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ON ADAPTIVE CONTROL FOR THE CONTINUOUS TIME-VARYING JLQG PROBLEM

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In this paper the adaptive control problem for a continuous infinite time-varying stochastic control system with jumps in parameters and quadratic cost is investigated. It is assumed that the unknown coefficients of the system have limits as time tends to infinity and the boundary system is absolutely observable and stabilizable. Under these assumptions it is shown that the optimal value of the quadratic cost can be reached based only on the values of these limits, which, in turn, can be estimated through strongly consistent estimators.

Keywords: time varying systems, adaptive control, JLQG problem

1. Introduction

The problem of finding control that minimizes an ergodic, quadratic cost functional for a linear system with unknown parameters is probably the most well-known stochastic adaptive control problem. There is an extensive literature devoted to this problem. The latest publications dealing with this class of adaptation are by Duncan *et al.*, (1990), Guo (1996), and Prandini (2001).

In this paper a variation of this problem for systems with jump parameters is investigated. These models are characterized by their hybrid state space. To the usual Euclidean space, on which we model the basic dynamics x, we append a finite set S. Let r be a discrete Markov chain with a state space S. In applications, r, called the mode, is a labeling process indicating the context within which x evolves. Important research on these models is motivated by significant applications. This class of processes has been used successfully to model air traffic (Blom, 1990), manufacturing systems (Boukas and Harie, 1990), power systems (Sworder and Rogers, 1986), fault tolerant systems (Świerniak *et al.*, 1998), and multiplex redundant systems (Siljak, 1980), cf. also references therein.

For systems with jump parameters, adaptive control can be understood in two ways. In the first one we assume that the states of the Markov chain cannot be observed directly, but only partially through some noisy channel. This approach is presented in (Dufour and Elliot, 1998; Pan and Bar-Shalom, 1996). In this paper we consider a version of an alternative situation. More precisely, the objective of the paper is to discuss the jump linear quadratic problem for continuous time-varying systems with noise on an infinite time interval. We take into account a system with unknown coefficients having limits as functions of time when the time tends to infinity and the convergence is sufficiently fast in a sense, and we assume that the limits are known. First we show that the control for this system can be realized in the form of time-invariant feedback with the feedback matrix equal to the one for the time invariant system with coefficients equal to the limits of the time-varying system. To this end, we show that the solution of the time-varying differential Riccati equation converges in a certain sense to the solution of some time-invariant algebraic Riccati equation under very natural conditions. Based on this result, we solve an adaptive version of the problem. Similar results for a system without jumps are obtained in (Czornik, 1998; 1999) and for discrete time systems with jumps in (Czornik, 2001; 2002; 2004).

The problem of finding control that minimizes an ergodic, quadratic cost functional for a jump linear system with known parameters has been intensively studied in the literature and there now exists voluminous literature on the LQ control problem for systems with jumping parameters. Various formulations of the jump linear quadratic problem are considered in (Costa and Fragoso, 1995; Chizeck *et al.*, 1998; Ghosh, 1995; Griffiths, 1985; Ji and Chizeck, 1989; Mariton, 1987; Pan and Bar-Shalom, 1986; Rami El Ghaoui, 1996; Sworder, 1969; Sworder and Robinson, 1983), and coupled Riccati equations connected with this problem are studied in (Abou-Kandil *et* *al*. 1994; 1995; Czornik, 2000; Ji and Chizeck, 1988).

This paper is organized as follows: In Section 2 the jump linear quadratic problem with noise on a finite and an infinite time interval is revisited. Some properies of time-varying coupled differential Riccati equations related to the JLQ problem are discussed in Section 3. Finally, in Section 4 we use the results from Sections 2 and 3 to solve the adaptive control problem, and Section 5 contains concluding remarks.

2. Preliminaries

Consider the following stochastic differential equation:

$$\mathrm{d}x(t) = \Big(A\big(t,r(t)\big)x(t) + B\big(t,r(t)\big)u(t)\Big)\mathrm{d}t + C\big(t,r(t)\big)\mathrm{d}w(t), \tag{1}$$

for $t \geq 0$. Here $r(t)$ is a continuous time Markov chain taking values in a finite set S, with a generator $\Lambda = [q_{ij}(t)]_{i,j\in S}$ ($q_{ii}(t) = -q_i(t)$). Moreover $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are the state and the input of the process, $\{w(t), F_t\}$ is a standard Wiener process in $\mathbb{R}^n$ independent of $r(t)$. Let the initial values $x(0)$ and $r(0)$ be independent random variables; $x(0)$ is also independent of the σ-algebra generated by $\{r(t) : t \in [0,\infty]\}$. Moreover, we assume that for each $i \in S$ the functions $A(\cdot,i) : [0,\infty) \to \mathbb{R}^{n\times n}$, $B(\cdot,i) : [0,\infty) \to \mathbb{R}^{m\times n}$, $C(\cdot,i) : [0,\infty) \to \mathbb{R}^{n\times n}$ are locally integrable. The control is assumed to be in a feedback form:

$$u(t) = f\big(t, x(t), r(t)\big),$$

where $f : \mathbb{R}\times\mathbb{R}^n\times S \to \mathbb{R}^m$ is such that for some constant k,

$$\|f(t,x,i) - f(t,y,i)\| \leq k\,\|x-y\|,$$
$$\|f(t,x,i)\| \leq k\,(1+\|x\|),$$

for all $t\in\mathbb{R}$, $x\in\mathbb{R}^n$, $i\in S$. The cost to be minimized is given by

$$\begin{aligned} &J(x_0, i_0, T, u) \\ &= E\Big(\int_0^T \Big[\langle Q(t,r(t))x(t), x(t)\rangle \\ &+ \langle R(t,r(t))\,u(t), u(t)\rangle\Big]\mathrm{d}t \Big| x(0)=x_0, r(0)=i_0\Big) \\ &+ E\,\langle F(r(T))\,x(T), x(T)\rangle, \end{aligned} \tag{2}$$

where, for each $i \in S$, the functions $Q(\cdot,i) : [0,\infty) \to \mathbb{R}^{n\times n}$, $R(\cdot,i) : [0,\infty) \to \mathbb{R}^{m\times m}$ are locally integrable and the matrices $Q(t,i), F(i)$ are nonnegative and $R(t,i)$ are positive definite for each $t\in[0,\infty)$ and $i\in S$. Using standard methods (see, e.g., Mariton, 1990), we can establish the following result:

Theorem 1. *The optimal control law for the problem (1)–(2) is given by*

$$u(t,i) = -L(t,i)x(t), \tag{3}$$

where

$$L(t,i) = R^{-1}(t,i)B'(t,i)K(T,t,i,F(i)) \tag{4}$$

for $r(t)=i$ *and* $K(T,\cdot,i,F(i)) : [0,T] \to \mathbb{R}^{n\times n}, i \in S$ *is the unique solution of the coupled differential Riccati equation*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}K(T,t,i,F(i)) \\ &= Q(T-t,i) + K(T,t,i,F(i))\,A(T-t,i) \\ &\quad + A'(T-t,i)\,K(T,t,i,F(i)) \\ &\quad + K(T,t,i,F(i))\,B(T-t,i)\,R^{-1}(T-t,i) \\ &\quad \times B'(T-t,i)\,K(T,t,i,F(i)) \\ &\quad - q_i(T-t)K(T,t,i,F(i)) \\ &\quad + \sum_{j\neq i} q_{ij}(T-t)K(T,t,j,F(j)), \end{aligned} \tag{5}$$

with the terminal conditions

$$K(T,0,i,F(i)) = F(i), \quad i\in S. \tag{6}$$

The value of the optimal cost is given by

$$\langle K(T,T,i_0,F(i_0))\,x_0, x_0\rangle + \mu(T,T,i_0), \tag{7}$$

where

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mu(T,t,i) \\ &= \mathrm{tr}\Big(C'(T-t,i)K\big(T,t,i,F(i)C(T-t,i)\big)\Big) \\ &\quad + \sum_{j\in S} q_{ij}(T-t)\mu(T,t,j) \end{aligned}$$

with the initial conditions $\mu(T,0,i)=0,\ i\in S$.

Taking $B(\cdot,i)\equiv 0, i\in S$, from the last part of the above theorem we obtain that for any $M(t,i)>0$, $i\in S$, $t\in[0,\infty)$ the following formula is true:

$$\begin{aligned} &E\Big(\int_0^t \langle M(s,r(s))\,x(s), x(s)\rangle\,\mathrm{d}s \Big| x(0)=x_0, \\ &r(0)=i_0\Big) = \Big\langle \widetilde{K}(T,t,i_0)x_0, x_0\Big\rangle + \widetilde{\mu}(T,t,i_0), \end{aligned} \tag{8}$$

where x is the solution of (1) with $B(\cdot,i) \equiv 0, i \in S$ and $\widetilde{K}(T,t,i,0)$, $\widetilde{\mu}(T,t,i)$, $i \in S$ are the solutions of

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{K}(T,t,i,0)\\&= M(T-t,i) + \widetilde{K}(T,t,i,0)A(T-t,i)\\&\quad +A'(T-t,i)\widetilde{K}(T,t,i,0)\\&\quad -q_i(T-t)\widetilde{K}(T,t,i,0) + \sum_{j\neq i} q_{ij}(T-t)\widetilde{K}(T,t,j),\\&K(T,0,i,0)=0,\end{aligned}$$

and

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mu}(T,t,i)\\&\qquad = \operatorname{tr}\Big(C'(T-t,i)\widetilde{K}(T,t,i,0)C(T-t,i)\Big)\\&\qquad\quad + \sum_{j\in S} q_{ij}(T-t)\widetilde{\mu}(T,t,j),\\&\widetilde{\mu}(T,0,i)=0.\end{aligned}$$

The next lemma concerns a simple property of the solution of the coupled differential Riccati equation which will be used in our further discussion.

Lemma 1. *For each $i \in S$ and $T > 0$, if $0 \le t_1 < t_2 \le T$, then $K(T,t_1,i,0) \le K(T,t_2,i,0)$.*

Proof. Let $C(\cdot,i) \equiv 0, i \in S$, then the proof follows easily from (2) and (7). ■

Now consider the problem with the infinite-time interval $[0,\infty)$. As the cost functional for this case we take

$$\begin{aligned}&J(x_0,i_0,u)\\&=\lim_{T\to\infty}\frac{1}{T}E\Big(\int_0^T\Big[\langle Q(t,S(t))x(t),x(t)\rangle \qquad (9)\\&\quad + \big\langle R\big(t,S(t)\big)u(t),u(t)\big\rangle\Big]\mathrm{d}t\Big|x(0)=x_0,r(0)=i_0\Big).\end{aligned}$$

For the noise-free system ($C(i)=0, i\in S$), as the cost functional on the infinite time interval we take

$$\begin{aligned}&J(x_0,i_0,u)\\&=\lim_{T\to\infty}E\Big(\int_0^T\Big[\big\langle Q\big(t,S(t)\big)x(t),x(t)\big\rangle \qquad (10)\\&\quad + \big\langle R\big(t,S(t)\big)u(t),u(t)\big\rangle\Big]\mathrm{d}t\Big|x(0)=x_0,r(0)=i_0\Big).\end{aligned}$$

Definition 1. We call the noise-free system

$$\{A(t,i),\ B(t,i),\ Q(t,i),\ R(t,i), q_{ij}(t) : i,j\in S\}$$

optimizable if, for every $(x_0,i_0)\in\mathbb{R}^n\times S$, there exists control u such that $J(x_0,i_0,u)<\infty$.

Lemma 2. *If the system*

$$\{A(t,i),\ B(t,i),\ Q(t,i),\ R(t,i), q_{ij}(t); i,j\in S\}$$

is optimizable, then there exists a constant $c>0$ such that

$$\|K(T,t,i,0)\| < c \qquad (11)$$

for any $T>0, t\in[0,T]$ and $i\in S$.

Proof. Fix $(x_0,i_0)\in\mathbb{R}^n\times S$ and let $\widetilde{u}$ be such that $J(x_0,i_0,\widetilde{u}) < \infty$. Consider the control problem (1) and (9). Then the proof is a straightforward consequence of Lemma 1 and the inequality

$$\langle K(T,T,i_0,0)x_0,x_0\rangle \le J(x_0,i_0,\widetilde{u}).$$ ■

Consider now the time-invariant control problem with

$$A(t,i)\equiv A(i),\quad B(t,i)\equiv B(i),\quad C(t,i)\equiv C(i),$$
$$Q(t,i)\equiv Q(i),\quad R(t,i)\equiv R(i),\quad q_{ij}(t)=q_{ij}.$$

In this case we can omit the index T in $K(T,t,i,F(i))$ and $\mu(T,t,i)$.

From the definition of optimalizability it is clear that it is a necessary condition for the existence of a solution to the JLQ problem on the infinite time interval. The next theorem shows that it is also a sufficient condition.

Theorem 2. *If the system*

$$\{A(i),\ B(i),\ Q(i),\ R(i), q_{ij};\quad i,j\in S\}$$

is optimizable, then the coupled algebraic Riccati equation

$$\begin{aligned}&Q(i)+K(i)A(i)+A'(i)K(i)\\&\quad -K(i)B(i)R^{-1}(i)B'(i)K(i)-q_iK(i)\\&\quad +\sum_{j\neq i}q_{ij}K(j)=0 \qquad (12)\end{aligned}$$

has a positive-semidefinite solution. In the set of positive-semidefinite solutions of (12) there exists a minimal solution $\{K_0(i): i\in S\}$. Moreover, $K_0(i)$ is the limit of $K(t,i,0)$ as $t\to\infty$ and the optimal control for the problem (1)–(9) is given by

$$\widetilde{u}(t) = -R^{-1}(i)B'(i)K_0(i)x(t), i\in S. \qquad (13)$$

Proof. From Lemmas 1 and 2 we conclude that the limit

$$\lim_{t\to\infty} K(t,i,0) = K_0(i) \tag{14}$$

exists. Taking the limit in both sides of (5) we see that $K_0(i), i \in S$ is indeed a solution of (12).

Now we show that for two given positive-semidefinite matrices X and Y such that $X \leq Y$, the inequality

$$K(t,i,X) \leq K(t,i,Y) \tag{15}$$

holds for all $t \geq 0$ and $i \in S$. For this purpose, consider the system (1) with $C(i) = 0$ and write $J_1(x_0, i_0, T, u)$ and $J_2(x_0, i_0, T, u)$ for the cost functional (2) with $F(i) = X$ and $F(i) = Y, i \in S$. Using $X \leq Y$ it follows that

$$J_1(x_0, i_0, T, u) \leq J_2(x_0, i_0, T, u)$$

for any control u. Hence, using Theorem 1, from (15) we conclude (7). Note that (14) and (15) make it obvious that $\{K_0(i) : i \in S\}$ is a minimal solution of (12).

It remains to solve the optimal control problem. For every control u we have

$$\frac{1}{T} J(x_0, i_0, T, u) \leq J(x_0, i_0, u),$$

where $J(x_0, i_0, T, u)$ is given by (2) with $F(i) = 0$ and $J(x_0, i_0, u)$ defined by (9). Consequently,

$$\begin{aligned}\inf_u J(x_0, i_0, u) &\geq \inf_u \frac{1}{T} J(x_0, i_0, T, u)\\ &= \frac{1}{T}\left(\langle K(T, i_0, 0)\, x_0, x_0\rangle + \mu(T, i_0)\right).\end{aligned}$$

Since the above inequality is true for all T, we may take the limit as T tends to infinity, which gives

$$\inf_u J(x_0, i_0, u) \geq \operatorname{tr}(C'(i_0)\, K(i_0)\, C(i_0)),$$

because from Lemma 1 we know that the functions $K(\cdot, i, 0), i \in S$ are bounded on $[0, \infty)$. An easy computation of (8) shows that for the control given by (13) the cost functional takes the value equal to $\operatorname{tr}(C'(i_0)\, K(i_0)\, C(i_0))$. ■

Remark 1. The control given by (13) is also optimal for the noise-free system with the cost functional (10), c.f. (Czornik and Świerniak, 2004).

3. Coupled Differential Riccati Equation

For the proof of the next lemma, we refer the reader to (Wonham, 1971).

Lemma 3. *Let $\overline{K}(T,t,i,F(i))$ be the solution of the equation*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\overline{K}(T,t,i,F(i))\\ &= \left(A(T-t,i) - B(T-t,i)L(t,i) - \frac{1}{2}q_i(T-t)I\right)'\\ &\quad \times \overline{K}(T,t,i,F(i)) + \overline{K}(T,t,i,F(i))\\ &\quad \times \left(A(T-t,i) - B(T-t,i)L(t,i) - \frac{1}{2}q_i(T-t)I\right)\\ &\quad + L'(t,i)R(T-t,i)L(t,i)\\ &\quad + \sum_{j\neq i} q_{ij}(T-t)\overline{K}(T,t,j,F(j))\\ &\quad + Q(T-t,i), \quad i \in S, \qquad (16)\\ &\overline{K}(T,0,i,F(i)) = F(i),\end{aligned}$$

where $L(\cdot,i) : [0,\infty) \to \mathbb{R}^{m\times n}$ is a locally integrable function. Then

$$\overline{K}(T,t,i,F(i)) \leq K(T,t,i,F(i)), \tag{17}$$

where $K(T,t,i,F(i))$ is the solution of (5).

Consider the time invariant noise-free system

$$\mathrm{d}x(t) = (A(S(t))\, x(t) + B(S(t))\, u(t))\, \mathrm{d}t, \tag{18}$$

with the output equation

$$y(t) = C(r(t))x(t). \tag{19}$$

The next definition of the stochastic stability and observability of (18)–(19) is given in (Ji and Chizeck, 1990). More details on this topic can be found in (Feng *et al.*, 1992).

Definition 2. We say that the system $\{A(i), B(i), q_{ij} : i,j \in S\}$ is *stochastically stabilizable* if, for all $(x_0, r_0) \in \mathbb{R}^n \times S$, there exists a function $L : S \to \mathbb{R}^{m\times n}$ such that for the control

$$u(t) = L(r(t))x(t)$$

there exists a symmetric positive definite matrix M satisfying

$$\lim_{T\to\infty} E\left(\int_0^T \|x(t)\|\, \mathrm{d}t \,\Big|\, x(0) = x_0, r(0) = i_0\right) \leq \langle M x_0, x_0\rangle.$$

We say that the system $\{A(i), C(i), r(t), q_{ij} : i, j \in S\}$ is *observable* if, and only if, for each $i \in S$, the pair $(C(i), A(i))$ is observable.

The following result on the convergence of a solution of the differential Riccati equation to the solution of the algebraic Riccati equation is a slight generalization of the results from (Abou-Kandil *et al.*, 1994; Ji and Chizeck, 1990).

Theorem 3. *If the system $\{A(i), B(i), q_{ij} : i, j \in S\}$ is stochastically stabilizable and the system $\{A(i), C(i), q_{ij} : i, j \in S\}$ is observable, then for any initial values*

$$\{F_0(i) : i \in S\}$$

we have

$$\lim_{t\to\infty} K\left(t, i, F_0(i)\right) = K(i),$$

where $\{K\left(t, i, F_0(i)\right), i \in S\}$ is the solution of the time invariant Riccati equation (5) and $\{K\left(i\right), i \in S\}$ is the unique positive-definite solution of the coupled algebraic Riccati equation (12), and the convergence is uniform on the set

$$\{F_0(i) : \|F_0(i)\| < c, i \in S\}$$

for any positive constant $c > 0$.

Proof. The uniqueness and positive definiteness of the solution is guaranteed by Theorem 5 in (Ji and Chizeck, 1990).

Consider again the system (1) with $C(i) = 0$. Let $\widetilde{x}$ be the solution of (1) corresponding to the optimal control $\widetilde{u}$ for the problem (1), (10) given by (13). In (Ji and Chizeck, 1990) it is shown that

$$\lim_{t\to\infty} E\left(\langle \widetilde{x}(t), \widetilde{x}(t)\rangle \middle| x(0) = x_0, r(0) = i_0\right) = 0. \quad (20)$$

Fix $c > 0$, set $\{F_0(i) : \|F_0(i)\| < c, i \in S\}$ and consider two control problems (1)–(2): the first with $F(i) = F_0(i), i \in S$ and the second with $F(i) = 0, i \in S$. Let $J_l\left(x_0, i_0, T, u\right)$ and $\widetilde{u}^{(l)}$, $l = 1, 2$, denote cost functionals and optimal control, respectively, for these two problems. According to the definition of the cost functional, we have

$$\begin{aligned}
&J_1\left(x_0, i_0, T, \widetilde{u}^{(1)}\right)\\
&\leq J_2\left(x_0, i_0, T, \widetilde{u}^{(2)}\right)\\
&\leq \Big\{E\Big(\int_0^T \Big[\langle Q\big(r(t)\big)\widetilde{x}(t), \widetilde{x}(t)\rangle\\
&+ \langle R\big(r(t)\big)\widetilde{u}(t), \widetilde{u}(t)\rangle\Big]\mathrm{d}t\Big| x(0) = x_0, r(0) = i_0\Big)\\
&+ E\left\langle F\big(r(T)\big)\widetilde{x}(T), \widetilde{x}(T)\right\rangle\Big\}.
\end{aligned}$$

From Theorems 1 and 2 taken in conjunction with (20) we know that the right and left-hand sides of the above inequality tend to $K(i_0)$ uniformly on the set $\{F_0(i) : \|F_0(i)\| < c, i \in S\}$. This proves the theorem. ■

The next theorem contains the main result of this section.

Theorem 4. *Assume that the system*

$$\Big\{A\left(t, i\right), \; B\left(t, i\right), \; C\left(t, i\right), \; Q\left(t, i\right), \; R\left(t, i\right), q_{ij}(t); \; t \in [0, \infty), i, j \in S\Big\}$$

is such that

1. $\int_0^\infty \|A(t, i) - A(i)\|\,\mathrm{d}t < \infty,$

   $\int_0^\infty \|B(t, i) - B(i)\|\,\mathrm{d}t < \infty,$

   $\int_0^\infty \|C(t, i) - C(i)\|\,\mathrm{d}t < \infty,$

   $\int_0^\infty \|Q(t, i) - Q(i)\|\,\mathrm{d}t < \infty,$

   $\int_0^\infty \|R(t, i) - R(i)\|\,\mathrm{d}t < \infty,$

   $\int_0^\infty |q_{ij}(t) - q_{ij}|\,\mathrm{d}t < \infty;$

2. *$R(i) > 0$, $i \in S$, where $R(i) = \lim_{t\to\infty} R(t, i)$;*

3. *$\{A(i), B(i), q_{ij}, i, j \in S\}$ is stochastically stabilizable, where*

   $$A(i) = \lim_{t\to\infty} A(t, i), \quad B(i) = \lim_{t\to\infty} B(t, i),$$
   $$q_{ij} = \lim_{t\to\infty} q_{ij}(t);$$

4. *$\Big\{A(i), \sqrt{Q(i)}, q_{ij}, i, j \in S\Big\}$ is observable, where $Q(i) = \lim_{t\to\infty} Q(t, i)$.*

Then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T K\left(T, t, i, F(i)\right)\mathrm{d}t = K(i), \quad (21)$$

for any initial condition $\{F(i) : F(i) > 0, i \in S\}$, where $K\left(T, t, i, F(i)\right), i \in S$ are given by (5) and $K(i), i \in S$ are the unique solutions of (12).

Proof. For the simplicity of our future notation, we write

$$
\begin{aligned}
&L\left(A, B,\left\{K_i, q_{ij}: i, j \in S\right\}, Q, R\right)\\
&\quad= Q(i)+K(i) A(i)+A^{\prime}(i) K(i)\\
&\qquad+K(i) B(i) R^{-1}(i) B^{\prime}(i) K(i)-q_i K(i)\\
&\qquad+\sum_{j \neq i} q_{ij} K(j).
\end{aligned}
$$

Using this notation, we can rewrite (5) and (12) as

$$
\begin{aligned}
&K(T, t, i, F(i))\\
&\quad= F(i)+\int_0^t L\Big(A(T-s, i), B(T-s, i),\\
&\qquad\Big\{K(T, s, i, F(i)), q_{ij}(T-s): i, j \in S\Big\},\\
&\qquad Q(T-s, i), R(T-s, i)\Big) \mathrm{d} s
\end{aligned} \tag{22}
$$

and

$$
L\left(A(i), B(i),\left\{K(i), q_{ij}: i, j \in S\right\}, Q(i), R(i)\right)=0. \tag{23}
$$

Together with (22) and (23) we will consider the following time invariant coupled Riccati equation:

$$
\begin{aligned}
&K(t, i, F(i))\\
&\quad= F(i)+\int_0^t L\Big(A(i), B(i),\\
&\qquad\Big\{K(t, i, F(i)), q_{ij}: i, j \in S\Big\}, Q(i), R(i)\Big) \mathrm{d} t.
\end{aligned} \tag{24}
$$

Let $\overline{K}(T, t, i, F(i))$ be the solution of (16) with $L(t, i) \equiv L(i)=R^{-1}(i) B^{\prime}(i) K(i)$. Then

$$
\begin{aligned}
&\overline{K}(T, t, i, F(i))\\
&\quad= \Phi^{\prime}(T, T-t, i) F(i) \Phi(T, T-t, i)\\
&\qquad+\int_0^t \Phi^{\prime}(T-s, T-t, i)\Big(L^{\prime}(i) R(T-s, i) L(i)\\
&\qquad+\sum_{j \neq i} q_{ij}(T-s) \overline{K}(T, s, j, F(j))\\
&\qquad+Q(T-s, i)\Big) \Phi(T-s, T-t, i) \mathrm{d} s\Big),
\end{aligned} \tag{25}
$$

where $\Phi(t, s, i),\ i \in S$ is the transition matrix of the equation

$$
\frac{\mathrm{d}}{\mathrm{d} t} z(t)=\Big(A(t, i)-B(t, i) L(i)-\frac{1}{2} q_i(t) I\Big) z(t). \tag{26}
$$

In (Ji and Chizeck, 1990) it was shown that the matrices $A(i)-B(i) R^{-1}(i) B^{\prime}(i) K(i)-\frac{1}{2} q_i I,\ i \in S$ are stable.

By Assumption 1 we have

$$
\begin{aligned}
&\int_0^{\infty}\Big\|\Big(A(t, i)-B(t, i) L(i)-\frac{1}{2} q_i(t) I\Big)\\
&\quad-\Big(A(i)-B(i) R^{-1}(i) B^{\prime}(i) K(i)-\frac{1}{2} q_i I\Big)\Big\| \mathrm{d} t<\infty.
\end{aligned}
$$

Consequently (see, Afanas’ev *et al.*, 1989), (26) is stable and

$$
\|\Phi(t, s, i)\| \leq a e^{-b(t-s)}
$$

for some positive constants a and b. From this fact and (25) we can conclude that

$$
\|S(T, t)\| \leq c_1+c_2 \int_0^t e^{-2 b(t-s)} \mathrm{d} s.
$$

For any set of the initial values $\{F(i): F(i)>0, i \in S\}$ there is a constant c such that

$$
\begin{aligned}
&\|K(T, t, i, F(i))\|<c,\\
&\|K(s, i, K(T, t, i, F(i)))\|<c
\end{aligned} \tag{27}
$$

for any $T, s>0,\ t \in[0, T]$ and $i \in S$. Assumption 1 makes it obvious that

$$
\begin{aligned}
&\lim_{t \rightarrow \infty}\|A(t, i)-A(i)\|=0,\\
&\lim_{t \rightarrow \infty}\|B(t, i)-B(i)\|=0,\\
&\lim_{t \rightarrow \infty}\|C(t, i)-C(i)\|=0,\\
&\lim_{t \rightarrow \infty}|q_{ij}(t)-q_{ij}|=0,\\
&\lim_{t \rightarrow \infty}\|Q(t, i)-Q(i)\|=0,\\
&\lim_{t \rightarrow \infty}\|R(t, i)-R(i)\|=0.
\end{aligned} \tag{28}
$$

From this and (27) it follows that for any $\varepsilon>0$ we can find $T_0(\varepsilon)$ such that

$$
\int_0^t\|\Delta(T, s, i)\| \mathrm{d} s<\varepsilon, \tag{29}
$$

for $T, t>0, T-t>T_0(\varepsilon)$, where

$$
\begin{aligned}
&\Delta(T, s, i)\\
&\quad= L\Big(A(i), B(i),\\
&\qquad\Big\{K(T, s, i, F(i)), q_{ij}: i, j \in S\Big\},\\
&\qquad Q(i), R(i)\Big)-L\Big(A(T-s, i), B(T-s, i),\\
&\qquad\Big\{K(T, s, i, F(i)), q_{ij}(T-s): i, j \in S\Big\},\\
&\qquad Q(T-s, i), R(T-s, i)\Big).
\end{aligned} \tag{30}
$$

We next show that for each $\varepsilon > 0$ and $t > 0$ there is a constant $c(t) > 0$ (depending only on t) such that

$$\|K\,(T, t+\tau, i, F(i)) - K\,(t, i, K\,(T, \tau, i, F(i)))\| < \varepsilon c(t), \quad (31)$$

for $T, \tau > 0, T - \tau - t > T_0(\varepsilon)$. Using the notation (30), we can rewrite (22) as

$$\begin{aligned} &K\,(T, t+\tau, i, F(i)) \\ &= F(i) + \int_0^{t+\tau} \Big(L\big(A(i), B(i), \\ &\Big\{K\big(T, s, i, F(i)\big), q_{ij} : i, j \in S\Big\} Q(i), R(i)\big), \\ &-\Delta\,(T, s, i)\Big)\mathrm{d}s. \end{aligned} \quad (32)$$

Then from (29) it follows that

$$\begin{aligned} &\Big\|K\big(T, t+\tau, i, F(i)\big) - K\big(t, i, K\big(T, \tau, i, F(i)\big)\big)\Big\| \\ &\le \Big\|F(i) + \int_0^{t+\tau} \big(L\big(A(i), B(i), \Big\{K\big(T, s, i, F(i)\big), \\ &\quad q_{ij} : i, j \in S\Big\}, Q(i), R(i)\big) - \Delta\big(T, s, i\big)\big)\mathrm{d}s \\ &\quad - \int_0^t L\big(A(i), B(i), \Big\{K\big(s, i, K\big(T, \tau, i, F(i)\big)\big), \\ &\quad q_{ij} : i, j \in S\Big\}, Q(i), R(i)\big)\mathrm{d}s - K\big(T, \tau, i, F(i)\big)\Big\| \\ &= \Big\|\int_0^{t+\tau} \big(L\big(A(i), B(i), \Big\{K\big(T, s, i, F(i)\big), \\ &\quad q_{ij} : i, j \in S\Big\}, Q(i), R(i)\big) - \Delta\big(T, s, i\big)\big)\mathrm{d}s \\ &\quad - \int_0^t L\big(A(i), B(i), \Big\{K\big(s, i, K\big(T, \tau, i, F(i)\big)\big), \\ &\quad q_{ij} : i, j \in S\Big\}, Q(i), R(i)\big)\mathrm{d}s \\ &\quad - \int_0^\tau L\big(A(T-s, i), B(T-s, i), \Big\{K\big(T, s, i, F(i)\big), \\ &\quad q_{ij}(T-s) : i, j \in S\Big\} Q(T-s, i), R(T-s, i)\big)\mathrm{d}s\Big\| \\ &\le \varepsilon + \Big\|\int_0^{t+\tau} L\big(A(i), B(i), \{K\big(T, s, i, F(i)\big), \\ &\quad q_{ij} : i, j \in S\Big\}, Q(i), R(i)\big)\mathrm{d}s \\ &\quad - \int_0^t L\big(A(i), B(i), \Big\{K\big(s, i, K\big(T, \tau, i, F(i)\big)\big), \\ &\quad q_{ij} : i, j \in S\Big\}, Q(i), R(i)\big)\mathrm{d}s \\ &\quad - \int_0^\tau L\big(A(T-s, i), B(T-s, i), \Big\{K\big(T, s, i, F(i)\big), \\ &\quad q_{ij}(T-s) : i, j \in S\Big\} Q(T-s, i), R(T-s, i)\big)\mathrm{d}s\Big\|. \end{aligned}$$

From (29) and (30) it is clear that

$$\begin{aligned} &\Big\|K\big(T, t+\tau, i, F(i)\big) - K\big(t, i, K\big(T, \tau, i, F(i)\big)\big)\Big\| \\ &\le \varepsilon + \Big\|\int_0^{t+\tau} L\big(A(i), B(i), \Big\{K\big(T, s, i, F(i)\big), \\ &\quad q_{ij} : i, j \in S\Big\}, Q(i), R(i)\big)\mathrm{d}s \\ &\quad - \int_0^t L\big(A(i), B(i), \Big\{K\big(s, i, K\big(T, \tau, i, F(i)\big)\big), \\ &\quad q_{ij} : i, j \in S\Big\}, Q(i), R(i)\big)\mathrm{d}s \\ &\quad - \int_0^\tau \big(L\big(A(i), B(i), \Big\{K\big(T, s, i, F(i)\big), \\ &\quad q_{ij} : i, j \in S\Big\}, Q(i), R(i)\big) - \Delta\big(T, s, i\big)\big)\mathrm{d}s\Big\| \\ &\le 2\varepsilon + \Big\|\int_0^{t+\tau} L\big(A(i), B(i), \Big\{K\big(T, s, i, F(i)\big), \\ &\quad q_{ij} : i, j \in S\Big\}, Q(i), R(i)\big)\mathrm{d}s \\ &\quad - \int_0^t L\big(A(i), B(i), \Big\{K\big(s, i, K\big(T, \tau, i, F(i)\big)\big), \\ &\quad q_{ij} : i, j \in S\Big\}, Q(i), R(i)\big)\mathrm{d}s \\ &\quad - \int_0^\tau L\big(A(i), B(i), \Big\{K\big(T, s, i, F(i)\big), \\ &\quad q_{ij} : i, j \in S\Big\}, Q(i), R(i)\big)\mathrm{d}s\Big\| \\ &= 2\varepsilon + \Big\|\int_\tau^{t+\tau} L\big(A(i), B(i), \Big\{K\big(T, s, i, F(i)\big), \\ &\quad q_{ij} : i, j \in S\Big\}, Q(i), R(i)\big)\mathrm{d}s \\ &\quad - \int_0^t L\big(A(i), B(i), \Big\{K\big(s, i, K\big(T, \tau, i, F(i)\big)\big), \\ &\quad q_{ij} : i, j \in S\Big\}, Q(i), R(i)\big)\mathrm{d}s\Big\| \\ &= 2\varepsilon + \Big\|\int_0^t L\big(A(i), B(i), \Big\{K\big(T, s+\tau, i, F(i)\big), \\ &\quad q_{ij} : i, j \in S\Big\}, Q(i), R(i)\big)\mathrm{d}s \end{aligned}$$

$$- \int_0^t L\big(A(i), B(i), \Big\{K\big(s, i, K\big(T, \tau, i, F(i)\big)\big), q_{ij} : i, j \in S\Big\}, Q(i), R(i)\big)\mathrm{d}s\Big\|. \quad (33)$$

It is not difficult to check that there is a constant $\widetilde{c} > 0$ such that

$$\big\|L\big(A, B, \{U_i, q_{ij} : i, j \in S\}, Q, R\big) - L\big(A, B, \{V_i, q_{ij} : i, j \in S\}, Q, R\big)\big\| \le \widetilde{c} \max_{j \in S} \|U_j - V_j\| \quad (34)$$

for any sets $\{U_i : i \in S\}$, $\{V_i : i \in S\}$ of symmetric positive definite matrices for which

$$\max_{j \in S} \|U_j - V_j\| < c,$$

where c is the constant from (27). Using the bound (34) to (33), we have

$$\big\|K(T, t+\tau, i, F(i)) - K(t, i, K(T, \tau, i, F(i)))\big\| \le 2\varepsilon + \widetilde{c} \max_{j \in S} \Big\| \int_0^t \big(K(T, s+\tau, j, F(j)) - K(s, j, K(T, \tau, j, F(j)))\big)\mathrm{d}s\Big\|$$

and

$$\max_{j \in S} \Big\|K(T, t+\tau, i, F(i)) - K(t, i, K(T, \tau, i, F(i)))\Big\| \le 2\varepsilon + \widetilde{c} \max_{j \in S} \Big\| \int_0^t \big(K(T, s+\tau, j, F(j)) - K(s, j, K(T, \tau, j, F(j)))\big)\mathrm{d}s\Big\|. \quad (35)$$

Applying Gronwall's lemma to (35), we obtain

$$\max_{j \in S} \Big\|K\big(T, t+\tau, i, F(i)\big) - K\big(t, i, K(T, \tau, i, F(i))\big)\Big\| \le 2\varepsilon e^{\widetilde{c}t},$$

and the proof of (31) is complete.

Fix $\delta > 0$. By Theorem 4 we can take $t_0 > 0$ such that

$$\Big\|K\big(t, i, \widetilde{F}(i)\big) - K(i)\Big\| < \frac{\delta}{2}, \quad (36)$$

for $t > t_0$, $i \in S$ and $\max_{i \in S} \Big\|\widetilde{F}(i)\Big\| < c$. For $\varepsilon = \delta/2c(t_0)$, where $c(t_0)$ is the constant from (31), take $T_0(\varepsilon)$ according to (29). From (31) and (36) we have

$$\begin{aligned}
&\Big\|K\big(T, t+t_0, i, F(i)\big) - K(i)\Big\| \\
&\le \Big\|K\big(T, t+t_0, i, F(i)\big) - K\big(t_0, i, K(T, t, i, F(i))\big)\Big\| \\
&\quad + \Big\|K\big(t_0, i, K(T, t, i, F(i))\big) - K(i)\Big\| \\
&\le c(t_0)\frac{\delta}{c(t_0)} + \frac{\delta}{2} = \delta,
\end{aligned} \quad (37)$$

for $T, t > 0$ such that $T - t > T_0(\varepsilon) + t_0$. Then, taking into account (27) and (37), we obtain

$$\begin{aligned}
&\lim_{T\to\infty} \sup \frac{1}{T}\Big\| \int_0^T \big(K\big(T, s, i, F(i)\big) - K(i)\big)\mathrm{d}s\Big\| \\
&\le \lim_{T\to\infty} \sup \frac{1}{T} \int_0^{t_0} \Big\|K\big(T, s, i, F(i)\big) - K(i)\Big\|\mathrm{d}s \\
&\quad + \lim_{T\to\infty} \sup \frac{1}{T} \int_{t_0}^{T - T_0(\varepsilon)} \Big\|K\big(T, s, i, F(i)\big) - K(i)\Big\|\mathrm{d}s \\
&\quad + \lim_{T\to\infty} \sup \frac{1}{T} \int_{T - T_0(\varepsilon)}^{T} \Big\|K\big(T, s, i, F(i)\big) - K(i)\Big\|\mathrm{d}s \\
&= \lim_{T\to\infty} \sup \frac{1}{T} \int_{t_0}^{T - T_0(\varepsilon)} \Big\|K\big(T, s, i, F(i)\big) - K(i)\Big\|\mathrm{d}s \\
&= \lim_{T\to\infty} \sup \frac{1}{T} \int_0^{T - T_0(\varepsilon) - t_0} \Big\|K\big(T, s+t_0, i, F(i)\big) - K(i)\Big\|\mathrm{d}s \\
&\le \lim_{T\to\infty} \sup \frac{T - T_0(\varepsilon) - t_0}{T}\delta = \delta
\end{aligned}$$

which, owing to the arbitrariness of δ, yields the desired conclusion. ■

4. Adaptive Control for the Time–Varying JLQ

The main result of this section is given by the following theorem:

Theorem 5. *Assume that the assumptions of Theorem 3 hold and that the coefficients* $A(t, r(t))$ *and* $B(t, r(t))$

of the system (1) are unknown and their limits

$$A(i) = \lim_{t\to\infty} A(t,i), \quad B(i) = \lim_{t\to\infty} B(t,i)$$

are known. Then the optimal adaptive control law for the time-varying control problem (1), (9) is given by

$$\widetilde{u}(t) = -R^{-1}(i)B'(i)K(i)x(t), \quad i \in S, \tag{38}$$

where $\{K(i), i \in S\}$ *is the unique solution of the algebraic coupled Riccati equation (12).*

Proof. From (8) in Theorem 1, for any control u we have

$$\begin{aligned} &J(x_0, u) \\ &\geq \lim_{T\to\infty} \frac{1}{T}\Big[\langle K(T,t,i_0,F(i_0))\,x_0, x_0\rangle \\ &+ \int_0^T \mathrm{tr}\Big(C'(T-s)K(T,s,i_0,F(i_0))\,C(T-s)\Big)\mathrm{d}s\Big]. \end{aligned} \tag{39}$$

From Lemma 4, Theorem 3 and (39) we conclude that

$$J(x_0, u) \geq \mathrm{tr}\left(C'(i_0)\,K(i_0)\,C(i_0)\right),$$

where $C(i) = \lim_{t\to\infty} C(t,i)$. The last inequality shows that the cost functional has a value not smaller than $tr\left(C'(i_0)\,K(i_0)\,C(i_0)\right)$. From (8), using Theorem 3 again, it is easy to check that for the control law given by (38) the cost functional takes this value. ■

5. Conclusion

In this paper we considered a version of adaptive control for jump linear systems with quadratic cost functionals. We took into account a system with unknown coefficients having limits as functions of time when the time tends to infinity and the limits are known. First we showed that the optimal control for this system can be realized in the form of time-invariant feedback with the feedback matrix equal to that for the time invariant system with coefficients equal to the limits of the time-varying system. To this end, we showed that the solution of the time-varying differential Riccati equation converges in some sense to the solution of some time-invariant algebraic Riccati equation under very natural conditions. Based on this result, we solved the adaptive version of the linear quadratic problem.

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References

Abou-Kandil H., Freiling G. and Jank G. (1994): *Solution and asymptotic behavior of coupled Riccati equations in jump linear systems*. — IEEE Trans. Automat. Contr., Vol. 39, No. 8, pp. 1631–1636.

Abou-Kandil H., Freiling G. and Jank G. (1995): *On the solution of discrete-time Markovian jump linear quadratic control problems*. — Automatica, Vol. 31, No. 5, pp. 765–768.

Afanas'ev V.N., Kolmanovskij V.B. and Nosov V.R. (1989): *Mathematical Theory of Control System Design*. — Moscow: Vyssha Shkola, (in Russian).

Blom H.A.P.(1990): *Bayesian estimation for decision-directed stochastic control*. — Amsterdam, The Netherlands: Technical Univ. Delft Press.

Boukas E.K. and Haurie A. (1990): *Manufacturing flow control and preventive maintenance: A stochastic control approach* — IEEE Trans. Automat. Contr., Vol. 35, No. 9, pp. 1024–1031.

Costa L.V. and Fragoso M.D. (1995): *Discrete-time LQ-optimal control problems for infinite markov jump parameter systems*. — IEEE Trans. Automat. Contr., Vol. 40, pp. 2076–2088.

Chizeck H.J., Willsky A.S. and Castanon D. (1998): *Discrete-time markovian-jump linear quadratic optimal control*. — Int. J. Contr., Vol. 43, No. 2, pp. 213–231.

Czornik A. (1998): *On time-varying LQG*. — Proc. IFAC Conf. *Syst. Struc. Contr.*, Nantes, France, pp. 427–432.

Czornik A. (1999): *On discrete-time linear quadratic-control*. — Syst. Contr. Lett., Vol. 36, No. 2, pp. 101–107.

Czornik A. (2000): *Continuity of the solution of the Riccati equations for continuous time JLQP*. — IEEE Trans. Automat. Contr., Vol. 45, No. 5, pp. 934–937.

Czornik A. and Świerniak A. (2001): *On the discrete JLQ and JLQG problems*. — Nonlin. Anal., Vol. 47, No. 1, pp. 423–434.

Czornik A. and Świerniak A. (2002): *On the discrete time-varying JLQG problem*. — Int. J. Appl. Math. Comput. Sci., Vol. 12, No. 2, pp. 101–105.

Czornik A. (2004): *Adaptive control for jump linear system with quadratic cost*. — Contr. Cybern., Vol. 33, No. 1, pp. 51–71 .

Czornik A. and Świerniak A. (2004): *On the continuous time-varying JLQ problem*. — Europ. J. Contr, Vol. 10, No. 3, pp. 264–272.

Duncan T.E., Guo L. and Pasik-Duncan B. (1999): *Adaptive continous-time linear quadratic gaussian control*. — IEEE Trans. Automat. Contr., Vol. 44, No. 9, pp. 1653–1662.

Dufour F. and Elliott R. (1998): *Adaptive control for linear systems with Markov perturbations*. — IEEE Trans. Automat. Contr., Vol. 43, No. 3, pp. 351–372.

Feng X., Loparo K.A., Ji Y. and Chizeck H.J. (1992): *Stochastic stability properties of jump linear systems*. — IEEE Trans. Automat. Contr., Vol. 37, No. 3, pp. 38–53.

Ghosh M.K. (1995): *On an LQG regulator with Markovian switching and pathwise average cost*. — IEEE Trans. Automat. Contr., Vol. 40, No. 4, pp. 1919–1921.

Griffiths B.E. and Loparo K. (1985): *Optimal control of jump-linear Gaussian systems*. — Int. J. Contr., Vol. 42, No. 5, pp.791–819.

Guo L. (1996): *Self-convergence of weighted least-squares with applications to stochastic adaptive control*. — IEEE Trans. Automat. Contr., Vol. 41, No. 1, pp. 79–89.

Ji Y. and Chizeck H.J. (1988): *Controllability, observability and discrete-time Markovian jump linear quadratic control*. — Int. J. Contr., Vol. 48, No. 6, pp.481–498.

Ji Y. and Chizeck H.J. (1989): *Optimal quadratic control of jump linear systems with seperately controlled transition probabilities*. — Int. J. Contr., Vol. 49, No. 2, pp. 481–491.

Ji Y. and Chizeck H.J. (1990): *Controllability, stability, and continuous-time Markovian jump linear quadratic control*. — IEEE Trans. Automat. Contr., Vol. 35, No. 7, pp. 777–788.

Mariton M. (1987): *Jump linear quadratic control with random state discontinuities*. — Automatica, Vol. 23, No. 3, pp. 237–140.

Mariton M. (1990): *Jump Linear Systems in Automatic Control*. — New York: Marcel Dekker.

Pan G. and Bar-Shalom Y. (1996): *Stabilization of jump linear Gaussian systems without mode observations*. — Int. J. Contr., Vol. 64, No. 7, pp. 631–666.

Prandini M. and Campi M.C. (2001): *Adaptive LQG control of input-output systems: A cost-biased approach*. — SIAM J. Contr. Optim., Vol. 39, No. 5, pp. 1499–1519.

Rami Ait M. and El Ghaoui L. (1996): *LMI optimization for non-standard Riccati equations arising in stochastic control*. — IEEE Trans. Automat. Contr., Vol. 41, No. 10, pp. 1666–1671.

Siljak D.D. (1980): *Reliable control Using Multiple Control Systems*. — Int. J. Contr., Vol. 31, No. 2, pp. 303–329.

Świerniak A., Simek K. and Boukas E.K. (1998): *Intelligent robust control of fault tolerant systems*, In: Artificial Intelligence in Real-Time Control (H.E. Rauch, Ed.). — Oxford: Elsevier, pp. 245–248.

Sworder D.D. and Rogers R.O. (1983): *An LQ-solution to a control problem associated with solar thermal central receiver*. — IEEE Trans. Automat. Contr., Vol. 28, No. 4, pp. 971–978.

Sworder D.D. (1969): *Feedback control of a class of linear systems with jump parameters*. — IEEE Trans. Automat. Contr., Vol. 14, No. 4, pp. 9–14.

Sworder D.D. and Robinson V.G. (1973): *Feedback regulators for jump parameter systems with state and control dependent transition rates*. — IEEE Trans. Automat. Contr., Vol. 18, No. 3, pp. 355–359.

Wonham W.M. (1971): *Random differential equations in control theory*, In: Probabilistic Methods in Applied Mathematics, Vol. 2 (A.T. Bharucha-Reid, Ed.). — New York: Academic Press.

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