9|, \\ &\alpha_1 + \alpha_2 + \alpha_3 = 1, \end{aligned} \tag{20}$$

in the presence of the following side conditions:

1. **State equations**. The state equations are set down in Table 1. In the analyzed problem, the total volume of the beam is known. After introducing an additional variable $V(x) = \int_0^x A(\xi)\,\mathrm{d}\xi$, the condition of the constant volume can be written down in the form of the differential equation $V(x)' = A(x)$ with the boundary conditions $V(0) = 0$ and $V(l) = V_0$.
2. **Boundary conditions and internal point conditions for state variables.** These are set down in Table 3.
3. **Geometrical conditions:**
   - on control
   $$\begin{aligned} &u(x) \equiv B(x), \quad U_1 = B_{\min} \le B(x) \le B_{\max} = U_2, \\ &B_{\min} = 0.4\,\mathrm{m}, \quad B_{\max} = 0.6\,\mathrm{m}, \end{aligned} \tag{21}$$
   - condition of the constant volume which allows us to compare the optimal solution with the solution for fixed $B(x) = B_0$.
4. **Constraints on state variables.** These constraints result from the assumed cost function (20) and have the form (Bulirsch *et al.*, 1991):
   $$\begin{aligned} &g_1 = p - \alpha_1|y_1| - \alpha_2|y_5| - \alpha_3|y_9| \ge 0, \\ &p' = 0. \end{aligned} \tag{22}$$

The Hamilton function (6) of the task considered has the following detailed form:

$$\begin{aligned} H = {} & \lambda_1 y_2 + \lambda_2 \frac{y_3}{EI} + \lambda_3 y_4 + \lambda_4\big(-q_I + \gamma A(x)\big) \\ & + \lambda_5 y_6 + \lambda_6 \frac{y_7}{EI} + \lambda_7 y_8 + \lambda_8\big(-q_{II} + \gamma A(x)\big) \\ & + \lambda_9 y_{10} + \lambda_{10} \frac{y_{11}}{EI} + \lambda_{11} y_{12} \\ & + \lambda_{12}\big(-q_{III} + \gamma A(x)\big) \\ & + \mu A(x) + \eta(p - \alpha_1|y_1| - \alpha_2|y_5| - \alpha_3|y_9|). \end{aligned} \tag{23}$$

The boundary conditions and the internal point conditions for the adjoint functions (7), resulting from the transversality conditions, are set down in Table 4.

From the condition $\partial H/\partial u = 0$ we will determine the optimal control $u(x)$ if constraints are not active. The Hamilton function (6) is piecewise constant and has jumps at fixed intermediate supports,

$$\begin{aligned} &H(y^+, \lambda^+, u^+, 16) = H(y^-, \lambda^-, u^-, 16) + D_1, \\ &H(y^+, \lambda^+, u^+, 32) = H(y^-, \lambda^-, u^-, 32) + D_2. \end{aligned} \tag{24}$$

Table 3. Boundary conditions and internal point conditions.

Variant	Variable	$x=0$	$x=16$	$x=32$	$x=48$
I	y_1	0	0	0	0
	y_2	—	$y_2(l_1^-)=y_2(l_1^+)$	$y_2(l_1^-)=y_2(l_1^+)$	—
	y_3	0	$y_3(l_1^-)=y_3(l_1^+)$	$y_3(l_1^-)=y_3(l_1^+)$	0
	y_4	—	$y_4(l_1^-)=y_4(l_1^+)+Q_1$	$y_4(l_1^-)=y_4(l_1^+)+Q_2$	—
II	y_5	0	0	0	0
	y_6	—	$y_6(l_1^-)=y_6(l_1^+)$	$y_6(l_1^-)=y_6(l_1^+)$	—
	y_7	0	$y_7(l_1^-)=y_7(l_1^+)$	$y_7(l_1^-)=y_7(l_1^+)$	0
	y_8	—	$y_8(l_1^-)=y_8(l_1^+)+Q_3$	$y_8(l_1^-)=y_8(l_1^+)+Q_4$	—
III	y_9	0	0	0	0
	y_{10}	—	$y_{10}(l_1^-)=y_{10}(l_1^+)$	$y_{10}(l_1^-)=y_{10}(l_1^+)$	—
	y_{11}	0	$y_{11}(l_1^-)=y_{11}(l_1^+)$	$y_{11}(l_1^-)=y_{11}(l_1^+)$	0
	y_{12}	—	$y_{12}(l_1^-)=y_{12}(l_1^+)+Q_5$	$y_{12}(l_1^-)=y_{12}(l_1^+)+Q_6$	—

Table 4. Boundary conditions and internal point conditions for adjoint functions.

Variant	Variable	$x=0$	$x=16$	$x=32$	$x=48$
I	λ_1	—	$\lambda_1(l_1^+)=\lambda_1(l_1^-)+L_1$	$\lambda_1(l_2^+)=\lambda_2(l_1^-)+L_2$	—
	λ_2	0	—	—	0
	λ_3	0	—	—	0
	λ_4	0	0	0	0
II	λ_5	—	$\lambda_5(l_1^+)=\lambda_5(l_1^-)+L_3$	$\lambda_5(l_2^+)=\lambda_5(l_1^-)+L_4$	—
	λ_6	0	—	—	0
	λ_7	0	—	—	0
	λ_8	0	0	0	0
III	λ_9	—	$\lambda_9(l_1^+)=\lambda_9(l_1^-)+L_5$	$\lambda_9(l_2^+)=\lambda_9(l_1^-)+L_6$	—
	λ_{10}	0	—	—	0
	λ_{11}	0	—	—	0
	λ_{12}	0	0	0	0

The constraints on the state variables (22) are only active at an isolated point. The adjoint variables λ_1, λ_5 and λ_9 are discontinuous at that point,

$$\lambda_i(x_b^+) = \lambda_i(x_b^-) + C_j, \quad i = 1, 5, 9. \tag{25}$$

4. Numerical Solutions

After setting down the necessary conditions of optimality, the optimal design task was reduced to the solution of Multipoint Boundary Value Problems (MPBVPs) for the system of the differential equations (19) with the boundary conditions and the internal point conditions given in Tables 1–4. It is necessary to determine 13 state variables y_i, 13 adjoint variables λ_i, 6 constants Q_i responsible for the jumps of state variables, 6 constants L_i connected with the discontinuity of adjoint functions, 1 multiplier η, 2 constants D_j, 3 constants C_j, so that, all in all, 44 parameters have to be determined. The solution of the formulated optimal design problem is possible in a numerical way. Numerical results were obtained by using the programs `Dircol-2.1` (Von Stryk, 2002) and `BNDSCO` (Oberle and Grimm, 1989) for optimization. After solving MPBVPs, the following structure of the optimal solution with 13 points of change in the control $u(x)$ was obtained:

$$\begin{cases} U_1 & \text{when} \quad x \in (0., 1.23), \\ U_{\text{opt}} & \text{when} \quad x \in (1.23, 2.46), \\ U_2 & \text{when} \quad x \in (2.46, 12.30), \\ g_1 = 0 & \text{when} \quad x = 7.13, \\ U_{\text{opt}} & \text{when} \quad x \in (12.30, 13.53), \\ U_1 & \text{when} \quad x \in (13.53, 14.76), \\ U_{\text{opt}} & \text{when} \quad x \in (14.76, 15.00), \\ U_2 & \text{when} \quad x \in (15.00, 20.00), \\ U_{\text{opt}} & \text{when} \quad x \in (20.00, 21.33), \end{cases} \tag{26a}$$

$$\begin{cases} U_1 & \text{when} \quad x \in (21.33, 36.36), \\ U_{\text{opt}} & \text{when} \quad x \in (36.36, 37.81), \\ U_2 & \text{when} \quad x \in (37.81, 39.27), \\ U_{\text{opt}} & \text{when} \quad x \in (39.27, 43.63), \\ U_1 & \text{when} \quad x \in (43.63, 48.00), \end{cases} \tag{26b}$$

for which the cost function (22) is $p = 0.8889 \cdot 10^{-2}$. As the optimization result, the optimal distribution of the control variable was obtained and the cost function (22) was decreased by 26.6% for the same volume of the beam.

The structure of the solution is given in the relations (26a) and (26b)

Table 5. Comparison of results.

u	p (22)	volume
$u = 0.50$	$p = 0.1195 \cdot 10^{-1}$	$V_0 = 1.728$
u^{opt}	$p = 0.8889 \cdot 10^{-2}$	$V_0 = 1.728$

5. Optimization of Elastic Arches under Multiple Loads

In this section we consider the formulation and solution of optimal design problems of elastic arches in terms of multiple load systems. This class of loads is crucial for the statement of optimization and control problems since solutions for one kind of forces are not optimal for another. In the design practice, however, we are looking for the best solution in terms of the most disadvantageous locations of loads. The maximum displacement, maximum stress or the volume (weight) is the cost function. In the optimization task the height or the width of the rectangular cross section of the arch and the position of intermediate supports become control variables. The constraints are of geometrical nature, and they refer to control variables and to the complete volume (weight) of the arch.

5.1. Optimal Design Problem

The subject of the analysis is a continuous arch with boundary conditions defined on both ends of the arch. Besides, an additional intermediate support is considered. The state variables existing in every state of loading are: deflections, the angle of deflection, the bending moment, normal and shear forces. For the continuous two-span arch, three various states of a useful load (Fig. 3) are taken into consideration: Case 1—the useful load distributed on the whole length of the arch, the respective state variables are y_i, $i = 1, \dots, 6$; Case 2—the useful load in the interval $(0, \xi_p)$, the state variables are y_i, $i = 8, \dots, 13$. Case 3—the useful load in the interval (ξ_p, l), the state variables are y_i, $i = 14, \dots, 19$. It is assumed that the dimensionless height of the rectangular cross-section is the control $U(\xi)$. The equilibrium state of the arch is defined by a system of ordinary differential equations together with the appropriate initial and boundary conditions for the unknowns $N, Q, M, \overline{u}, \overline{w}$ and β. These variables are functions of x and represent geometrical and mechanical quantities: N – longitudinal force, Q – shear force, M – bending moment, β – angle, $\overline{u}$ and $\overline{w}$ – displacements in normal and tangential directions, respectively.

System of state equations:

if $\quad \xi \in (0, 1), \quad$ then

$$y_1' = \frac{8a_1 y_2}{Z1} + a_2 U 4a_1(1-2\xi) + \frac{(p_0 - q_0)4a_1(1-2\xi)}{\sqrt{Z1}},$$

$$y_2' = \frac{-8a_1 y_1}{Z1} - a_2 U - \frac{p_0 + q_0(16a_1^2(1-2\xi)^2)}{\sqrt{Z1}},$$

$$y_3' = y_2\sqrt{Z1},$$

$$y_4' = \frac{y_1 a_3 \sqrt{Z1}}{U} - \frac{8a_1 y_5}{Z1},$$

$$y_5' = \frac{8a_1 y_2}{Z1} - y_6\sqrt{Z1}$$

$$y_6' = \frac{y_3\sqrt{Z1}}{U^3},$$

$$y_8' = \frac{8a_1 y_9}{Z1} + a_2 U 4a_1(1-2\xi) + \frac{(p_0 - q_0)4a_1(1-2\xi)}{\sqrt{Z1}},$$

$$y_9' = \frac{-8a_1 y_8}{Z1} - a_2 U - \frac{p_0 + q_0(16a_1^2(1-2\xi)^2)}{\sqrt{Z1}},$$

$$y_{10}' = y_9\sqrt{Z1},$$

$$y_{11}' = \frac{y_8 a_3 \sqrt{Z1}}{U} - \frac{8a_1 y_{12}}{Z1},$$

$$y_{12}' = \frac{8a_1 y_9}{Z1} - y_{13}\sqrt{Z1}$$

$$y_{13}' = \frac{y_{10}\sqrt{Z1}}{U^3},$$

$$y_{14}' = \frac{8a_1 y_{15}}{Z1} + a_2 U 4a_1(1-2\xi) + \frac{(-q_0)4a_1(1-2\xi)}{\sqrt{Z1}},$$

$$y_{15}' = \frac{-8a_1 y_{14}}{Z1} - a_2 U - \frac{q_0(16a_1^2(1-2\xi)^2)}{\sqrt{Z1}},$$

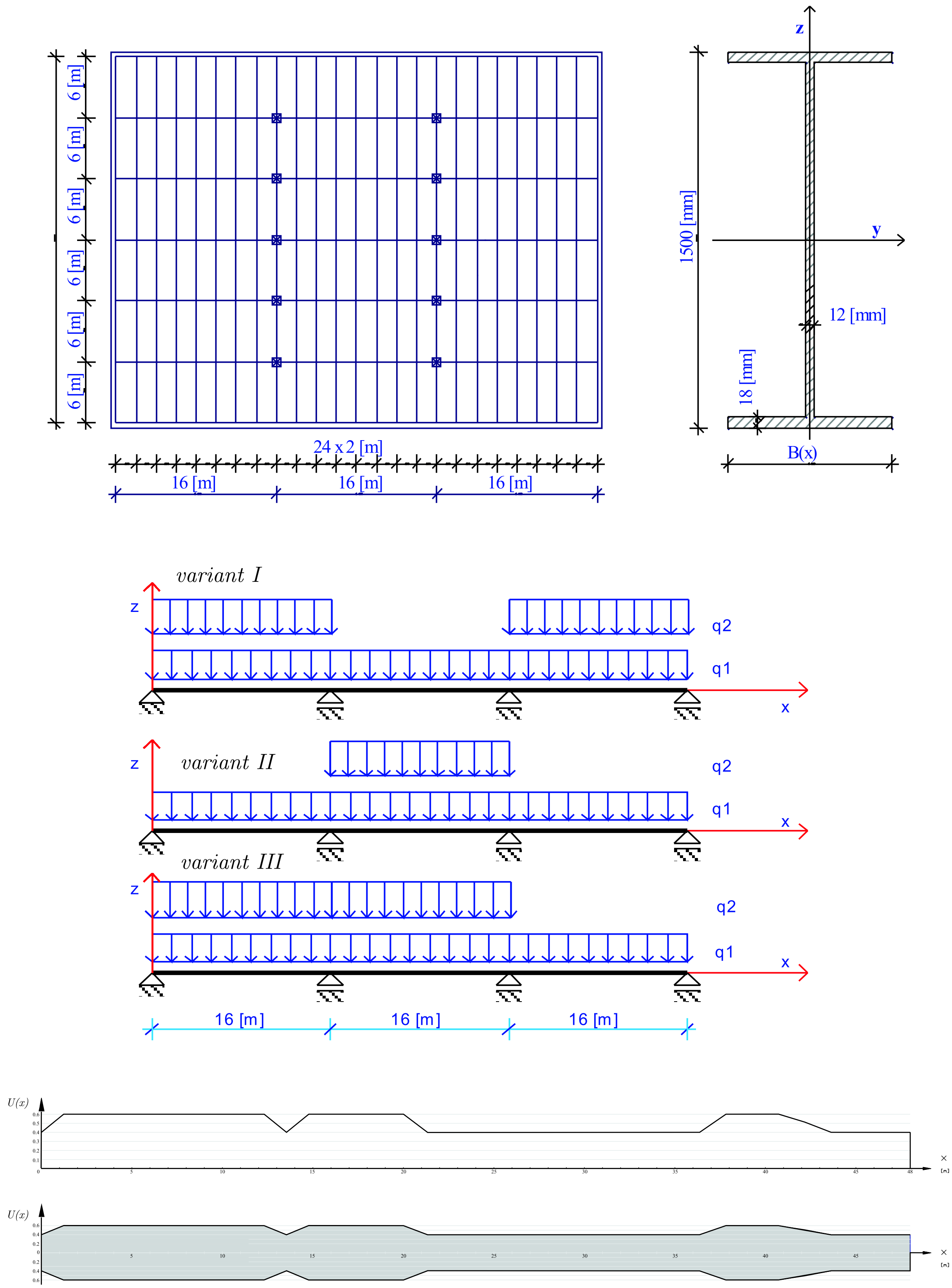


Fig. 3. Schemes of the beam with multiple loads, optimal control.

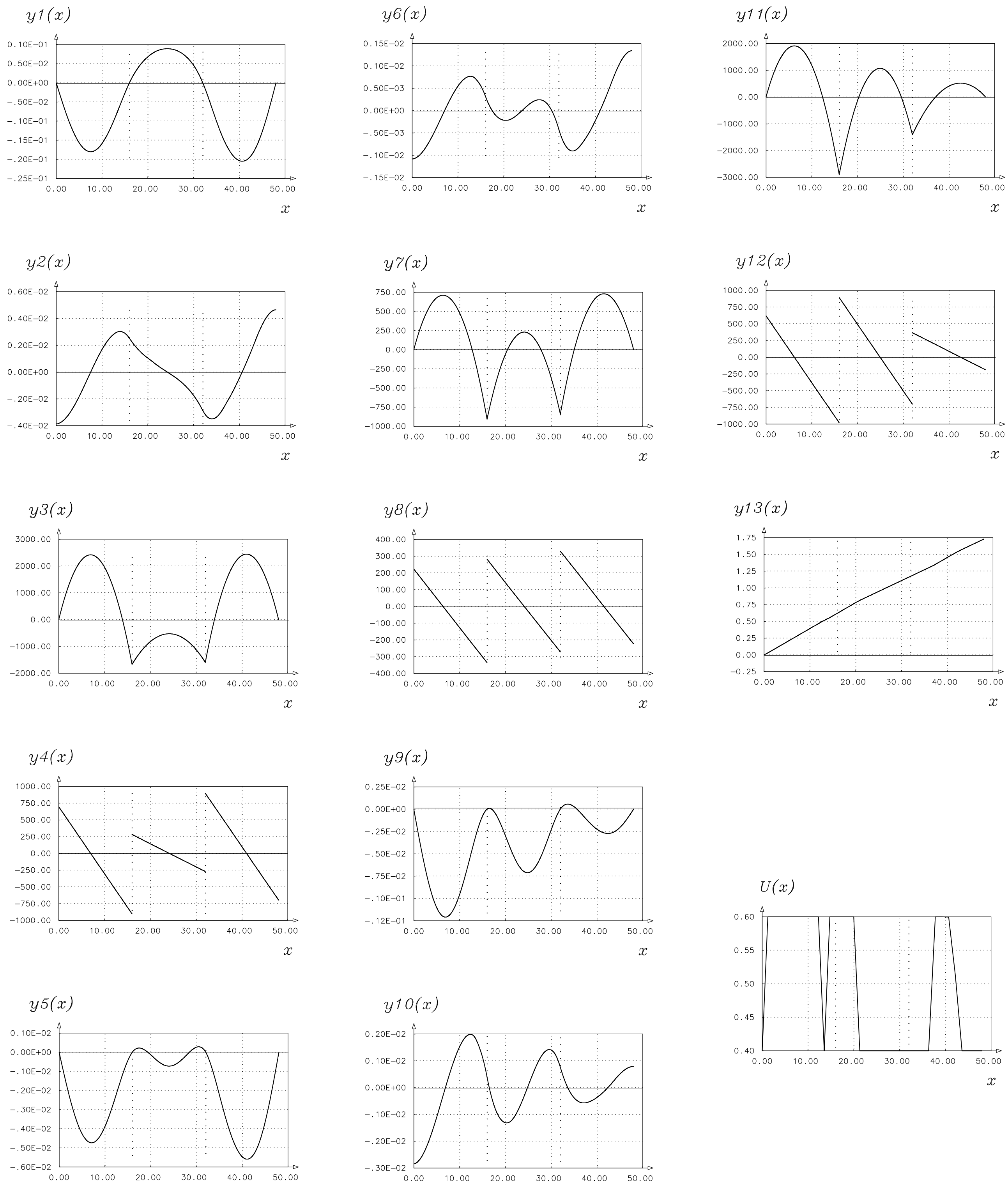


Fig. 4. State variable y_j, optimal control $u(x)$.

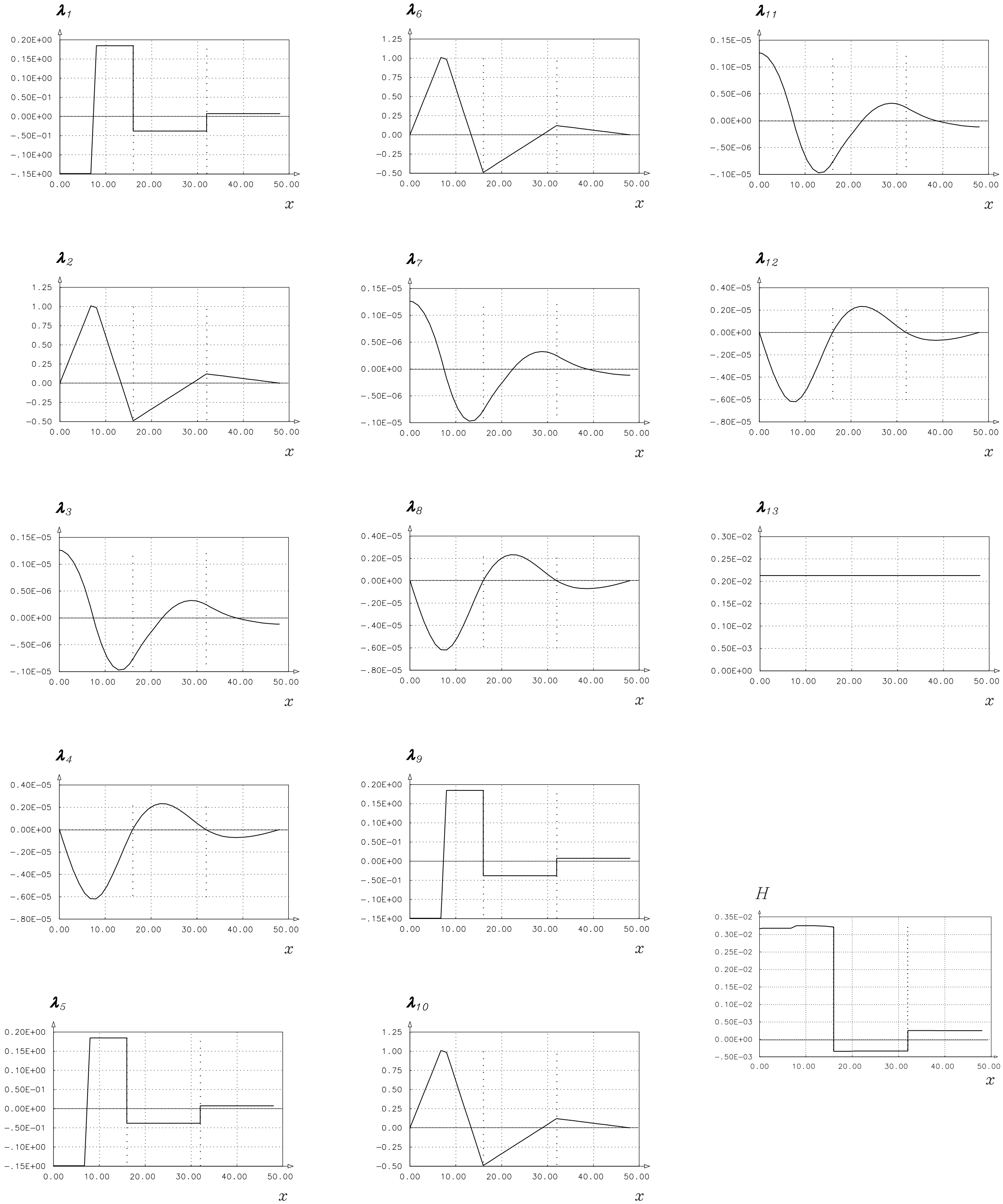


Fig. 5. Adjoint variables.

$$y'_{16} = y_{15}\sqrt{Z1},$$

$$y'_{17} = \frac{y_{14}a_3\sqrt{Z1}}{U} - \frac{8a_1y_{18}}{Z1},$$

$$y'_{18} = \frac{8a_1y_{15}}{Z1} - y_{19}\sqrt{Z1}$$

$$y'_{19} = \frac{y_{16}\sqrt{Z1}}{U^3}, \tag{27}$$

if $\xi \in (1,2)$, then

$$y'_1 = \frac{8a_1y_2}{Z2} + a_2U4a_1(1-2\xi) + \frac{(p_0-q_0)4a_1(1-2\xi)}{\sqrt{Z2}},$$

$$y'_2 = \frac{-8a_1y_1}{Z2} - a_2U - \frac{p_0+q_0(16a_1^2(1-2\xi)^2)}{\sqrt{Z2}},$$

$$y'_3 = y_2\sqrt{Z2},$$

$$y'_4 = \frac{y_1a_3\sqrt{Z2}}{U} - \frac{8a_1y_5}{Z2},$$

$$y'_5 = \frac{8a_1y_2}{Z2} - y_6\sqrt{Z2}$$

$$y'_6 = \frac{y_3\sqrt{Z2}}{U^3},$$

$$y'_8 = \frac{8a_1y_9}{Z2} + a_2U4a_1(1-2\xi)$$

$$y'_9 = \frac{-8a_1y_8}{Z2} - a_2U$$

$$y'_{10} = y_9\sqrt{Z2},$$

$$y'_{11} = \frac{y_8a_3\sqrt{Z2}}{U} - \frac{8a_1y_{12}}{Z2},$$

$$y'_{12} = \frac{8a_1y_9}{Z1} - y_{13}\sqrt{Z2}$$

$$y'_{13} = \frac{y_{10}\sqrt{Z2}}{U^3},$$

$$y'_{14} = \frac{8a_1y_{15}}{Z2} + a_2U4a_1(1-2\xi) + \frac{(p_0)4a_1(1-2\xi)}{\sqrt{Z2}},$$

$$y'_{15} = \frac{-8a_1y_{14}}{Z2} - a_2U - \frac{p_0}{\sqrt{Z2}},$$

$$y'_{16} = y_{15}\sqrt{Z2},$$

$$y'_{17} = \frac{y_{14}a_3\sqrt{Z2}}{U} - \frac{8a_1y_{18}}{Z2},$$

$$y'_{18} = \frac{8a_1y_{15}}{Z2} - y_{19}\sqrt{Z2}$$

$$y'_{19} = \frac{y_{16}\sqrt{Z2}}{U^3}. \tag{28}$$

We normalize x and transform the following dependent variables: $\xi = x/l$, $y_1, y_8, y_{14} = n = Nl^2/EI_0$, $y_2, y_9, y_{15} = t = Ql^2/EI_0$, $y_3, y_{10}, y_{16} = m = Ml/EI_0$, $y_4, y_{11}, y_{17} = u = \overline{u}/l$, $y_5, y_{12}, y_{18} = w = \overline{w}/l$, $y_6, y_{13}, y_{19} = \beta$, $U = h/h_0$,

$$a_1 = \frac{f}{l}, \quad a_2 = \frac{\gamma l^3 b_0 h_0}{EI_0}, \quad a_3 = \frac{I_0}{b_0h_0l^2}, \quad p_0 = \frac{Pl^3}{EI_0}, \quad q_0 = \frac{Sl^3}{EI_0}, \tag{29}$$

with $I_0 = b_0h_0^3/12$ – the moment of inertia, l – the span of the arch, $P(\xi)$ – the vertical component of the load, $S(\xi)$ – the horizontal component of the load, E – Young's modulus.

The boundary conditions are as follows:

$$\begin{aligned} &y_3(0)=0, \quad y_4(0)=0, \quad y_5(0)=0,\\ &y_4(2)=0, \quad y_5(2)=0, \quad y_6(2)=0,\\ &y_{10}(0)=0, \quad y_{11}(0)=0, \quad y_{12}(0)=0,\\ &y_{11}(2)=0, \quad y_{12}(2)=0, \quad y_{13}(2)=0,\\ &y_{16}(0)=0, \quad y_{17}(0)=0, \quad y_{18}(0)=0,\\ &y_{17}(2)=0, \quad y_{18}(2)=0, \quad y_{19}(2)=0. \end{aligned} \tag{30}$$

For the intermediate support ξ_{p1}, 12 additional conditions are imposed:

$$\begin{aligned} y_4(\xi_{p1})=0, &\quad y_1(\xi_{p1}^+) = y_1(\xi_{p1}^-)+C_1,\\ y_5(\xi_{p1})=0, &\quad y_2(\xi_{p1}^+) = y_2(\xi_{p1}^-)+C_2,\\ y_{11}(\xi_{p1})=0, &\quad y_8(\xi_{p1}^+) = y_8(\xi_{p1}^-)+C_3,\\ y_{12}(\xi_{p1})=0, &\quad y_9(\xi_{p1}^+) = y_9(\xi_{p1}^-)+C_4,\\ y_{17}(\xi_{p1})=0, &\quad y_{14}(\xi_{p1}^+) = y_{14}(\xi_{p1}^-)+C_5,\\ y_{18}(\xi_{p1})=0, &\quad y_{15}(\xi_{p1}^+) = y_{15}(\xi_{p1}^-)+C_6. \end{aligned} \tag{31}$$

Constraints and cost functions

Constraints define the set of admissible controls. For the control variable, we introduce geometrical constraints:

$$\begin{aligned} &U_1 \le U(\xi) \le U_2,\\ &y'_7 = U(\xi)\sqrt{Z1} \quad \text{if} \quad \xi \in (0,\xi_{p1}),\\ &y'_7 = U(\xi)\sqrt{Z2} \quad \text{if} \quad \xi \in (\xi_{p1},2),\\ &y_7(0)=0, \qquad y_7(2)=V_0,\\ &Z1 = 1+16a_1^2\big((1-2\xi)^2\big),\\ &Z2 = 1+16a_1^2\big((3-2\xi)^2\big). \end{aligned} \tag{32}$$

The cost function is defined by the functional

$$J(U) = \alpha_1 \max_{\xi} \left(\sqrt{y_4^2 + y_5^2}\right) + \alpha_2 \max_{\xi} \left(\sqrt{y_{11}^2 + y_{12}^2}\right) + \alpha_3 \max_{\xi} \left(\sqrt{y_{17}^2 + y_{18}^2}\right), \tag{33}$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 1,$$

or

$$J(U) = \alpha_1 \max_{\xi} \left\{C_{11}\frac{|y_3|}{U^2} + C_{22}\frac{|y_1|}{U}\right\} + \alpha_2 \max_{\xi} \left\{C_{11}\frac{|y_{10}|}{U^2} + C_{22}\frac{|y_8|}{U}\right\} + \alpha_3 \max_{\xi} \left\{C_{11}\frac{|y_{16}|}{U^2} + C_{22}\frac{|y_{14}|}{U}\right\}. \tag{34}$$

We wish to determine $U(\xi)$ which corresponds to the minimum of the functional (33) or (34) satisfying the state equations (27), (28) with the proper boundary (30), and jump conditions (31) and the constraints (32).

5.2. General Solution

The min-max objectives such as

$$J[U,p] = \min_{U,p} \left\{\alpha_1 \max_{\xi} \left(\sqrt{y_4^2 + y_5^2}\right) + \alpha_2 \max_{\xi} \left(\sqrt{y_{11}^2 + y_{12}^2}\right) + \alpha_3 \max_{\xi} \left(\sqrt{y_{17}^2 + y_{18}^2}\right)\right\} \tag{35}$$

can be transformed into Mayer-type problems by introducing an additional control parameter p satisfying

$$p = \alpha_1 \max_{\xi} \left(\sqrt{y_4^2 + y_5^2}\right) + \alpha_2 \max_{\xi} \left(\sqrt{y_{11}^2 + y_{12}^2}\right) + \alpha_3 \max_{\xi} \left(\sqrt{y_{17}^2 + y_{18}^2}\right). \tag{36}$$

Then the resulting constrained Mayer-type problem is

$$J[U,p^*] = p \Longrightarrow \text{min !} \tag{37}$$

subject to the additional inequality constraint

$$p' = 0,$$

$$g(U,y,p) = p - \left\{\left(\alpha_1\sqrt{y_4^2 + y_5^2}\right) + \left(\alpha_2\sqrt{y_{11}^2 + y_{12}^2}\right) + \left(\alpha_3\sqrt{y_{17}^2 + y_{18}^2}\right)\right\} \geq 0. \tag{38}$$

Optimal control theory provides necessary conditions for the trajectory $y(\xi)$ and the control function $U(\xi)$ associated with it (Buss *et al.*, 2002). The control $U(\xi)$ was calculated from the minimum principle. Defining the Hamilton function H,

$$\begin{aligned} H &= \sum_i \lambda_i\, y_i{}' && \text{for} \quad g(y,U,p) > 0, \\ H_1 &= H + \mu\, g(y,U,p) && \text{if} \quad g(y,U,p) = 0, \end{aligned} \tag{39}$$

the optimal solution $U(\xi)$ can be obtained in particular from the conditions

$$\frac{\partial H}{\partial U} = 0. \tag{40}$$

The adjoint functions λ_i are a solution of the equations

$$\lambda_i' = -\frac{\partial H}{\partial y_i}. \tag{41}$$

The adjoint functions $\lambda_k,\ k = 4,5,11,12,17,18$ are discontinuous at the intermediate support point

$$\begin{aligned} &\lambda_k(\xi_{p1}^+) = \lambda_k(\xi_{p1}^-) + B_k, && k = 4,5,11,12,17,18, \\ &\lambda_k(\xi_{p1}) = 0, && k = 1,2,8,9,14,15. \end{aligned} \tag{42}$$

The Hamiltonian H is discontinuous at the intermediate support point, where the position is fixed:

$$H(\xi_{p1}^+, y, \lambda, U) = H(\xi_{p1}^-, y, \lambda, U) + \sigma_1. \tag{43}$$

The problem of the optimal design of continuous arches thus reduces to solving the system of 38 differential equations (27), (28), (41) with the initial boundary conditions (30), (42) and conditions (31), (43) within the interval. The so-called multipoint boundary value problem (MPBVP) (Hiltman *et al.*, 1993) has 71 dimensions. The MPBVP of such a large number of dimensions is composed of 19 state variables y_i, 19 adjoint variables λ_i, 6 discontinuities in state variables C_j, 12 jumps of adjoint variables B_j, 13 points in change in control $U(\xi)$, 1 discontinuity of the Hamilton function σ_1. Optimal control theory does not give any information about the structure of the control, i.e., about the number of alterations of the right-hand sides of the equations. The essential condition is to assume *a priori* a certain structure of the control. The correctness of this assumption will be verified after solving the boundary problem.

The optimal solution depends on the values of factors α_i. The structure of the optimal control of the elastic two-span arch is shown in Fig. 8 for various sets of factors (versions W1–W4).

Table 6. Factors α_i for the cost function (33).

Version	α_i	p
W1	$\alpha_1 = 0.5, \alpha_2 = \alpha_3 = 0.25$	$0.182243 \cdot 10^{-3}$
W2	$\alpha_1 = 0.4, \alpha_2 = \alpha_3 = 0.3$	$0.214554 \cdot 10^{-3}$
W3	$\alpha_1 = 1/3, \alpha_2 = \alpha_3 = 1/3$	$0.236091 \cdot 10^{-3}$
W4	$\alpha_1 = 0.6, \alpha_2 = \alpha_3 = 0.2$	$0.149342 \cdot 10^{-3}$

Comparison of results, version **W4**

W4 $p = 0.149342 \cdot 10^{-3}, \quad U = U_{\text{opt}}$
W4 $p = 0.167442 \cdot 10^{-3}, \quad U = 1.0$

Table 7. Factors α_i for the cost function (34).

Version	α_i	p
W1	$\alpha_1 = 0.5, \alpha_2 = \alpha_3 = 0.25$	0.134480
W2	$\alpha_1 = 0.4, \alpha_2 = \alpha_3 = 0.3$	0.150201
W3	$\alpha_1 = 1/3, \alpha_2 = \alpha_3 = 1/3$	0.160229
W4	$\alpha_1 = 0.6, \alpha_2 = \alpha_3 = 0.2$	0.117407

Comparison of results, version **W4**

W4 $p = 0.117407, \quad U = U_{\text{opt}}$
W4 $p = 0.169452, \quad U = 1.0$

6. Conclusion

Optimal design problems for elastic bar systems are formulated as optimal control problems. Using the minimum principle, problems of the optimization of bar systems were reduced to boundary problems for ordinary differential equations. In general, multipoint boundary value problems (MPBVPs) were obtained in the tasks of the optimal design of bar systems. The dimensions of the resulting MPBVPs are very significant. In the present paper, a hybrid approach, i.e., the connection of the direct collocation method with that of the indirect multiple shooting, was applied. The indirect multiple shooting method exactly satisfies the necessary conditions of optimality and makes it possible to obtain a very accurate solution which satisfies the necessary conditions and becomes a candidate for the optimal solution. Checking the stability of the Hamilton function is an additional guarantee of correctness. At present, the majority of necessary conditions may be set down automatically, and preparatory works are facilitated. It is necessary to take into account certain limitations and imperfections of indirect methods:

– Prior to computations, preparation and setting down of MPBVPs (e.g., adjoint equations) require much effort.

– Methods are unusually sensitive to the estimated initial values for adjoint equations and controls.

– During the formulation of the MPBVPs it is necessary to know the sequence of the appearance of the right-hand sides of state equations. Obtaining a proper structure of the solution requires the use of homotopy in connection with the methods of collocation and multiple shooting. The numerical results confirm that optimal control methods may be successfully applied in the mechanics of bar structures.

References

Bulirsch R. and Montrone F. and Pesch H.J. (1991): *Abort landing in the presence of a windshear as a minimax optimal control problem. Part 1: Necessary conditions.* — Optim. Theory Applic., Vol. 70, No. 1, pp. 1–23.

Buss M., Glocker M., Hardt M., Von Stryk O., Bulirsch R., Schmidt G. (2002): *Nonlinear hybrid dynamical systems: Modeling, optimal control, and applications*, In: Modelling, Analysis and Design of Hybrid Systems: (S. Engell, G. Frehse, E. Schnieder, Eds.). — Berlin: Springer, pp. 311–335.

Hiltman P., Chudej K., Breitner M. (1993): *Eine modifizierte Mehrzielmethode zur Lösung von Mehrpunkt – Randwertproblemen – Benuzeranleitung.* — Sonderforschungsbereich 255 DFG, TU München, Report 14.

Hinsberger H. (1996): *Ein direktes Mehrschiessverfahren zur Lösung von Optimalsteuerungsproblemen – DIRMUS – Benutzeranleitung.* — TU Clausthal.

Malanowski K., Maurer,H. (1998): *Sensitivity analysis for optimal control problems subject to higher order state constraints.* — Echtzeit-Optimierung grosser Systeme, DFG, Preprint 98–5, 1–32,
available at: `www.zib.de/dfg-echtzeit`

Mikulski L. (1999): *Optimal design of elastic continuous structures.* — TU Cracow, Series Civil Engineering, Monograph 259.

Oberle H.J., Grimm, W. (1989): *BNDSCO – A program for the numerical solution of optimal control problems.* — Deutsche Forschungsanstalt für Luft und Raumfahrt, DLR IB 515-89/22, Oberpfaffenhofen.

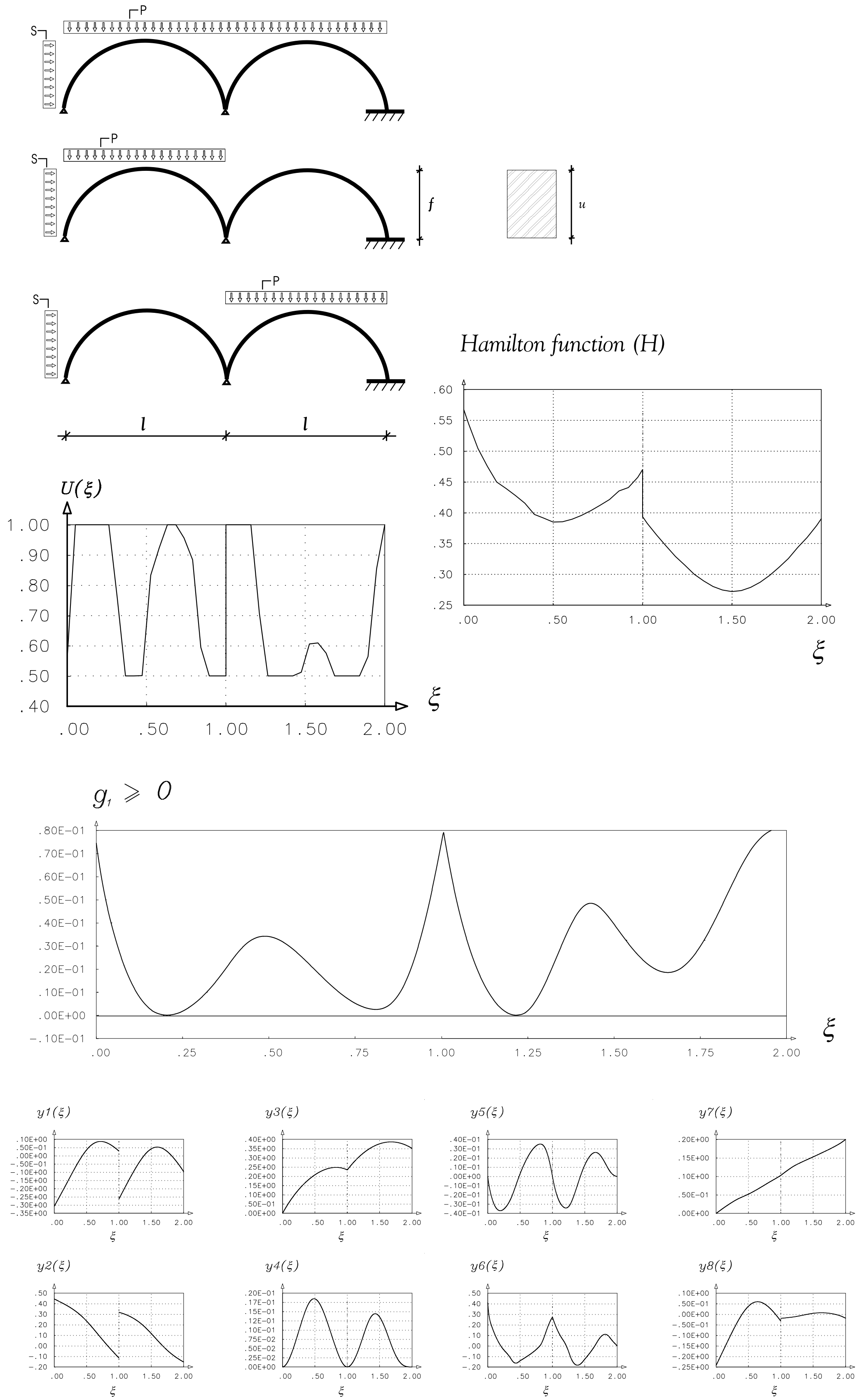


Fig. 6. Arch with multiple loads, optimal control.

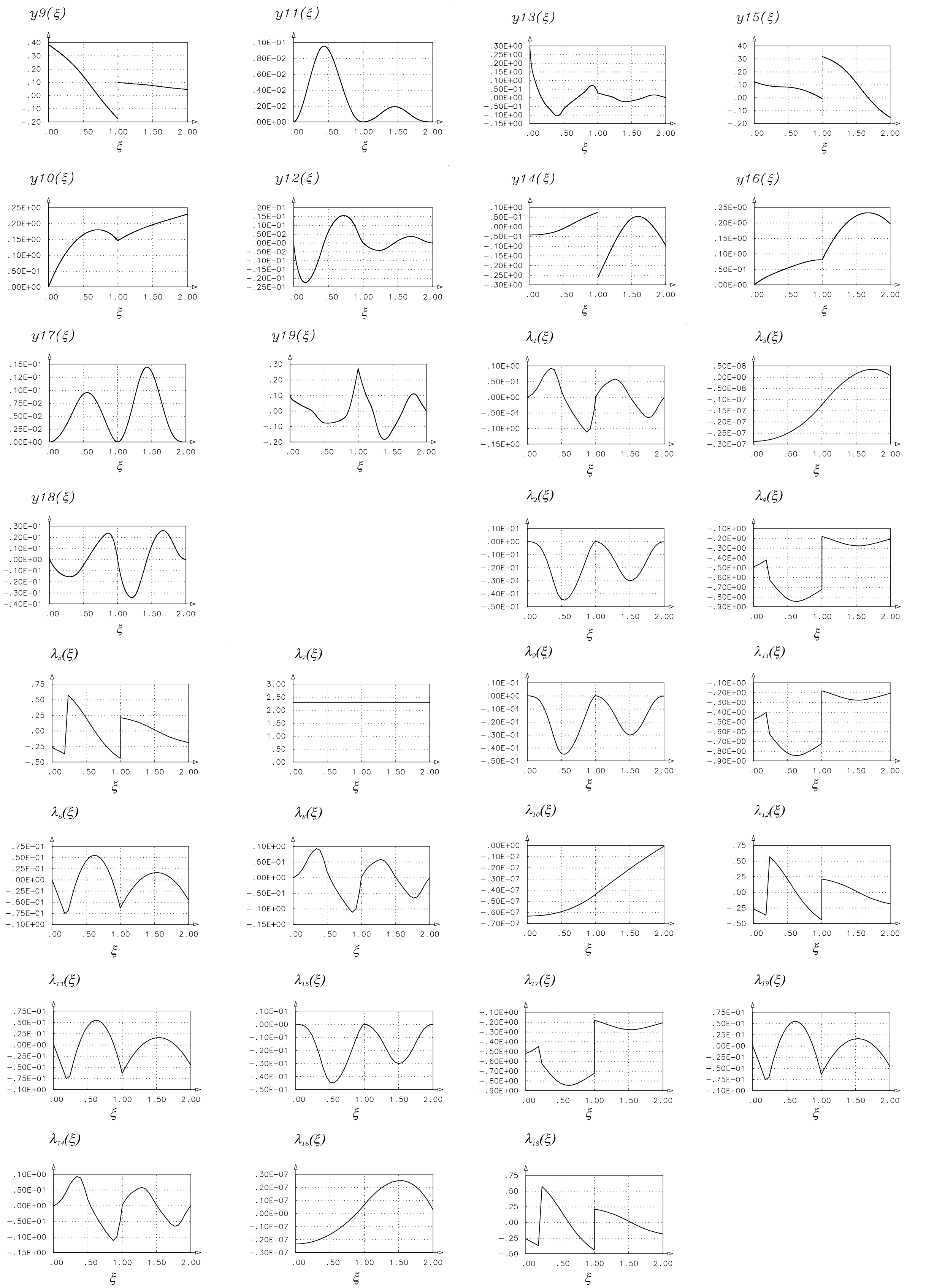
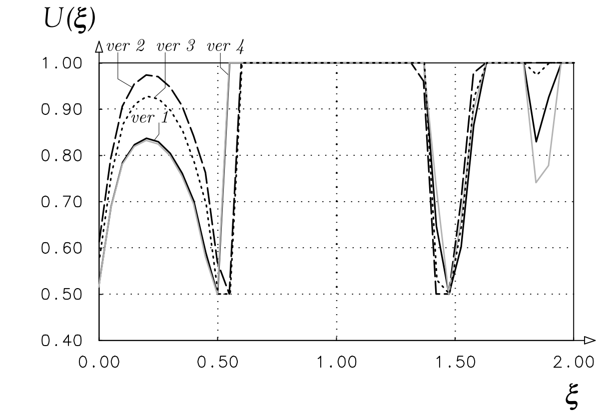


Fig. 7. Optimal solution of the arch y_j, λ_j.



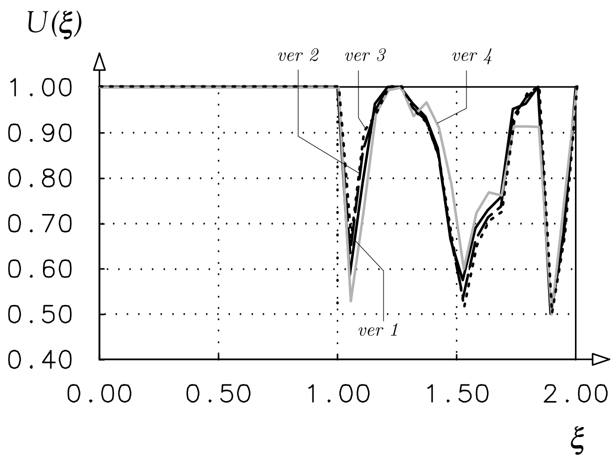


Fig. 8. Control structures dependent on the cost function and factors α_i.

Pesch H.J. (1994): *A practical guide to the solution of real-life optimal control problems*. — Contr. Cybern., Vol. 23, Nos. 1–2, pp. 7–60.

Pesch H.J. (2002): *Schlüssel Technologie Mathematik*. — Stuttgart-Leipzig-Wiesbaden: Teubner Verlag.

Von Stryk O. (2002): *User's guide DIRCOL — A direct collocation method for the numerical solution of optimal control problems*. — Technische Universität Darmstadt, Fachgebiet Simulation und Systemoptimierung (SIM), Ver. 2.1.