STABILITY OF A CLASS OF ADAPTIVE NONLINEAR SYSTEMS*

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This paper presents a research effort focused on the problem of robust stability of the closed-loop adaptive system. It is aimed at providing a general framework for the investigation of continuous-time, state-space systems required to track a (stable) reference model. This is motivated by the model reference adaptive control (MRAC) scheme, traditionally considered in such a setting. The application of differential inequlities results to the analysis of the Lyapunov stability for a class of nonlinear systems is investigated and it is shown how the problem of model following control may be tackled using this methodology.

Keywords: nonlinear systems, Lyapunov stability, adaptive systems

1. Introduction

Differential inequalities constitute a part of the qualitative theory of differential equations. In this paper we shall explore a part of this subject (Hatvany, 1975; Lakshmikantham and Leela, 1969a; 1969b; Rabczuk, 1976; Szarski, 1967; Walter, 1970) concerned with certain problems of ordinary differential equations (ODEs).

The qualitative theory of ODEs aims at investigating the properties of their solutions without explicit knowledge of their form. This is of significant importance in the nonlinear context, because then such knowledge is seldom available. What can be used, however, are the properties of the right-hand side (RHS) of the equation and information about its domain of definition.

Since the problem setting excluded quantitative (closed-form or numerical) knowledge about solutions, we must use some *qualitative* results about them, e.g., asymptotic properties (boundedness), the Lyapunov stability, monotonicity, etc. This (incomplete) list of examples shows that such results are of practical interest in the control engineering context. An especially interesting feature is that they may hold for families of solutions giving insights into robustness.

A well-known example of qualitative theory is the use of Lyapunov functions (Hahn, 1967) for inferring the Lyapunov stability. Another important, but not so well-known, example is the differential inequalities approach (Liu and Siegel, 1994). This may be applicable – among other things – to robust stability analysis (see Section 3), and hence it is of interest in nonlinear, adaptive control.

2. Differential Inequalities

We shall now briefly present some basic concepts and results pertaining to differential inequalities. Emphasis will be put on the scalar case, as it is directly relevant to the methods of Section 3 and much less involved than the vector ODE one.

The early history of the subject (Rabczuk, 1976) is related to the independent works of O. Perron and S.A. Chaplygin, and we explain the essence of the approach using Chaplygin's reasoning (Luzin, 1951).

Let us consider a scalar nonlinear ODE with known RHS f:

$$\dot{x} = f(x,t), \qquad x(t_0) = x_0, \tag{1}$$

defined in a domain Γ, such that $(x_0, t_0) \in \Gamma$ and (1) has, for each (x_0, t_0), a unique solution in Γ (e.g., f may be Lipschitz in Γ). Since f is nonlinear, we cannot, in general, integrate (1), so a method of approximate integration was proposed by Chaplygin (Luzin, 1951).

Suppose that $x = x(t)$ is the integral curve of (1) corresponding to the initial condition $(x_0, t_0) \in \Gamma$. Let us draw two differentiable curves $v = v(t)$ and $w = w(t)$, such that $v(t_0) = w(t_0) = x_0$ and

$$v(t) < x(t) < w(t) \tag{2}$$

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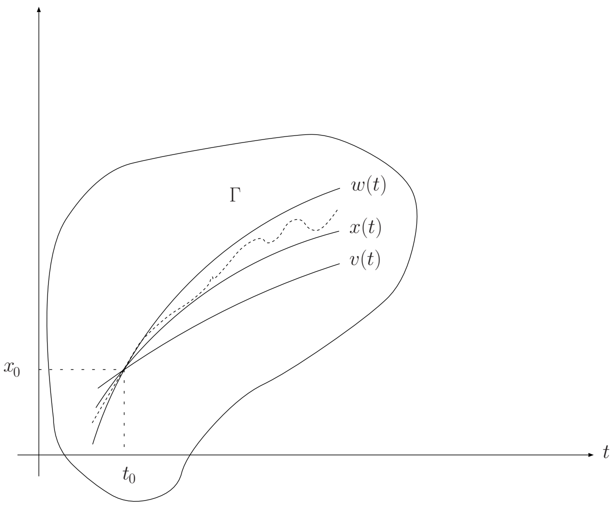

Fig. 1. Chaplygin's method of differential inequalities (comparison principle): $x(t)$ is the nominal, $v(t)$ a lower and $w(t)$ an upper solution. The ODE is $\dot{x} = f(x,t)$ with the initial condition (x_0, t_0) and the domain of definition Γ. The differential inequalities are $\dot{v} < f(v,t)$ and $f(w,t) < \dot{w}$ for $t > t_0$, implying $v(t) < x(t) < w(t)$ for $t > t_0$.

for all $t > t_0$ i t w Γ (see Fig. 1). If it is possible to arbitralily narrow the piece of Γ between $v(t)$ and $w(t)$, then the solution is approximated within any desired accuracy. Of course, $x(t)$ is unknown, so we must ask what conditions related to (1), $v(t)$ and $w(t)$) should be postulated so that (2) would hold.

A crucial observation is that since $x(t)$ is defined by a differential equation, it is characterised by its tangent for any t from Γ. In particular, this is true at $t = t_0$, so v and w also satisfy (1) at that point, as $v(t_0) = w(t_0) = x_0$. But f is at least continuous in Γ (to have the existence of $x(t)$), the equations

$$\begin{aligned} \dot{v} &= f(v,t), \\ \dot{w} &= f(w,t) \end{aligned} \tag{3}$$

imply that immediately to the right of t_0 the tangent of $v(t)$ is smaller than the tangent of $x(t)$, while the tangent of $w(t)$ is greater. In other words, $\dot{v} < \dot{x} < \dot{w} \Leftrightarrow f(v,t) < f(x,t) < f(w,t)$ for all t close to t_0 (with $t > t_0$), or the following *differential inequalities* hold:

$$\dot{v} < f(v,t), \tag{4}$$

$$\dot{w} > f(w,t) \tag{5}$$

locally to the right of t_0.

If (4) and (5) are true globally, i.e., for all $t > t_0$ (t in Γ), then the condition (2) is satisfied for all such t (see Fig. 1). Let us suppose that the differential inequalities hold and either $v(t)$ or $w(t)$ intersects $x(t)$ at $t_1 > t_0$ (when (2) fails an intersection must occur, as $v(t)$, $x(t)$ and $w(t)$ are continuous). Then, by the continuity of f, either (4) or (5) is violated in a neighbourhood of t_1, which is a contradiction.

The above reasoning leads to the following fundamental result (Luzin, 1951):

Theorem 1. *(Chaplygin) Let the scalar ordinary differential equation (1) be given with an initial condition $(x_0, t_0) \in \Gamma$, where Γ is a domain (open and connected set) of existence and uniqueness for (1). If the right-hand side f of (1) is continuous in Γ, $x(t)$ is the solution of (1) corresponding to (x_0, t_0) and the differential inequalities (4) and (5) hold for all $t > t_0$ (t in Γ) with $v(t_0) = w(t_0) = x_0$, then the inequalities (2) are also true for the same values of t.*

Note that the theorem gives an estimate (i.e., (2)) of the unknown solution $x(t)$ on the basis of known functions satisfying (4)–(5). It is a sufficient condition for that, as it requires that the upper (lower) estimate have a greater (smaller) tangent than $x(t)$ for all times $t > t_0$ (t in Γ). Clearly, the dotted curve in Fig. 1 satisfies (half of) (2) but not (5).

For a given f it may be nontrivial to find v and w for which (4) and (5) hold on a reasonably long interval of t. Therefore, the following simple consequence of Theorem 1 is useful:

Corollary 1. *Let (1) be as in Theorem 1. If there are functions f_1 and f_2, continuous in Γ, such that*

$$f_1(x,t) < f(x,t) < f_2(x,t) \tag{6}$$

for all $(x,t) \in \Gamma$ (where $t > t_0$) and ensuring for these (x,t) the existence and uniqueness of solutions of the scalar ordinary differential equations

$$\dot{v} = f_1(v,t), \qquad v(t_0) = x_0, \tag{7}$$

$$\dot{w} = f_2(w,t), \qquad w(t_0) = x_0, \tag{8}$$

then the solutions of (7) and (8) satisfy (2) for all $t > t_0$ (t in Γ).

Therefore, it suffices to find two ODEs whose right-hand sides bound the RHS of (1), as in (6). Of course, f_1 and f_2 should be chosen to be as simple as possible to allow an easy integration of (7) and (8). In particular, if f_1 and f_2 are linear, f is twice differentiable and we know (which is not always possible) the subset of Γ where $\partial^2 f/\partial x^2$ has a constant sign, then Corollary 1 leads to a powerful numerical method generating sequences $\{v_k(t)\}$ and $\{w_k(t)\}$ rapidly convergent to $x(t)$ (see Luzin, 1951).

Let us finally note that Theorem 1 and Corallary 1 deal with a nonlinear time-varying ODE, so that the setting is fairly general (confined, however, to the scalar case). This is particularly useful in the context of the Lyapunov stability (see Section 3), especially for adaptive control systems, as their closed-loop description explicitly involves time (unless we consider regulation).

The reasoning behind the proof of Theorem 1 cannot be repeated for vector ODEs and similar results are not so straightforward to obtain. In the linear case, however, some useful properties can be proved (Rabczuk, 1976). These results are interesting for linear time-varying systems, which are useful in the context of adaptive control.

Theorem 2. *Let A be an $n \times n$ constant matrix with real entries and $x : [0,\infty) \to \mathbb{R}^n$ a vector of functions satisfying the vector differential inequality*

$$\dot{x} \geq Ax, \qquad x(0) = x_0, \tag{9}$$

where the inequality is understood to be component-wise, i.e., $\dot{x}_i \geq \sum_{j=1}^{n} a_{ij}x_j$. A necessary and sufficient condition for a solution of (9) to be bounded from below by the solution of

$$\dot{v} = Av, \qquad v(0) = x_0 \tag{10}$$

is that $a_{ij} \geq 0$ for $i \neq j$.

Proof. (Necessity) Suppose that (9) holds. Then, defining $u(t) = \dot{x} - Ax$ (so that $u(t) \geq 0$ for all $t \geq 0$), (9) is equivalent to

$$\dot{x} = Ax + u(t), \qquad x(0) = x_0, \tag{11}$$

from which

$$x(t) = \mathrm{e}^{At} + \int_0^t \mathrm{e}^{A(t-s)}u(s)\,\mathrm{d}s. \tag{12}$$

In particular, if x_0=0, then $v(t) \equiv 0$, and thus we need to show that $x(t) \geq 0$ for all $t \geq 0$. From (12) we have $x(t) = \int_0^t \mathrm{e}^{A(t-s)}u(s)\mathrm{d}s$ with $u(s) \geq 0$ for all $0 \leq s \leq t$. Hence the proof of necessity reduces to a necessary condition for the nonnegativity of the matrix e^{At}, understood component-wise, i.e., $[\mathrm{e}^{At}]_{ij} \geq 0$ for all i, j. For small $t \geq 0$ we have

$$\mathrm{e}^{At} \approx I + At, \tag{13}$$

so that $a_{ij} \geq 0$ for $i \neq j$ gives the required property. (Note that a_{ii} may be negative, since – according to (13) – $[\mathrm{e}^{At}]_{ii} = 1 + a_{ii}t$ and thus choosing t small enough, $[\mathrm{e}^{At}]_{ii} \geq 0$.)

(Sufficiency) Let $A_{ij} \geq 0$ for $i \neq j$ hold. From (10) we have

$$v(t) = \mathrm{e}^{At}x_0 \tag{14}$$

and thus, by comparison with (12), we have to prove that $\int_0^t \mathrm{e}^{A(t-s)}u(s)\mathrm{d}s \geq 0$ for all $t \geq 0$. This, in the view of the nonnegativity of u, reduces to the demonstration of $\mathrm{e}^{At} \geq 0$ for all $t \geq 0$. Now, for any $m \in \mathbb{N}$, the identity

$$\mathrm{e}^{At} = (\mathrm{e}^{At/m})^m \tag{15}$$

holds trivially. For any arbitrarily large t there exists an $m \in \mathbb{N}$, such that (13) is valid for $\mathrm{e}^{At/m}$. But then (15) expresses e^{At} as a finite product of nonnegative matrices, and thus e^{At} is nonnegative as required. ■

This result is interesting for linear time-varying systems, which are useful in the context of adaptive control.

3. Lyapunov Functions via Differential Inequalities

The basics of differential inequalities have been explained in the previous section. Let us now turn to their applications to the Lyapunov stability. The approach presented here was pioneered by C. Corduneanu (1960; 1961), but for a comprehesive discussion see also (Lakshmikantham and Leela, 1969a).

Let us just recall that the Lyapunov direct method (Hahn, 1963) is of paramount importance for adaptive control systems (Narendra and Annaswamy, 1989). If the control objective is model reference tracking, then, even for the linear time-invariant model of the plant, the closed-loop system is nonlinear and nonautonomous, hence the importance of the uniform asymptotic stability of time varying systems (see Definition 5). This is a non-trivial problem and in the closed-loop configuration additionally complicated by the conflict of control action driving the tracking error to zero with the adaptation rule aiming at zeroing the parameter error. The nonlinear, time-varying behaviour should be considered in the presence of disturbances and unmodelled dynamics, and thus requires *robust* stability (see Definition 6). The differential inequalities approach to the Lyapunov stability deals precisely with this problem.

In order to make our discussion rigorous, we start with the necessary definitions. Let $K_{\rho,t_0} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} |\ \|x\| < \rho,\ t \geq t_0\}$ be a half-cylindrical neighbourhood of the t-axis in $\mathbb{R}^n \times \mathbb{R}$.

Definition 1. A function $\phi : [0,\rho] \to \mathbb{R}$ is said to belong to *the class* K if and only if it is continuous, monotonically increasing and $\phi(0) = 0$.

Definition 2. A function $V : K_{\rho,t_0)} \to \mathbb{R}$ is called *positive definite* if and only if $V(0,t) \equiv 0$ and there exists a function $\phi : [0,\rho] \to \mathbb{R}$ of the class K such that

$$V(x,t) \geq \phi(\|x\|) \tag{16}$$

in K_{ρ,t_0}. The function $-V$ is called *negative definite*.

Note that (16) implies (by Definition 1) that V is positive *uniformly* in t (for all $t \geq t_0$).

Definition 3. A function $V : K_{\rho,t_0)} \to \mathbb{R}$ is called *decrescent* if and only if there exists a function $\psi : [0, \rho] \to \mathbb{R}$ of the class K such that

$$V(x,t) \leq \psi(\|x\|) \tag{17}$$

in K_{ρ,t_0}.

The function ψ in (17) limits the growth of V from above *uniformly* in t (for all $t \geq t_0$).

Definition 4. A function $V : K_{\rho,t_0)} \to \mathbb{R}$ is called *radially unbounded* if and only if (16) holds for any ρ, where $\phi(\rho) \to \infty$ as $\rho \to \inf$.

Radial unboundedness is important for global stability results.

Definition 5. Consider the vector differential equation

$$\dot{x} = f(x,t), \qquad f(0,t) \equiv 0 \tag{18}$$

having a unique solution $x(t; t_0, x_0)$ in a domain $\Gamma \subset \mathbb{R}^n \times \mathbb{R}$ for each initial condition $(x_0, t_0) \in \Gamma$. We say that the trivial solution of (18) is

1. *stable* iff for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t; t_0, x_0)\| < \varepsilon$ for all $t \geq t_0$;
2. *uniformly stable* iff for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t; t_0, x_0)\| < \varepsilon$ for all $t \geq t_0$;
3. *asymptotically stable* iff it is stable and there exists $\delta^\star = \delta^\star(t_0) > 0$ such that $\|x_0\| < \delta^\star$ implies $\lim_{t\to\infty} \|x(t; t_0, x_0)\| = 0$;
4. *uniformly asymptotically stable* iff it is uniformly stable and there exists $\delta^\star > 0$ such that for any $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that $\|x_0\| < \delta^\star$ implies $\|x(t; t_0, x_0)\| < \varepsilon$ for all $t \geq t_0 + T(\varepsilon)$.

Definition 5 encapsulates the essential notions of the Lyapunov stability of the zero equilibrium of (18) needed in our context. The uniform stability (Concept 2 above) differs from the ordinary one (Concept 1) by being independent of the initial time t_0, despite the time-varying character of (18). The important strengthening in Concept 3 is that the trajectory $x(t; t_0, x_0)$ will asymptotically tend to zero, which is a critical property for error equations in adaptive systems. Finally, Concept 4 is the most desirable combination: the trivial solution of (18) is uniformly stable and it has a neighbourhood (defined by $\|x_0\| < \delta^\star$) such that invariably (i.e., for all times $t \geq t_0 + T(\varepsilon)$) the solution $x(t; t_0, x_0)$ will approach the equilibrium as closely as desired (within arbitrary precision ε).

Note that the notions of Definition 5 describe *local* properties, i.e., they postulate the existence of a neighbourhood of the t-axis (e.g., a half-cylinder K_{ρ,t_0}), in which the relevant conditions hold.

From the robustness point of view it is important to consider stability when the right-hand side of (18) is not known exactly and/or the system (18) operates in the presence of disturbances. This motivates the following definition:

Definition 6. Consider the vector differential equation

$$\dot{x} = f(x,t) + R(x,t), \qquad f(0,t) \equiv 0 \tag{19}$$

having a unique solution $x(t; t_0, x_0)$ in a domain $\Gamma \subset \mathbb{R}^n \times \mathbb{R}$ for each initial condition $(x_0, t_0) \in \Gamma$. We say that the trivial solution of (19) is

1. *integrally stable* iff, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\|x_0\| < \delta$ and
$$\int_{t_0}^{t} \sup_{\|x\|<\varepsilon} \|R(x,t)\| \,\mathrm{d}t < \delta \tag{20}$$
imply $\|x(t; t_0, x_0)\| < \varepsilon$ for all $t \geq t_0$;
2. *stable under perturbations bounded in the mean* iff, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\|x_0\| > \delta$ and
$$\sup_{t\geq t_0} \int_{t}^{t+1} \sup_{\|x\|<\varepsilon} \|R(x,t)\| \,\mathrm{d}t < \delta \tag{21}$$
imply $\|x(t; t_0, x_0)\| < \varepsilon$ for all $t \geq t_0$.

This definition deals with the robust stability of (19) and thus conditions are imposed on the perturbation term $R(x,t)$ in (19). In this framework Property 1 in Definition 5 is considered. Thus, Concept 1 in Definition 6 postulates that for all trajectories $x(t; t_0, x_0)$ (from a neighbourhood $\|x_0\| < \delta$), the contribution of $R(x,t)$ over time is small if so are the trajectories (see (20)). A similar idea is expressed in Part 2 of Definition 6 with one important difference: the smallness of $R(x,t)$ is measured by the boundedness in the mean, i.e., on all intervals $[t, t{+}1]$.

Let us now present the differential inequalities approach to the Lyapunov stability.

Lemma 1. *(Szarski, 1967) Let* $t_0 \geq 0$, $k_{r,t_0} = \{(y,t) \in \mathbb{R} \times \mathbb{R} |\ 0 \leq y < r \leq \infty,\ t \geq t_0\}$ *and* $\omega : k_{r,t_0} \to \mathbb{R}$ *be a continuous function. Consider the differential equation*

$$\dot{y} = \omega(y,t) \tag{22}$$

with $(y_0, t_0) \in k_{r,t_0}$*, and* J *the maximal interval of the existence of a solution* $y(t)$ *of (22) for* (y_0, t_0)*. If* $z(t)$ *is a continuous function for all* $t \in [t_0, t_1]$*, where* $J \subset [t_0, t_1]$*, such that*

$$D_+ z(t) \le \omega(z(t), t) \qquad \text{with} \qquad z(t_0) \le y_0 \tag{23}$$

for all $t \in [t_0, t_1]$*, then*

$$z(t) \le y(t) \tag{24}$$

for all $t \in [t_0, t_1]$*.*

Here

$$D_+ z(t) = \liminf_{h \to 0^+} [z(t+h) - z(t)]/h \tag{25}$$

is the lower right Dini derivative.

For the lemma to hold it suffices that ω is continuous, but then only the existence of solutions of (22) is guaranteed, and not necessarily the uniqueness. Moreover, nothing is said about the trivial solution of (22). In particular, we do not know if $y(t) \equiv 0$ satisfies (22). These additional assumptions must be made in order to proceed to the core results on stability (Corduneanu, 1960).

Theorem 3. *(Corduneanu) Let* ω *be as in Lemma 1, give rise to a unique solution of (22) for any* $(y_0, t_0) \in k_{r,t_0}$ *and* $\omega(0, t) \equiv 0$*. Moreover, suppose that the function* $V : K_{\rho,t_0} \to \mathbb{R}$*, associated with (18), is locally Lipschitz in* K_{ρ,t_0} *and satisfies*

$$V'(x, t) \le \omega(V(x, t), t) \tag{26}$$

for all $(x, t) \in K_{\rho,t_0}$*, where*

$$V'(x, t) = \liminf_{h \to 0^+} \frac{V(x + hf(x, t), t + h)}{h}. \tag{27}$$

1. *If the trivial solution of (22) is stable and* V *is positive definite, then the trivial solution of (18) is also stable.*

2. *If the trivial solution of (22) is uniformly stable and* V *is positive definite and decrescent, then the trivial solution of (18) is also uniformly stable.*

3. *If the trivial solution of (22) is asymptotically stable and* V *is positive definite, then the trivial solution of (18) is also asymptotically stable.*

4. *If the trivial solution of (22) is uniformly asymptotically stable and* V *is positive definite and decrescent, then the trivial solution of (18) is also uniformly asymptotically stable.*

5. *If the trivial solution of (22) is integrally stable,* V *is globally Lipschitz in* K_{ρ,t_0} *and positive definite, then the trivial solution of (18) is also integrally stable.*

6. *If the trivial solution of (22) is stable under perturbations bounded in the mean,* V *is globally Lipschitz in* K_{ρ,t_0} *and positive definite, then the trivial solution of (18) is also stable under perturbations bounded in the mean.*

Note that V is a Lyapunov function with (26) an analogue of the negative definiteness condition. However, it is more than that, as (26) allows considering families of positive definite (and decrescent, where appropriate) functions, and this is abetted by the possibility of choosing a convenient ω. This allows investigating robust stability, even via conditions 1–4 of Theorem 3. The claims 5–6 intrinsically deal with perturbations of the system (18).

In Theorem 3 the role of $z(t)$ of Lemma 1 is played by $V(x, t)$ evaluated along the solution $x(t; x_0, t_0)$. The lower right Dini derivative of (27), together with the (local or global) Lipschitz condition, corresponds to the total (along the trajectory) derivative of the differentiable Lyapunov function.

While the results of Theorem 3 are, by definition, local in nature, they can be extended to the global ones (stability in the large), if it is possible to find a sufficiently large ρ. If the required domain of stability is unbounded or equal to $\mathbb{R}^n$ (stability in the whole), V must additionally be radially unbounded. Again the robustness aspect of the method will be retained.

An analogous approach (Corduneanu, 1961) allows investigating instability.

4. MRAC via Special Techniques of the Lyapunov Stability

In this section we describe certain special techniques of the Lyapunov stability, which, coupled with the apparatus of the previous sections, can be used to tackle the problem of model following. They are also applicable in the model reference adaptive control (MRAC) context and offer an alternative perspective on the problem.

Section 4.1 presents a hardly known approach, originally proposed in a different context in mechanics. Here it is recast in the model following framework. Section 4.1.1 shows how a development in the Lyapunov theory which is independent of the results of Section 4.1 leads to a more general setting relevant in the MRAC setup. The techniques employ the differential inequalities machinery of Section 3, thus offering robustness.

4.1. Model Following: The Stability Approach

Early results are due to Makarov (1938), who considered the following problem:

Definition 7. Let us consider the system of two vector differential equations

$$\dot{r} = g(r,t), \qquad g(0,t) \equiv 0, \tag{28}$$

$$\dot{x} = f(x,r,t) \tag{29}$$

having in a domain $\Gamma \subset \mathbb{R}^n \times \mathbb{R}$, for all initial conditions (r_0,t_0),$(x_0,t_0) \in \Gamma$, a unique solution $r(t;t_0,r_0)$ of (28) and a unique solution $x(t;t_0,x_0)$ of (29). The system (28)–(29) is called *Makarov stable* if and only if for any $\varepsilon > 0$ there exist $\delta_1 = \delta_1(\varepsilon,t_0)$ and $\delta_2 = \delta_2(\varepsilon,t_0)$ such that $\|r_0\| < \delta_1$ and $\|r_0 - x_0\| < \delta_2$ imply $\|r(t;t_0,x_0)\| < \varepsilon$ and $\|r(t;t_0,x_0) - x(t;t_0,x_0)\| < \varepsilon$ for all $t \ge t_0$.

In essence, the definition postulates that the reference model (28) be stable (in the sense of Part 1 of Definition 5) and so be the error between the reference $r(t)$ and the state $x(t)$ of the system (29). By modifications similar to those in Parts 2–4 of Definition 5, we can get analogous versions of the Makarov stability.

Of course, introducing the new variable $e(t) = r(t) - x(t)$ the problem may be recast into the $(2n+1)$-dimensional classical (i.e., Part 1 of Definition 5) setting. However, Definition 7 allows using *a priori* knowledge of (29) and does not require $f(0,0,t) \equiv 0$. Also, it may be easier to work directly with (29) than to deal with the new system obtained from the concatenation of $r(t)$ and $e(t)$. In fact, the investigation of the Makarov stability can be done by considering *two* Lyapunov-like functions: one for $r(t)$, the other for $r(t) - x(t)$. To give the result, we need the following definition:

Definition 8. Let $M_{\rho,t_0} = \{(r,x,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mid \|r - x\| < \rho,\ t \ge t_0\}$. A function $V : M_{\rho,t_0} \to \mathbb{R}$ is called *Makarov positive definite* if and only if $v(r,r,t) \equiv 0$ and there exists a function $\phi : [0,\rho] \to \mathbb{R}$ of the class K such that

$$V(r,x,t) \ge \phi(\|r - x\|) \tag{30}$$

in M_{ρ,t_0}.

Recall that K_{ρ,t_0} dealt with the t-axis in $\mathbb{R}^n \times \mathbb{R}$. In Definition 8, M_{ρ,t_0} takes into account that the 'desired' trivial solution is with respect to the difference $r(t) - x(t)$.

Theorem 4. *(Makarov) Let there exist two differentiable functions $V_1 : K_{\rho,t_1} \to \mathbb{R}$ and $V_2 : M_{\rho,t_1} \to \mathbb{R}$, such that V_1 is positive definite and V_2 is Makarov positive definite. The system (28)–(29) is Makarov stable if the derivatives*

$$V_1'(r,t) = \lim_{h\to 0} \frac{V_1(r + hg(r,t), t+h)}{h} = \frac{\partial V_1}{\partial t} + \sum_{k=1}^{n} \frac{\partial V_1}{\partial r_k} g_k(r,t),$$

$$V_2'(r,x,t) = \lim_{h\to 0} \frac{V_2(r+hg(r,t), x+hf(x,r,t), t+h)}{h} = \frac{\partial V_2}{\partial t} + \sum_{k=1}^{n} \Big(\frac{\partial V_2}{\partial r_k} g_k(r,t) + \frac{\partial V_2}{\partial x_k} f_k(r,x,t) \Big)$$

are negative definite in K_{ρ,t_0} and M_{ρ,t_0} respectively.

Instead of the differentiability of V_1 and V_2 in Theorem 4, we could require them to be locally Lipschitz and consider the lower right Dini derivative, as in Theorem 3. However, differentiability permits the explicit formulae (31) for V_1' and V_2', which are helpful in understanding the essence of Makarov's approach. Instability results in the same vein can also be obtained.

The idea of considering two simultaneous systems of vector differential equations and analysing the stability of the difference of their solutions has been pursued in various contexts by several authors. Particularly interesting are methods utilising differential inequalities, as in the works of (Lakshmikantham, 1962a; 1962b; Lakshmikantham and Leela, 1969a). The dicrete-time analogoues were considered by Pachpatte (1971).

4.1.1. Partial Stability

Makarov's approach to model following, presented in Section 4.1, is closelly related to partial stability, because we may take for granted the stability of the reference model (28). Thus, the problem reduces to the issue of the stability of (29), i.e., it concerns only a part of the $(2n+1)$-dimensional space. This motivates the following definition (Corduneanu, 1964).

Definition 9. Let us consider the system of two vector differential equations

$$\dot{r} = G(r,e,t), \qquad G(0,0,t) \equiv 0, \tag{31}$$

$$\dot{e} = F(r,e,t), \qquad F(0,0,t) \equiv 0, \tag{32}$$

having in a domain $\Gamma \subset \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$ a unique solution $r(t;t_0,r_0,e_0)$ of (31) and a unique solution $e(t;t_0,r_0,e_0)$ of (32) for all initial conditions $(r_0,e_0,t_0) \in \Gamma$. The trivial solution $(r(t),e(t)) \equiv$

$(0,0) \in \mathbb{R}^m \times \mathbb{R}^n$ of (31)–(32) is called *partially stable* with respect to e if and only if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, t_0)$, such that $\|r_0\| + \|e_0\| < \delta$ implies $\|e(t, t_0, r_0, e_0)\| < \varepsilon$ for all $t \geq t_0$.

Let us note the special structure of the problem, emphasising the role of the solution $e(t) \equiv 0$ of (32). The setting does not require the stability with respect to r, only the existence of the trivial solution $(r(t), e(t)) \equiv (0,0)$ $(r(t), e(t)) \equiv 0$. The vector e in (31)–(32) may be interpreted in the context of MRAC as $e(t) = r(t) - x(t)$, provided $n = m$ (see remarks to Definition 7). Definition 9 is a general one, so solutions of (31) depend on e_0 (since G depends on e), but in the setting similar to Definition 7 this would not be relevant, as (31) would essentially be (28).

By standard modifications of Definition 9 we can get other notions of partial stability, analogous to Parts 2–4 of Definition 5. In this context we need the following analogues of Definitions 2 and 3, where $P_{\rho_0,t_0} = \{(r,e,t) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} | \; \|e\| < \rho, \; t \geq t_0\}$ emulates K_{ρ,t_0}:

Definition 10. A function $V : P_{\rho,t_0} \to \mathbb{R}$ is called *positive definite with respect to* e if and only if $V(0,0,t) \equiv 0$ and there exists a function $\phi : [0,\rho] \to \mathbb{R}$ of the class K, such that

$$V(r,e,t) \geq \phi(\|e\|) \tag{33}$$

in P_{ρ,t_0}.

Definition 11. A function $V : P_{\rho,t_0} \to \mathbb{R}$ is called *partially decrescent* if and only if there exists a function $\psi : [0,\rho] \to \mathbb{R}$ of the class K such that

$$V(r,e,t) \leq \psi(\|r\| + \|e\|) \tag{34}$$

in P_{ρ,t_0}.

The main result in the style of Theorem 3 is the following:

Theorem 5. *Let ω be as in Lemma 1, give rise to a unique solution of (22) for any $(y_0, t_0) \in k_{r,t_0}$ and $\omega(0,t) \equiv 0$. Moreover, suppose that the function $V : P_{\rho,t_0} \to \mathbb{R}$, associated with equations (31)–(32), is locally Lipschitz in P_{ρ,t_0} and satisfies*

$$V'(r,e,t) \leq \omega(V(r,e,t),t) \tag{35}$$

for all $(r,e,t) \in P_{\rho,t_0}$, where

$$V'(r,e,t) = \liminf_{h\to 0^+} \frac{V(r+hG(r,e,t), e+hF(r,e,t), t+h}{h}. \tag{36}$$

1. *If the trivial solution of (22) is stable and V is positive definite with respect to e, then the trivial solution $r(t) \equiv 0$, $e(t) \equiv 0$ of (31)–(32) is partially stable with respect to e.*

2. *If the trivial solution of (22) is uniformly stable, V is positive definite with respect to e and partially decrescent, then the trivial solution $r(t) \equiv 0$, $e(t) \equiv 0$ of (31)–(32) is uniformly stable with respect to e.*

3. *If the trivial solution of (22) is asymptotically stable and V is positive definite with respect to e, then the trivial solution $r(t) \equiv 0$, $e(t) \equiv 0$ of (31)–(32) is asymptotically stable with respect to e.*

4. *If the trivial solution of (22) is uniformly asymptotically stable and V is positive definite with respect to e and partially decrescent, then the trivial solution $r(t) \equiv 0$, $e(t) \equiv 0$ of (31)–(32) is uniformly, asymptotically stable.*

The reasoning can be carried over to Cases 5 and 6 of Theorem 3. The required modifications of Definition 6 are straightforward. If V in Theorem 5 is differentiable, as opposed to the weaker Lipschitz condition, then the similarity of (36) and (31) is explicit.

5. Conclusions

The approach presented in the paper is an attempt of applying the results from the quantitative theory of ordinary differential equations to the analysis of the robustness of the Lyapunov stability for nonlinear systems. This approach is similar to the one that has been used with success to the analysis of the BIBO stability of discrete-time, nonlinear systems using difference inequalities (Dzieliński, 2002a; 2002b). In the continuous case presented here, differential inequalities are used.

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