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ON THE STABILITY OF NEUTRAL–TYPE UNCERTAIN SYSTEMS WITH MULTIPLE TIME DELAYS

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The problems of both single and multiple delays for neutral-type uncertain systems are considered. First, for single neutral time delay systems, based on a Razumikhin-type theorem, some delay-dependent stability criteria are derived in terms of the Lyapunov equation for various classes of model transformation and decomposition techniques. Second, robust control for neutral systems with multiple time delays is considered. Finally, we demonstrate numerical examples to illustrate the effectiveness of the proposed approaches. Compared with results existing in the literature, our methods are shown to be superior to them.

Keywords: Razumikhin-type theorem, time-delay, neutral type uncertain systems

1. Introduction

The stability of time-delay systems has been widely investigated in the last two decades, because time delay often causes system instability (Castelan and Infante, 1979; Gu, 1997; Dugard and Verriest, 1997; He *et al.*, 2004). Many engineering systems can be modelled using functional differential equations of the neutral type, e.g., processes including steam or water pipes, heat exchangers etc. The introduction of the delay factor makes the systems analysis more complicated, in addition to the difficulties caused by perturbations or uncertainties. During the same period of time, a lot of work has been done in the field of numerical solution of delay-differential equation. This is because many engineering and control systems can be described by neutral differential equations. Thus, the stability of a neutral system with time delay has interested many researchers (Castelan and Infante, 1979; Dugard and Verriest, 1997; Han, 2002; Lien, 1999; Lien and Fan, 2000; Mahmound, 2000). Some results to verify the stability of the linear autonomous matrix neutral difference-differential equation with one delay,

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + C\dot{x}(t-\tau),$$

were proposed by Castelan and Infante (1979). In the literature, there are numerous methods to deal with the delay-independent problem of neutral time delay systems, such as the Riccati equation approach (Dugard and Verriest, 1997), the matrix pencil (Niculescu, 2001), the linear matrix inequality (Han, 2002; Lien, 1999; Lien and Fan, 2000), the characteristic equation (He and Cao, 2004), and so on. In general, delay-independent tests are more conservative than delay-dependent tests when the delay is small. However, for uncertain neutral time delay, few efforts (Han, 2002; Lien, 1999; Lien and Fan, 2000) have been made to analyze the problem of delay-dependent stability. For instance, the existence of time delay systems may make the stability analysis more difficult. However, until now the delay-dependent stability for neutral time delay systems has been rarely discussed. The goal of this paper is to develop a delay-dependent method for asymptotic stability of uncertain neutral time delay systems.

It is well known that there are several basic methods of studying the stability of delay differential equations. One is the Lyapunov functional method (Han, 2002; Yan, 2000), the others are the Razumikhin Lyapunov function method (Su and Huang, 1992), and the spectral radius method (Yang and Liu 2002). The Razumikhin Lyapunov function method, which takes delay into account, is often less conservative than the Lyapunov functional method. Most time-domain techniques make use of model transformations in order to derive delay-dependent stability results. A different idea exploited in the literature (Goubet *et al.*, 1997) is to find some decomposition of the "delayed" term B of the form $B = B_1 + B_2$ in order to improve delay bounds. In this paper, based on some model transformation and decomposition techniques, we propose delay-dependent criteria which are expressed in terms of the Razumikhin-type theorem and the Lyapunov equation. They ensure global uniform asymptotic stability for any time delay which is no greater than a given bound. Thus, in general, the proposed criteria are less conservative than some existing ones. Numerical examples show that the results obtained using the proposed criteria significantly improve the estimate of the delay time compared with some results existing in the literature. In this paper the following notation is adopted:

Notation:

$\mathbb{R}$	the real number field,
$\mathbb{R}^n$	the n-dimensional real vector space,
x	a vector, $x = [x_1 \ \ x_2 \ \ \cdots \ \ x_n]^T, \ \ x_i \in \mathbb{R}$
A^T	the transpose of a matrix A,
$\lambda_i(A)$	the i-th eigenvalue of a matrix A,
$\lambda_{\max}(A)$	the maximum eigenvalue of A,
$\lambda_{\min}(A)$	the minimum eigenvalue of A,
$\|A\|$	the norm of a matrix A defined as $\|A\| = \sqrt{\lambda_{\max}(A^TA)}$,
$\mu(A)$	the matrix measure of a matrix A defined as $\mu(A) = [\lambda_{\max}(A^T + A)]/2$,
$G_{n,\tau}$	the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence as $G_{n,\tau} = G([-\tau,0], \ \mathbb{R}^n)$.

2. Problem Statement

Consider the neutral uncertain system with time delays

$$\dot{x}(t) - C\dot{x}(t-\tau) = (A+\Delta A)x(t) + (B+\Delta B)x(t-\tau), \tag{1}$$

where $x(t) \in \mathbb{R}^n$, A, B, and C are known constant matrices, τ is time delay, and the system matrix A is assumed to be a Hurwitz matrix. That is, all the eigenvalues of A have negative real parts and $\|C\| < 1$. ΔA and ΔB are linear parametric uncertainties with bounds given as follows:

$$\|\Delta A\| \le \alpha, \tag{2a}$$

$$\|\Delta B\| \le \beta. \tag{2b}$$

To analyze the stability of the system described by (1), decompose the matrix $B = B_1 + B_2$, where B_1 and B_2 are constant matrices. Define the operator $L : G_{n,\tau} \mapsto \mathbb{R}^n$,

$$L(x_t) = x(t) + B_1 \int_{t-\tau}^{t} x(s)\,\mathrm{d}s - Cx(t-\tau), \tag{3}$$

which is also called the parameterized model transformation.

Let

$$\sigma = \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \tag{4}$$

$$\delta = \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}, \tag{5}$$

where $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalues of the matrix A, respectively, P is a symmetric positive-definite matrix and Q is a symmetric positive-semidefinite matrix involved in the following Lyapunov equation:

$$(A+B_1)^T P + P(A+B_1) = -Q. \tag{6}$$

Theorem 1. *Consider the uncertain neutral time delay system (1) and assume that $A + B_1$ is a Hurwitz stable matrix satisfying*

$$\begin{aligned}\sigma - \Big\{ \alpha + q\delta\big[\, &\big\|(A+B_1)^T C\big\| + \|C\| + \beta + \big\|B_2^T\big\| \\ &+ q\delta\big(\,\big\|B_2^T\big\| + \beta\big)\, \|C\| + \tau\, \big(\big\|(A+B_1)^T B_1\big\| \\ &+ \|B_1\|\,(\alpha+\beta) + \big\|B_2^T B_1\big\|\big)\,\big]\Big\} > 0. \end{aligned} \tag{7}$$

Then the uncertain neutral time delay system (1) is asymptotically stable, i.e., the uncertain parts of the nominal system can be tolerated for any delay time $\tau \ge 0$.

Proof. Consider the system (3) and take the following positive definite function as our Lyapunov function:

$$V\big(x(t)\big) = L^T(x_t) P L(x_t), \tag{8}$$

where x_t is the state at time t defined by

$$x_t(s) = x(t+s), \quad \forall s \in [-\tau, 0].$$

Taking the time derivative of $V(x(t))$ in (8) along the trajectories of the system (1), we have

$$\begin{aligned}\dot{V}(x(t)) &= \dot{L}^T(x_t)PL(x_t) + L^T(x_t)P\dot{L}(x_t) \\ &= \big[(A+B_1)x(t) + \Delta Ax(t) + B_2x(t-\tau) \\ &\quad + \Delta Bx(t-\tau)\big]^T P\Big[x(t) - Cx(t-\tau) \\ &\quad + B_1\int_{t-\tau}^{t} x(s)\,\mathrm{d}s\Big] \\ &\quad + \Big[x(t) - Cx(t-\tau) + B_1\int_{t-\tau}^{t} x(s)\,\mathrm{d}s\Big]^T \\ &\quad \times P\big[(A+B_1)x(t) \\ &\quad + \Delta Ax(t) + B_2x(t-\tau) + \Delta Bx(t-\tau)\big]\end{aligned}$$

$$
\begin{aligned}
&\le x^T(t)\big[(A+B_1)^T P + P(A+B_1)\big]x(t)\\
&\quad - 2x^T(t)(A+B_1)^T PCx(t-\tau)\\
&\quad + 2x^T(t)(A+B_1)^T PB_1 \int_{t-\tau}^{t} x(s)\,\mathrm{d}s\\
&\quad + 2x^T(t)\Delta A^T Px(t) - 2x^T(t)PCx(t-\tau)\\
&\quad + 2x^T(t)\Delta A^T PB_1 \int_{t-\tau}^{t} x(s)\,\mathrm{d}s\\
&\quad + 2x^T(t-\tau)B_2^T Px(t)\\
&\quad - 2x^T(t-\tau)B_2^T PCx(t-\tau)\\
&\quad + 2x^T(t-\tau)B_2^T PB_1 \int_{t-\tau}^{t} x(s)\,\mathrm{d}s\\
&\quad + 2x^T(t-\tau)\Delta B^T Px(t)\\
&\quad - 2x^T(t-\tau)\Delta B^T PCx(t-\tau)\\
&\quad + 2x^T(t-\tau)\Delta B^T\, P\, B_1 \int_{t-\tau}^{t} x(s)\,\mathrm{d}s. \qquad (9)
\end{aligned}
$$

Applying a Razumikhin-type theorem, we assume that for any positive number $q > 1$, the following inequality holds:

$$V\big(x(\xi)\big) < q^2 V\big(x(t)\big), \quad t - 2\tau \le \xi \le t. \qquad (10)$$

Then

$$\|x(\xi)\| < q\delta \,\|x(t)\|\,. \qquad (11)$$

Substituting (10) and (11) into (9), we have

$$\dot V\big[x(t)\big] \le -\omega \,\|x(t)\|^2\,, \quad \omega \in \mathbb{R}, \qquad (12)$$

where

$$
\begin{aligned}
\omega = \lambda_{\min}(Q) - 2\Big\{&\alpha + q\delta\Big[\,\big\|(A+B_1)^T C\big\| + \|C\|\\
&+ \beta + \big\|B_2^T\big\| + q\delta\big(\,\big\|B_2^T\big\| + \beta\big)\,\|C\|\\
&+ \tau\big(\,\big\|(A+B_1)^T B_1\big\| + \|B_1\|\,(\alpha+\beta)\\
&+ \big\|B_2^T B_1\big\|\,\big)\Big]\Big\}\lambda_{\max}(P).
\end{aligned}
$$

■

If the condition (7) of Theorem 1 is satisfied, then a sufficiently small $q > 1$ exists such that $\omega > 0$. Based on the results obtained in the proof of Theorem 1, we have $\dot V[x(t)] < 0$ if $\omega > 0$. Moreover, $\omega > 0$ if and only if (7) holds. This implies that the time delay system (1) is asymptotically stable. When $C = 0$, the system (1) reduces to

$$\dot x(t) = (A + \Delta A)x(t) + (B + \Delta B)x(t-\tau). \qquad (13)$$

Therefore, from Theorem 1, we have the following result for the delay-dependent stability criterion:

Corollary 1. *Consider the system (13) and assume that $A + B_1$ is a Hurwitz stable matrix. Then given a scalar $\tau \ge 0$, the system (13) is globally asymptotically stable for any constant time delay τ^* satisfying $0 \le \tau^* \le \tau$ if*

$$
\begin{aligned}
&0 \le \tau^* \le \tau\\
&= \frac{\sigma - \alpha - q\delta(\beta + \|B_2^T\|)}{q\delta[\|(A+B_1)^T B_1\| + \|B_1\|\,(\alpha+\beta) + \|B_2^T B_1\|]}. \qquad (14)
\end{aligned}
$$

The proof of Corollary 1 is similar to that of Theorem 1 and hence is omitted here.

Remark 1. The key point of Theorem 1 and Corollary 1 is the decomposition of the matrix B into $B = B_1 + B_2$, where B_1 is chosen such that $A + B_1$ is more stable than A (Goubet *et al.*, 1997). Roughly, this decomposition corresponds to the decomposition of the delayed terms into two groups: the stabilizing ones and destabilizing ones. This technique enables one to take the stabilizing effect of part of the delayed terms into account. The result of Gu and Niculescu (2000) derived for the retarded case is also valid for neutral systems.

3. Stability Analysis for Neutral Systems with Multiple Time Delays

Recently, the study of stability has been extended to neutral uncertain systems with multiple time delays of the form

$$
\begin{aligned}
\dot x(t) = (A + \Delta A)x(t) + \sum_{j=0}^{n} \big[&(B_j + \Delta B_j)x(t-\tau_j)\\
&+ C_j \dot x(t-\tau_j)\big], \qquad (15)
\end{aligned}
$$

where $x(t)$ is the state vector, A, B_j and C_j are known constant matrices, τ_j is positive constant time delay for $j = 0, 1, 2, , , n$, and the system matrix A is assumed to be a Hurwitz matrix, i.e., the eigenvalues of A have negative real parts. ΔA and ΔB_j are linear parametrical uncertainties with bounds as follows:

$$\|\Delta A\| \le \alpha, \qquad (16a)$$

$$\|\Delta B_j\| \le \beta_j. \qquad (16b)$$

Define the Lyapunov equation

$$\Big(A + \sum_{j=1}^{n} B_j\Big)^T P + P\Big(A + \sum_{j=1}^{n} B_j\Big) = -Q, \qquad (17)$$

where P is a symmetric positive-definite matrix and Q is a symmetric positive semi-definite matrix.

Let us decompose the matrix

$$\sum_{j=0}^{n} B_j = B_0 + \sum_{j=1}^{n} B_j,$$

where $B_j(j = 1, \dots, n)$ are constant matrices. In time domain, one of the simplest ideas to derive delay-dependent stability is to use the relation

$$L(x_t) = x(t) + \sum_{j=1}^{n} \int_{t-\tau_j}^{t} B_j x(s)\,\mathrm{d}s - \sum_{j=0}^{n} C_j x(t-\tau_j). \tag{18}$$

Our result is summarized in the following theorem:

Theorem 2. *Consider the neutral uncertain system with multiple time delays (15) and assume that* $A + \sum_{j=1}^{n} B_j$ *is a Hurwitz stable matrix. Then given a scalar* $\tau \geq 0$, *the system (15) is globally asymptotically stable for any constant time delay* τ_j^* *satisfying* $0 \leq \tau_j^* \leq \tau_j$ *if*

$$\begin{aligned}
\sigma - \Bigg\{ & \alpha\Big(1 + \Big\|\sum_{j=1}^{n} B_j\Big\|\Big) + \|B_0^T\| \\
&+ q\delta\Bigg[\Big\|\Big(A + \sum_{j=1}^{n} B_j\Big)^T \sum_{j=0}^{n} C_j\Big\| + \alpha\Big\|\sum_{j=0}^{n} C_j\Big\| \\
&+ \sum_{j=0}^{n}\beta_j + q\delta\Big(\|B_0^T\| + \sum_{j=0}^{n}\beta_j\Big)\Big\|\sum_{j=0}^{n} C_j\Big\|\Bigg] \\
&+ q\delta\sum_{j=1}^{n}\tau_j\Bigg[\Big\|\Big(A + \sum_{j=1}^{n} B_j\Big)^T \sum_{j=1}^{n} B_j\Big\| \\
&+ \Big(\alpha + \|B_0^T\| + \sum_{j=0}^{n}\beta_j\Big)\Big\|\sum_{j=1}^{n} B_j\Big\|\Bigg]\Bigg\} > 0.
\end{aligned} \tag{19}$$

Proof. We consider the system (15) and take the following positive definite function as our Lyapunov function (8). The time derivative of (8) along the trajectories of the system of Eqn. (15) is

$$\begin{aligned}
\dot V[x(t)] &= \dot L^T(x_t) P L(x_t) + L^T(x_t) P \dot L(x_t) \\
&= \Big[\Big(A+\sum_{j=1}^{n} B_j\Big)x(t) + \Delta A x(t) + B_0 x(t-\tau_0) \\
&\quad + \sum_{j=0}^{n} \Delta B_j x(t-\tau_j)\Big]^T P\Big[x(t) \\
&\quad + \sum_{j=1}^{n}\int_{t-\tau_j}^{t} B_j x(s)\,\mathrm{d}s - \sum_{j=0}^{n} C_j x(t-\tau_j)\Big] \\
&\quad + \Big[x(t) + \sum_{j=1}^{n}\int_{t-\tau_j}^{t} B_j x(s)\,\mathrm{d}s \\
&\quad - \sum_{j=0}^{n} C_j x(t-\tau_j)\Big]^T P\Big[\Big(A + \sum_{j=1}^{n} B_j\Big)x(t) \\
&\quad + \Delta A x(t) + B_0 x(t-\tau_0) + \sum_{j=0}^{n}\Delta B_j x(t-\tau_j)\Big] \\
&= x^T(t)\Big[\Big(A+\sum_{j=1}^{n} B_j\Big)^T P + P\Big(A+\sum_{j=1}^{n} B_j\Big)\Big]x(t) \\
&\quad + 2x^T(t)\Big(A+\sum_{j=1}^{n} B_j\Big)^T P\sum_{j=1}^{n}\int_{t-\tau_j}^{t} B_j x(s)\,\mathrm{d}s \\
&\quad - 2x^T(t)\Big(A + \sum_{j=1}^{n} B_j\Big)^T P\sum_{j=0}^{n} C_j x(t-\tau_j) \\
&\quad + 2x^T(t-\tau_0) B_0^T P x(t) \\
&\quad + 2x^T(t-\tau_0) B_0^T P\sum_{j=1}^{n}\int_{t-\tau_j}^{t} B_j x(s)\,\mathrm{d}s \\
&\quad - 2x^T(t-\tau_0) B_0^T P\sum_{j=0}^{n} C_j x(t-\tau_j) \\
&\quad + 2x^T(t)\Delta A^T P x(t) \\
&\quad + 2x^T(t)\Delta A^T P\sum_{j=1}^{n}\int_{t-\tau_j}^{t} B_j x(s)\,\mathrm{d}s \\
&\quad - 2x^T(t)\Delta A^T P\sum_{j=0}^{n} C_j x(t-\tau_j) \\
&\quad + 2x^T(t-\tau_j)\sum_{j=0}^{n} B_j^T P x(t) \\
&\quad + 2x^T(t-\tau_j)\sum_{j=0}^{n} B_j^T\sum_{j=1}^{n}\int_{t-\tau_j}^{t} B_j x(s)\,\mathrm{d}s \\
&\quad - 2x^T(t-\tau_j)\sum_{j=0}^{n} B_j^T\sum_{j=0}^{n} C_j x(t-\tau_j) \\
&\leq -x^T(t) Q x(t) + 2x^T(t)\Big(A + \sum_{j=1}^{n} B_j\Big)^T \\
&\quad \times P\sum_{j=1}^{n}\int_{t-\tau_j}^{t} B_j x(s)\,\mathrm{d}s - 2x^T(t)\Big(A + \sum_{j=1}^{n} B_j\Big)^T \\
&\quad \times P\sum_{j=0}^{n} C_j x(t-\tau_j) + 2x^T(t-\tau_0) B_0^T P x(t)
\end{aligned}$$

$$
\begin{aligned}
&+ 2x^T(t-\tau_0)B_0^T P \sum_{j=1}^{n} \int_{t-\tau_j}^{t} B_j x(s)\,\mathrm{d}s \\
&- 2x^T(t-\tau_0)B_0^T P \sum_{j=0}^{n} C_j x(t-\tau_j) \\
&+ 2x^T(t)\Delta A^T P x(t) + 2x^T(t)\Delta A^T \\
&\times P \sum_{j=1}^{n} \int_{t-\tau_j}^{t} B_j x(s)\,\mathrm{d}s \\
&- 2x^T(t)\Delta A^T P \sum_{j=0}^{n} C_j x(t-\tau_j) \\
&+ 2x^T(t-\tau_j) \sum_{j=0}^{n} \Delta B_j^T P x(t) \\
&+ 2x^T(t-\tau_j) \sum_{j=0}^{n} \Delta B_j^T \sum_{j=1}^{n} \int_{t-\tau_j}^{t} B_j x(s)\,\mathrm{d}s \\
&- 2x^T(t-\tau_j) \sum_{j=0}^{n} \Delta B_j^T \sum_{j=0}^{n} C_j x(t-\tau_j). \qquad (20)
\end{aligned}
$$

As in our Razumikhin-type theorem, we assume that for any positive number $q > 1$, the inequalities (10) and (11) hold. Substitute them into (20). Then we have

$$
\dot V[x(t)] \le -\omega_1 \|x(t)\|^2, \quad \omega \in \mathbb{R}, \qquad (21)
$$

where

$$
\begin{aligned}
\omega_1 = \lambda_{\min}(Q) - 2\Bigg\{ &\alpha\Big(1 + \Big\|\sum_{j=1}^{n} B_j\Big\|\Big) + \|B_0^T\| \\
&+ q\delta\Bigg[\Big\|\Big(A+\sum_{j=1}^{n}B_j\Big)^T \sum_{j=0}^{n}C_j\Big\| + \alpha\Big\|\sum_{j=0}^{n}C_j\Big\| + \sum_{j=0}^{n}\beta_j \\
&+ q\delta\Big(\|B_0^T\| + \sum_{j=0}^{n}\beta_j\Big)\Big\|\sum_{j=0}^{n} C_j\Big\|\Bigg] \\
&+ q\delta \sum_{j=1}^{n}\tau_j \Bigg[\Big\|\Big(A + \sum_{j=1}^{n} B_j\Big)^T \sum_{j=1}^{n} B_j\Big\| \\
&+ \Big(\alpha + \|B_0^T\| + \sum_{j=0}^{n}\beta_j\Big)\Big\|\sum_{j=1}^{n} B_j\Big\|\Bigg]\Bigg\}\lambda_{\max}(P).
\end{aligned}
$$

■

Based on the results obtained in the proof of Theorem 2, we have $\dot V[x(t)] < 0$ if $\omega_1 > 0$. Furthermore, $\omega_1 > 0$ if and only if (19) holds. Therefore, the time delay system (15) is asymptotically stable.

Consider the following time delay system with multiple time delays, which is a special case of the system (15):

$$
\dot x(t) = Ax(t) + \sum_{j=0}^{n} B_j x(t-\tau_j). \qquad (22)
$$

Consider another operator $L : G_{n,\tau} \mapsto \mathbb{R}^n$, with

$$
L(x_t) = x(t) + \sum_{j=1}^{n} \int_{t-\tau_j}^{t} B_j x(s)\,\mathrm{d}s. \qquad (23)
$$

The time derivative of (23) is

$$
\dot L(x_t) = \Big(A + \sum_{j=1}^{n} B_j\Big)x(t) + B_0 x(t-\tau_0), \qquad (24)
$$

where $A + \sum_{j=1}^{n} B_j$ is a Hurwitz matrix. Choose the matrix $P > 0$ to be the solution of the Lyapunov equation

$$
\Big(A + \sum_{j=1}^{n} B_j\Big)^T P + P\Big(A + \sum_{j=1}^{n} B_j\Big) = -Q,
$$

where $Q > 0$.

So, directly from Theorem 2, we can obtain the following result:

Corollary 2. *Suppose that $A+\sum_{j=1}^{n} B_j$ is a Hurwitz stable matrix. Then the system (22) is asymptotically stable for any constant time delay τ_j^* satisfying $0 \le \tau_j^* \le \tau_j$ if*

$$
\begin{aligned}
\sigma - q\delta\Bigg\{ \|B_0^T\| + \sum_{j=0}^{n}\tau_j \Bigg[\Big\|\Big(A + \sum_{j=1}^{n} B_j\Big)^T \sum_{j=0}^{n} B_j\Big\| \\
+ \Big\|B_0^T \sum_{j=1}^{n} B_j\Big\|\Bigg]\Bigg\} > 0. \qquad (25)
\end{aligned}
$$

The proof of Corollary 2 is similar to that of Theorem 2 and hence is omitted.

4. Examples

To demonstrate the applicability of the present schemes, we give five examples.

Example 1. Consider the linear neutral delay-differential system

$$
\dot x(t) = (A + \Delta A)x(t) + (B + \Delta B)x(t-\tau) + C\dot x(t-\tau), \qquad (26)
$$

where

$$
A = \begin{bmatrix} -2 & 0.5 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}, \quad C = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix},
$$

Table 1. Delay time τ for various values of c when $\alpha = \beta = 0$, $Q = \text{diag}\,\{0.1, 0.1\}$.

$\lvert c\rvert$	0.9	0.7	0.5	0.3	0.1	0
Han (2002)	0.99	2.73	3.62	4.1	4.33	4.35
Our result	3.3657	3.6632	3.9607	4.2583	4.5558	4.7046

$$|c| < 1, \quad \Delta A = \Delta B = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

Find an upper bound on the maximum delay time τ to make the above system (26) asymptotically stable.

Solution. The objective is to obtain a maximum delay bound such that the uncertain system (26) is asymptotically stable, and to show the superiority of this method over the case of the proposed criteria. We set $c = 0.15$,

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

Applying the Lyapunov equation (6) and Theorem 1, we get

$$\lambda_{\min}(P) = 0.2564, \ \lambda_{\max}(P) = 0.5885, \ \lambda_{\min}(Q) = 1,$$

$$\sigma = 0.8497, \quad \delta = 0.6601, \quad \alpha_0 = 0.1, \quad \beta = 0.1,$$

$$\left\|(A + B_1)^T C\right\| = 0.2972, \qquad \|B_2\| = 0.3,$$

$$\left\|(A + B_1)^T B_1\right\| = 0.1981, \quad \left\|B_2^T B_1\right\| = 0.03,$$

$$0 \le \tau = 1.6143.$$

Our method guarantees the stability of the system (26) for all delays which are less than or equal to $\tau = 1.6143$. The upper bound given by Lien and Fan (2000) is $\tau < 0.8750$. Hence, for this example, the robust stability criterion of this paper is less conservative than the existing result by Lien and Fan (2000). ♦

Example 2. Consider the system (1) subject to perturbations of system matrices, so that the state equation is

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)x(t - \tau) + C\dot{x}(t - \tau), \tag{27}$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

$$C = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}, \qquad |c| < 1,$$

ΔA and ΔB are unknown matrices satisfying $\|\Delta A\| \le \alpha$ and $\|\Delta B\| \le \beta$. When $c = 0$, the system (27) reduces to that discussed by Gu (1997). Now, we wish to find the range of the delay time τ to guarantee that the above system (27) is asymptotically stable. ♦

Solution. The objective of this example is to obtain a maximum delay bound such that the uncertain system (27) is asymptotically stable, and to show the superiority of this method in the case of the proposed criteria. The matrix B is naturally decomposed as $B = B_1 + B_2$, where

$$B_1 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.9 & 0 \\ -1 & -0.9 \end{bmatrix}.$$

Set $c = \alpha = \beta = 0$,

$$Q = \begin{bmatrix} 0.0705 & 0 \\ 0 & 0.0705 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}.$$

Applying the Lyapunov equation (6) and Corollary 1, we get

$$P = \begin{bmatrix} 0.0168 & 0 \\ 0 & 0.0352 \end{bmatrix},$$

and $0 < \tau \le 6.1726$. The maximum value of time delay for this case to have guaranteed stability is $\tau = 6.1726$. The same conclusion can be drawn for Example 2 in (Gu, 1997; Gu and Niculescu, 2000; Mahmound, 2000).

For $\alpha = \beta = 0$, Table 1 gives the maximum delay bound τ by the criteria considered in (Han, 2002) and this paper. It is clear that the proposed stability criterion significantly improves the estimate of the stability limit over the results by Han (2002).

For $\alpha = \beta = 0$, the maximum value of the delay time τ is listed in Table 2 for various values of c. As c increases, the delay time τ decreases.

For $c = 0$, we now consider the effect of uncertainty bounds α and β on the maximum time-delay for stability τ. Table 3 illustrates numerical results for various α and β. We can see that the delay time τ decreases as α and β increase.

As $\alpha = \beta$ increases from 0 to 0.25, the delay time τ decreases from 4.5558 to 1.9163. It is clear that when $\alpha = \beta$ increases, τ decreases. Hence we can see that, in this case, our approach produces less conservative results than those obtained by the method for uncertain system with time delay.

Table 2. Upper bound on the delay time τ for various values of c when $\alpha = \beta = 0.2$.

$\lvert c\rvert$	0.40	0.35	0.30	0.25	0.20	0.15	0.10	0.05	0
Han (2002)	0.37	0.59	0.79	0.98	1.16	1.33	1.48	1.63	1.77
He *et al.*, (2004)	1.16	1.33	1.50	1.66	1.81	1.96	2.11	2.25	2.39
Our result	1.7008	1.7920	1.8832	1.9744	2.0656	2.1568	2.2480	2.3392	2.4304

Table 3. Delay time τ v. $\alpha = \beta$ when $c = 0.1,\ Q = \mathrm{diag}\,\{0.1, 0.1\}$.

$\alpha = \beta$	0.25	0.2	0.15	0.1	0.05	0
Han (2002)	0.77	1.48	2.19	2.9	3.61	4.33
Our result	1.9163	1.9236	2.8772	3.4242	3.7309	4.5558

Example 3. Consider the neutral system with time delay investigated in (Lien, 1999):

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{2} B_i x(t-\tau_i) + \sum_{i=1}^{2} C_i \dot{x}(t-\tau_i), \quad t \geq 0 \tag{28}$$

where

$$A = \begin{bmatrix} -2 & -0.6 \\ -0.5 & -2 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -1 & 0.2 \\ 0.5 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Here we wish to determine the delay time for making (28) asymptotically stable. ♦

Solution. Using the condition (19) of Theorem 2, we obtain $\tau_1 = \tau_2 < 7.6652$. It is obvious that the system (28) is asymptotically stable for delay times satisfying $\tau_1 = \tau_2 < 7.6652$.

Remark 2. Consider the case where the system (28) is with a single time delay, i.e., take $i = 1$. If the delay-dependent criterion given by Lien (1999) is employed, then the system (28) is asymptotically stable provided that

$$\|C_1\| + \tau_1 \|B_1\| < 1 \tag{29a}$$

and

$$\mu(A+B_1) + \|C_1\| \, \|A+B_1\| + \tau_1 \left\|(A+B_1)^T B_1\right\| < 0. \tag{29b}$$

Now the criterion (29) is applied in Example 3. It is interesting to note that

$$\|C_1\| + \tau_1 \|B_1\| = 10.6337 > 1$$

and

$$\mu(A+B_1) + \|C_1\| \, \|A+B_1\| + \tau_1 \left\|(A+B_1)^T B_1\right\| = 27.1336 > 0.$$

Hence, the criterion in (Lien, 1999) cannot be satisfied.

Example 4. Consider the following neutral system with time delay as discussed in (Lien, 1999; Yang and Liu, 2002):

$$\dot{x}(t) = Ax(t) + \sum_{j=0}^{1} \left[B_j x(t-\tau_j) + C_j \dot{x}(t-\tau_j)\right], \quad t \geq 0, \tag{30}$$

where

$$A = \begin{bmatrix} -2 & -0.6 \\ -0.5 & -2 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} -1 & 0.2 \\ 0.5 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.1 & 0.1 \\ 0.3 & -0.1 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$\tau = \tau_0 = \tau_1.$$

Here we determine the delay time to make the system (30) asymptotically stable. ♦

Solution. By employing the condition (19) of Theorem 2, we obtain $\tau = 5.9193$. The result is that (30) is asymptotically stable for any constant time delay $\tau \leq 5.9193$. The upper bound on the delays that guarantees the stability of this system is $\tau \leq 2.1015$, which is 181.67% larger than the result given by Yang and Liu (2002).

Remark 3. If the delay-dependent criterion of (Lien, 1999) is employed, then the system (30) is asymptotically stable provided that

$$\sum_{i=1}^{m} \left[\, \|C_i\| + \tau_i \|B_i\| \,\right] < 1 \tag{31a}$$

Table 4. Upper bound on the delay time τ for various parameters α, $Q = \text{diag}\{0.3, 0.3\}$.

α	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35
Han (2004)	1.61	1.51	1.41	1.30	1.19	1.08	0.96	0.83
Our result	2.6609	2.4591	2.2756	2.1081	1.9546	1.8133	1.6829	1.5622

and

$$\mu\Big(A+\sum_{i=0}^{m} B_i\Big) + \sum_{i=0}^{m}\Big[\|C_i\|\Big\|A+\sum_{i=0}^{m} B_i\Big\| + \tau_i\Big\|\Big(A+\sum_{i=0}^{m} B_i\Big)^T B_i\Big\|\Big] < 0. \tag{31b}$$

Consequently, compute

$$\sum_{i=0}^{2}\big[\|C_i\| + \tau_i\,\|B_i\|\big] = 10.4727 > 1 \tag{32a}$$

and

$$\mu\Big(A+\sum_{i=0}^{1} B_i\Big) + \sum_{i=0}^{1}\Big[\|C_i\|\Big\|A+\sum_{i=0}^{1} B_i\Big\| + \tau_i\Big\|\Big(A+\sum_{i=0}^{1} B_i\Big)^T B_i\Big\|\Big] = 28.5547 > 0. \tag{32b}$$

Therefore, the criterion of (Lien, 1999) cannot be satisfied.

Example 5. Consider the following neutral time delay uncertain system as disussed in (Han, 2004):

$$\dot{x}(t) = \big(A + \Delta A(t)\big)x(t) + \big(B + \Delta B(t)\big)x(t-\tau) + C\dot{x}(t-\tau) \tag{33}$$

where

$$A = \begin{bmatrix} -0.9 & 0.2 \\ 0 & 0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -1.1 & 0.2 \\ -0.1 & -1.1 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{bmatrix},$$

$$\|\Delta A\| \le \alpha \ \text{ and } \ \|\Delta B\| \le \alpha, \quad \alpha \ge 0, \ \forall t.$$

♦

Solution. Decompose B using

$$B_1 = \begin{bmatrix} -0.283 & -0.182 \\ -0.08 & -0.260 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.817 & -0.018 \\ -0.02 & -0.84 \end{bmatrix}.$$

For $\|\Delta A\| \cong 0$ and $\|\Delta B\| \cong 0$ (i.e., nominal system), (33) reduces to the system discussed in Lien *et al.* (2000). Using the criterion in this paper, the maximum value of the delay time τ for the nominal system to be asymptotically stable is $\tau = 2.6609$. By the criteria in (Han, 2004; Lien *et al.*, 2000; Niculescu, 2000), the nominal system is asymptotically stable for any τ satisfying $\tau \le 0.3$, $\tau \le 0.71$, and $\tau \le 1.61$, respectively. This example shows again that the stability criterion produces a substantially less conservative result.

The effect of the uncertainty bound α on the maximum time delay for stability τ is listed in Table 4.

Table 4 illustrates the numerical results for various α. We can see that as $\alpha \to 0$, the stability limit for delay approaches the uncertainty-free case. As α increases, τ decreases.

Some comparisons have been made with the same examples that appear in many recent papers. Again, our result has shown to be less conservative.

5. Conclusion

In this paper, delay-dependent asymptotic stability criteria for parametrically perturbed neutral systems with single and multiple time delays were investigated. A Razumikhin-type theorem, the Lyapunov equation approach, and model transformation with decomposition techniques are employed to investigate the stability conditions. The objective of this paper was to guarantee an allowable bound on the delay time such that if time delays are less than the obtained constant delay bounds, neutral systems with time delays can be tolerated. Some comparisons were made with the same examples that appear in many recent papers. Significant improvements over those results are to be noted. Therefore, there is a stronger possibility that the proposed criterion is less conservative than those in the literature.

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