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NONLINEAR STABILIZING CONTROL OF AN UNCERTAIN BIOPROCESS MODEL

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In this paper we consider a nonlinear model of a biological wastewater treatment process, based on two microbial populations and two substrates. The model, described by a four-dimensional dynamic system, is known to be practically verified and reliable. First we study the equilibrium points of the open-loop system, their stability and local bifurcations with respect to the control variable. Further we propose a feedback control law for asymptotic stabilization of the closed-loop system towards a previously chosen operating point. A numerical extremum seeking algorithm is designed to stabilize the dynamics towards the maximum methane output flow rate in the presence of coefficient uncertainties. Computer simulations in Maple are reported to illustrate the theoretical results.

Keywords: wastewater treatment model, local bifurcations of steady states, asymptotic stabilization, extremum seeking, uncertain data.

1. Introduction

Biological wastewater treatment using anaerobic digestion is a process where microorganisms decompose the organic compounds inside the effluent to reduce the pollutant concentration in the outlet stream below a specified value, usually fixed by environmental and safety rules. At the same time, this process can also produce valuable energy (methane). This complex ecosystem involves plenty of bacterial species, whose dynamics are difficult to grasp due to physiological reasons. The operation of such processes poses a number of practical problems, since anaerobic digestion is a complex nonlinear system, which is known to be unstable, see, e.g., (Antonelli *et al.*, 2003) and the references there. One of the main drawbacks in the modelling and control of anaerobic digestion lies in the difficulty to monitor on-line the key biological variables of the process. A second drawback is the high uncertainty of the models describing the biological process; this is due to the fact that in most cases the reproducibility of laboratory and practical experiments is not guaranteed, hence the experimental data are noisy and the reason for the noise is difficult to specify. Thus developing control systems only based on simple measurements that guarantee the stability of the process in the presence of uncertainties is of primary importance. For further information about different

control approaches see, e.g., (Antonelli *et al.*, 2003; Heinzle *et al.*, 1993; Mailert *et al.*, 2004) and the references there.

The present paper is devoted to studying a four-dimensional nonlinear control system that models a wastewater treatment process. In the previous work (Dimitrova and Krastanov, 2006), the authors designed an adaptive stabilizing feedback control law for the same model in the presence of parameter uncertainties. This adaptive feedback depends on observable state variables s_1 and x_1 (see the definitions below) and stabilizes asymptotically the closed-loop system towards an (unknown) equilibrium point such that its projection on the s_1-axis is equal to a previously chosen operating point s_1^*. Here we propose a feedback law for asymptotic stabilization that is more practically oriented, i.e., it depends only on on-line measurable quantities.

The paper is organized as follows: Section 2 presents shortly the dynamic model of the biological wastewater treatment process. Section 3 is devoted to studying the local one-parameter bifurcations of the equilibrium points of the open-loop system: it is shown that the dynamics undergo transcritical bifurcations with respect to the control input (the dilution rate) u. Assuming that the model parameters are unknown but bounded within intervals, asymptotic stabilization of the dynamic system towards

a previously chosen operating point (called also reference or set point) is studied in Section 4. In order to prove that the closed-loop system is asymptotically stable, a suitable Lyapunov function is constructed explicitly. Choosing in a proper way different operating points, an algorithm is used in Section 5 to stabilize the dynamic system towards the equilibrium point where the maximum production of biogas (methane) is achieved. This problem is known as extremum (peak) seeking. In the literature, extremum seeking algorithms are usually designed in the form of block diagrams (schemes) that are implemented on a bioreactor to tune the dilution rate of the open-loop system, see, e.g., (Marcos *et al.*, 2004; Simeonov *et al.*, 2004; 2007; Wagner *et al.*, 1999). The algorithm presented here, in contrast to the above mentioned, is numerical; it is designed to stabilize the closed-loop system towards the desired maximum operating point. Computer simulations illustrating the robustness of the theoretical results are reported in Section 6. The extremum seeking algorithm is sketched in Appendix.

2. Model description

We consider a model of an anaerobic digestion process, based on two main reactions (Bernard *et al.*, 2001; Hess and Bernard, 2008): (a) acidogenesis, where the organic substrate (denoted by s_1) is degraded into volatile fatty acids (VFA, denoted by s_2) by acidogenic bacteria (x_1); (b) methanogenesis, where VFA are degraded into methane CH_4 and carbon dioxide CO_2 by methanogenic bacteria (x_2). We assume that the methane flow rate is the measurable output and denote it by Q. The mass balance model in a continuously stirred tank bioreactor is described by the following nonlinear system of ordinary differential equations:

$$\frac{\mathrm{d}s_1}{\mathrm{d}t} = u(s_1^i - s_1) - k_1 \cdot \mu_1(s_1) \cdot x_1, \tag{1}$$

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = (\mu_1(s_1) - \alpha u) \cdot x_1, \tag{2}$$

$$\frac{\mathrm{d}s_2}{\mathrm{d}t} = u(s_2^i - s_2) + k_2 \cdot \mu_1(s_1) \cdot x_1 - k_3 \cdot \mu_2(s_2) \cdot x_2, \tag{3}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = (\mu_2(s_2) - \alpha u) \cdot x_2, \tag{4}$$

$$Q = k_4 \cdot \mu_2(s_2) \cdot x_2, \tag{5}$$

where

$$\mu_1(s_1) = \frac{\mu_{\max} s_1}{k_{s_1} + s_1},$$

$$\mu_2(s_2) = \frac{\mu_0 s_2}{k_{s_2} + s_2 + \left(\frac{s_2}{k_I}\right)^2}$$

are model functions of the specific growth rates of the microorganisms. The first one, $\mu_1(s_1)$, presents the Monod law; the second one, $\mu_2(s_2)$, presents the Haldane law

Table 1. Definition of the model variables and parameters.

s_1	concentration of chemical oxygen demand (COD) [g/l]
s_2	concentration of volatile fatty acids (VFA) [mmol/l]
x_1	concentration of acidogenic bacteria [g/l]
x_2	concentration of methanogenic bacteria [g/l]
u	dilution rate [day^{-1}]
s_1^i	influent concentration s_1 [g/l]
s_2^i	influent concentration s_2 [mmol/l]
k_1	yield coefficient for COD degradation [g COD/(g x_1)]
k_2	yield coefficient for VFA production [mmol VFA/(g x_1)]
k_3	yield coefficient for VFA consumption [mmol VFA/(g x_2)]
k_4	coefficient [l^2/g]
$\mu_{\max}$	maximum acidogenic biomass growth rate [day^{-1}]
μ_0	maximum methanogenic biomass growth rate [day^{-1}]
k_{s_1}	saturation parameter associated with s_1 [g COD/l]
k_{s_2}	saturation parameter associated with s_2 [mmol VFA/l]
k_I	inhibition constant associated with s_2 [(mmol VFA/l)$^{1/2}$]
α	proportion of dilution rate reflecting process heterogeneity
Q	methane gas flow rate

and exhibits substrate inhibition, that is, there is a point $\tilde{s}_2$ such that $\mu_2(s_2)$ achieves its maximum (Bastin and Dochain, 1990),

$$\tilde{s}_2 = k_I \sqrt{k_{s_2}}. \tag{6}$$

The state variables s_1, s_2 and x_1, x_2 denote substrate and biomass concentrations, respectively: s_1 represents the organic substrate, characterized by its chemical oxygen demand (COD), s_2 denotes the volatile fatty acids (VFA), x_1 and x_2 are the acidogenic and methanogenic bacteria. The definition of the model parameters is given in Table 1.

The parameter $\alpha \in [0, 1]$ represents the proportion of bacteria that are affected by the dilution; $\alpha = 0$ and $\alpha = 1$ correspond to an ideal fixed bed reactor and to an ideal continuous stirred tank reactor, respectively (cf., for example, (Alcaraz-González *et al.*, 2002; Antonelli *et al.*, 2003; Bernard *et al.*, 2000; 2001; Grognard and Bernard, 2006; Schoefs *et al.*, 2003)). The dilution rate u is considered as a control variable.

3. Bifurcation analysis of the equilibrium points of the open-loop system

In this section we shall consider the open-loop system (1)–(4), that is, we assume that the control input u is a positive parameter.

3.1. Equilibrium points. The equilibrium points are solutions of the nonlinear algebraic system obtained from (1)–(4) by setting the right-hand side functions to zero, that is,

$$u \cdot (s_1^i - s_1) - k_1 \cdot \mu_1(s_1) \cdot x_1 = 0, \tag{7}$$

$$(\mu_1(s_1) - \alpha u) \cdot x_1 = 0, \tag{8}$$

$$u \cdot (s_2^i - s_2) + k_2 \cdot \mu_1(s_1) \cdot x_1 - k_3 \cdot \mu_2(s_2) \cdot x_2 = 0, \tag{9}$$

$$(\mu_2(s_2) - \alpha u) \cdot x_2 = 0. \tag{10}$$

We are looking for nonnegative solutions s_j and x_j, $j = 1, 2$, of the above system. For that reason, we assume that the following inequalities are valid (Bastin and Dochain, 1990; Hess and Bernard, 2008):

$$s_1 \leq s_1^i, \quad s_2 \leq s_2^i + \frac{k_2}{k_1} s_1^i. \tag{11}$$

The second inequality in (11) is motivated by the fact that the imbalance between acidogenesis and methanogenesis might lead to the accumulation of VFA (s_2) and therefore to the acidification ($x_2 = 0$) of the bioreactor; thus $s_2^i + (k_2/k_1)s_1^i$ can be considered as a worst-case upper bound of the total concentration s_2 (Hess and Bernard, 2008). As mentioned before, the Haldane model function $\mu_2(s_2)$ achieves a maximum at the point $\tilde{s}_2$ (see (6)). We assume that

$$\tilde{s}_2 \leq s_2^i, \tag{12}$$

otherwise $\mu_2(s_2)$ would be monotonically increasing for $s_2 \geq 0$ as the Monod law $\mu_1(s_1)$ for $s_1 \geq 0$ does.

Equations (7) and (8) are uncoupled with respect to s_2 and x_2. They possess a nontrivial solution

$$s_1^{(1)}(u) = \frac{\alpha u k_{s_1}}{\mu_{\max} - \alpha u}, \quad x_1^{(1)}(u) = \frac{s_1^i - s_1(u)}{\alpha k_1},$$
$$u \in \left(0, \frac{1}{\alpha}\mu_1(s_1^i)\right) \tag{13}$$

and a trivial solution (called the wash-out state with respect to x_1)

$$s_1^{(2)}(u) = s_1^i, \quad x_1^{(2)}(u) = 0 \text{ for all } u > 0. \tag{14}$$

Write

$$u_1 = \frac{1}{\alpha}\mu_1(s_1^i).$$

Note that $u < u_1$ implies $s_1^{(1)}(u) < s_1^i$. For $u = u_1$ we get

$$s_1^{(1)}(u_1) = s_1^{(2)}(u_1) = s_1^i, \quad x_1^{(1)}(u_1) = x_1^{(2)}(u_1) = 0.$$

Consider (10) and assume first that $x_2 \neq 0$. Then $\mu_2(s_2) - \alpha u = 0$ is equivalent to the following quadratic equation with respect to s_2:

$$\frac{\alpha u}{k_I^2} s_2^2 + (\alpha u - \mu_0)s_2 + \alpha u k_{s_2} = 0, \tag{15}$$

for which the discriminant

$$\Delta(u) = \alpha^2 \left(1 - 4\frac{k_{s_2}}{k_I^2}\right) u^2 - 2\alpha\mu_0 u + \mu_0^2$$

vanishes at the points

$$u_2 = \frac{\mu_0}{\alpha\left(1 + 2\sqrt{k_{s_2}}/k_I\right)}, \quad u^{(2)} = \frac{\mu_0}{\alpha\left(1 - 2\sqrt{k_{s_2}}/k_I\right)}.$$

Obviously, $u_2 > 0$ and $\text{sign}\, u^{(2)} = \text{sign}\left(1 - 2\sqrt{k_{s_2}}/k_I\right)$.

Let first $u \in (0, u_2]$. Then $\Delta(u) \geq 0$ and $\alpha u < \mu_0$ are valid. Thus (15) possesses two positive roots,

$$s_2^{(1)}(u) = \frac{2\alpha u k_{s_2}}{\mu_0 - \alpha u + \sqrt{\Delta(u)}}, \tag{16}$$

$$s_2^{(2)}(u) = \frac{\mu_0 - \alpha u + \sqrt{\Delta(u)}}{2\alpha u/k_I^2}. \tag{17}$$

If $u^{(2)} > 0$ is fulfilled, then $\Delta(u) \geq 0$ for $u \in [u^{(2)}, +\infty)$. In this case, $\alpha u > \mu_0$ holds true and the two roots (16) and (17) are negative, which is physically impossible. Therefore the admissible interval for the steady states (16), (17) is $(0, u_2]$. It is straightforward to see that

$$u_2 = \frac{1}{\alpha}\mu_2(\tilde{s}_2).$$

Further, solving (9) with $s_1 = s_1^{(1)}(u)$, $x_1 = x_1^{(1)}(u)$, we obtain the roots

$$x_2^{(1)}(u) = \frac{s_2^i - s_2^{(1)}(u) + \alpha k_2 x_1^{(1)}(u)}{\alpha k_3},$$

$$x_2^{(2)}(u) = \frac{s_2^i - s_2^{(2)}(u) + \alpha k_2 x_1^{(1)}(u)}{\alpha k_3}.$$

Substituting $x_1^{(1)}(u)$ by $x_1^{(2)}(u) = 0$ into the above expressions for $x_2^{(1)}(u)$ and $x_2^{(2)}(u)$, we obtain

$$x_2^{(3)}(u) = \frac{s_2^i - s_2^{(1)}(u)}{\alpha k_3}, \quad x_2^{(4)}(u) = \frac{s_2^i - s_2^{(2)}(u)}{\alpha k_3}.$$

Assume now that $x_2 = 0$ in (10) holds true, that is, we set

$$x_2^{(5)} = 0.$$

Then solving (9) with $s_1 = s_1^{(1)}(u)$ and $x_1 = x_1^{(1)}(u)$ implies

$$s_2^{(3)}(u) = s_2^i + \frac{k_2}{k_1}\left(s_1^i - s_1^{(1)}(u)\right) \text{ for } u \in (0, u_1).$$

Finally, solving (9) with $s_1 = s_1^i$, $x_1 = 0$ leads to

$$s_2^{(4)}(u) = s_2^i.$$

All components of the equilibrium points should be nonnegative. This condition determines the admissible values for the parameter u. First let us note that $s_2^{(1)}(u)$ is a monotonically increasing function of u whereas $s_2^{(2)}(u)$ is monotonically decreasing on $(0, u_2]$. Moreover,

$$s_2^{(1)}(0) = 0, \quad s_2^{(1)}(u_2) = s_2^{(2)}(u_2) = \tilde{s}_2 \leq s_2^i,$$
$$\lim_{u \to 0} s_2^{(2)}(u) = +\infty.$$

Therefore, $x_2^{(3)}(u) \geq 0$ for $u \in (0, u_2]$. Further,

$$x_2^{(4)}(u) \geq 0 \iff s_2^{(2)}(u) \leq s_2^i \iff u \leq \frac{1}{\alpha} \mu_2(s_2^i).$$

Write

$$u_3 = \frac{1}{\alpha} \mu_2(s_2^i).$$

According to the assumption (11), $s_2^{(2)}(u)$ should satisfy

$$s_2^{(2)}(u) \leq s_2^i + \frac{k_2}{k_1} s_1^i,$$

which is possible if and only if

$$u \geq \frac{1}{\alpha} \mu_2 \left(s_2^i + \frac{k_2}{k_1} s_1^i \right).$$

Write

$$u_4 = \frac{1}{\alpha} \mu_2 \left(s_2^i + \frac{k_2}{k_1} s_1^i \right).$$

Further, $x_2^{(2)}(u)$ is a monotonically increasing function of u, $\lim_{u \to 0} x_2^{(2)}(u) = -\infty$ and $x_2^{(2)}(u_2) > 0$. Thus there is a unique point

$$u_5 \in (0, u_2) \text{ such that } x_2^{(2)}(u_5) = 0. \qquad (18)$$

Then $x_2^{(2)}(u) \geq 0$ if and only if $u \geq u_5$.

Obviously, $\tilde{s}_2 \leq s_2^i$ (see (12)) implies

$$u_4 < u_3 \leq u_2. \qquad (19)$$

Finally, we have the following six equilibrium points

$$E_1(u) = \left(s_1^{(1)}(u), x_1^{(1)}(u), s_2^{(1)}(u), x_2^{(1)}(u) \right),$$
$$u \in (0, \min\{u_1, u_2\}],$$
$$E_2(u) = \left(s_1^{(1)}(u), x_1^{(1)}(u), s_2^{(2)}(u), x_2^{(2)}(u) \right),$$
$$u \in [\max\{u_4, u_5\}, \min\{u_1, u_2\}],$$
$$E_3(u) = \left(s_1^{(1)}(u), x_1^{(1)}(u), s_2^{(3)}(u), 0 \right), \quad u \in (0, u_1],$$
$$E_4(u) = \left(s_1^i, 0, s_2^{(1)}(u), x_2^{(3)}(u) \right), \quad u \in (0, u_3],$$
$$E_5(u) = \left(s_1^i, 0, s_2^{(2)}(u), x_2^{(4)}(u) \right), \quad u \in [u_3, u_2],$$
$$E_6(u) = \left(s_1^i, 0, s_2^i, 0 \right), \quad u \geq 0.$$

Remark 1. The equilibrium point $E_2(u)$ does not exist if $u_1 < \max\{u_4, u_5\}$.

Remark 2. It follows from (9) and (7) that $x_2 = 0$, $s_2 = s_2^i$ always leads to $x_1 = 0$, $s_1 = s_1^i$.

Figure 1 presents the curves (branches) of the steady state components as functions of the parameter u.

3.2. Local one-parameter bifurcations of the equilibrium points. Denote, for simplicity, $z = (s_1, x_1, s_2, x_2)$ and by $G = G(z; u) = (G_1, G_2, G_3, G_4)^T$ the vector of right-hand side functions of (1)–(4). It is known that if an equilibrium point is hyperbolic, that is, when the linearization (Jacobian matrix) $DG(z_0; u_0) = D_z G(z_0; u_0)$ at some equilibrium point z_0 for some value of $u = u_0$ does not possess eigenvalues on the imaginary axis, then $(z_0; u_0)$ is linearly stable or unstable. This means that varying u slightly in a neighborhood of u_0 will not change the nature of the stability of the steady state (Carr, 1981; Wiggins, 1990). When $(z_0; u_0)$ is not hyperbolic, that is, when $DG(z_0; u_0)$ has some eigenvalues on the imaginary axis, then for u close to u_0 a new dynamical behavior can occur. In what follows we shall consider the simplest way in which an equilibrium point can be nonhyperbolic, namely, when $DG(z_0; u_0)$ possesses a single zero eigenvalue with the remaining eigenvalues having nonzero real parts.

The Jacobian matrix has the form

$$DG(z; u) = \begin{pmatrix} \dfrac{\partial G_1}{\partial s_1} & \dfrac{\partial G_1}{\partial x_1} & 0 & 0 \\[2mm] \dfrac{\partial G_2}{\partial s_1} & \dfrac{\partial G_2}{\partial x_1} & 0 & 0 \\[2mm] \dfrac{\partial G_3}{\partial s_1} & \dfrac{\partial G_3}{\partial x_1} & \dfrac{\partial G_3}{\partial s_2} & \dfrac{\partial G_3}{\partial x_2} \\[2mm] 0 & 0 & \dfrac{\partial G_4}{\partial s_2} & \dfrac{\partial G_4}{\partial x_2} \end{pmatrix}.$$

Taking into account the particular expressions of the right-hand side functions $G_i(z; u)$, $i = 1, \ldots, 4$, the determinant $|DG(z; u)|$ of $DG(z; u)$ can be formed as a product $|DG(z; u)| = |D^1 G(z; u)| \cdot |D^2 G(z; u)|$, where

$$|D^1 G(z; u)|$$
$$= \frac{\partial G_1}{\partial s_1} \frac{\partial G_2}{\partial x_1} - \frac{\partial G_1}{\partial x_1} \frac{\partial G_2}{\partial s_1}$$
$$= -u \left(\mu_1(s_1) - \alpha u \right) + \alpha k_1 u x_1 \frac{\mathrm{d}}{\mathrm{d}s_1} \mu_1(s_1),$$

$$|D^2 G(z; u)|$$
$$= \frac{\partial G_3}{\partial s_2} \frac{\partial G_4}{\partial x_2} - \frac{\partial G_3}{\partial x_2} \frac{\partial G_4}{\partial s_2}$$
$$= u \left(\alpha k_3 x_2 \frac{\mathrm{d}}{\mathrm{d}s_2} \mu_2(s_2) - (\mu_2(s_2) - \alpha u) \right).$$
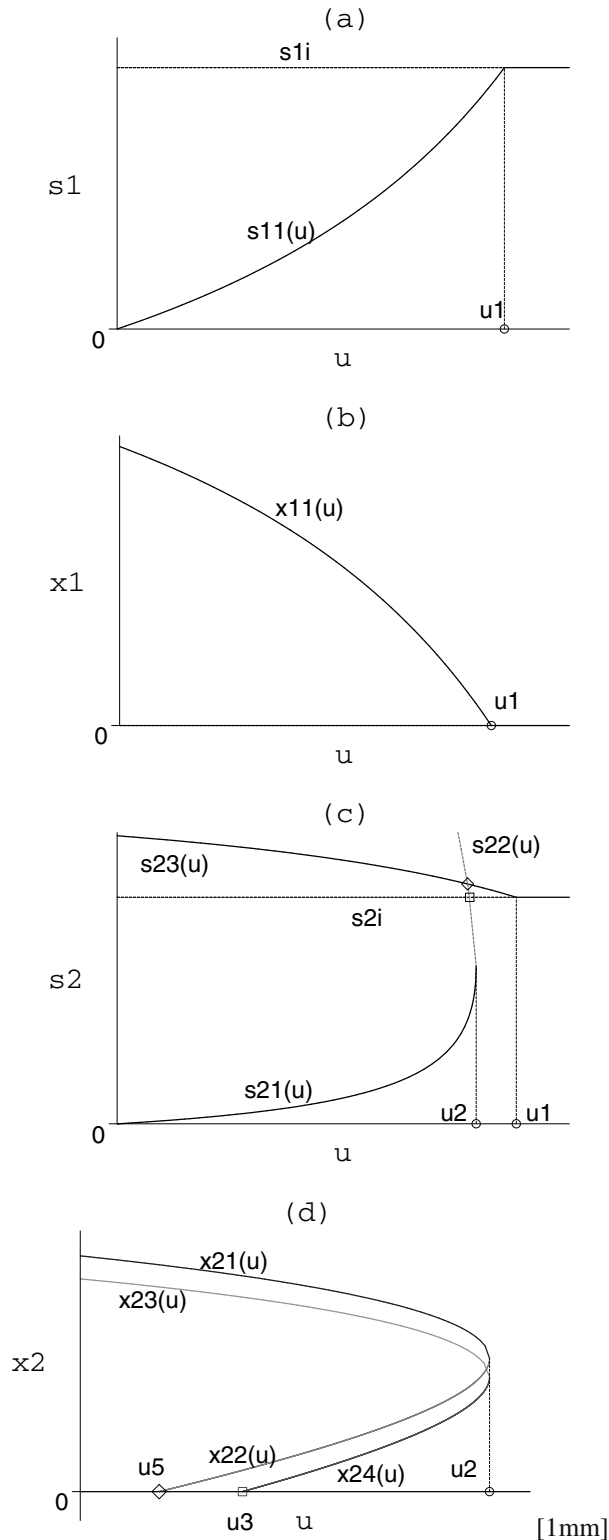
(a)

(b)

(c)

(d)

Fig. 1. Branches of equilibrium points vs. u in the case when $u_2 < u_1$: (a) s_1-components, (b) x_1-components, (c) s_2-components, (d) x_2-components. Notation: (a) s1i :=s_1^i, s11(u):=$s_1^{(1)}(u)$; (b) x11(u):=$x_1^{(1)}(u)$; (c) s2i:=s_2^i, s2j(u):=$s_2^{(j)}(u)$, $j = 1, 2, 3$; (d) x2j(u):=$x_2^{(j)}(u)$, $j = 1, 2, 3, 4$.

In what follows we shall assume that $|D^1G(z; u)|$ and $|D^2G(z; u)|$ do not vanish simultaneously at an equilibrium point.

Case 1. $|D^1G(z; u)| = 0$, $|D^2G(z; u)| \neq 0$.

In this case we have the following nonhyperbolic equilibrium points:

$$H_1 = \left(s_1^i, 0, s_2^{(1)}(u_1), \frac{s_2^i - s_2^{(1)}(u_1)}{\alpha k_3}; u_1\right)$$
$$\text{if } u_1 < u_3 \leq u_2,$$
$$H_2 = \left(s_1^i, 0, s_2^{(2)}(u_1), \frac{s_2^i - s_2^{(2)}(u_1)}{\alpha k_3}; u_1\right)$$
$$\text{if } u_3 < u_1 < u_2,$$
$$H_3 = \left(s_1^i, 0, s_2^i, 0; u_1\right) \text{ if } u_1 < u_2, \quad u_1 \neq u_3.$$

Case 2. $|D^1G(z; u)| \neq 0$, $|D^2G(z; u)| = 0$.

The nonhyperbolic equilibrium points are

$$H_4 = \left(s_1^i, 0, s_2^i, 0; u_3\right) \text{ if } u_1 < u_3,$$
$$H_5 = \left(s_1^i, 0, \tilde{s}_2, \frac{s_2^i - \tilde{s}_2}{\alpha k_3}; u_2\right) \text{ if } u_1 < u_2, \ u_2 \neq u_3,$$
$$H_6 = \left(s_1^{(1)}(u_2), x_1^{(1)}(u_2), \tilde{s}_2, \right.$$
$$\left. \frac{s_2^i - \tilde{s}_2 + \alpha k_2 x_1^{(1)}(u_2)}{\alpha k_3}; u_2\right) \text{ if } u_2 < u_1,$$
$$H_7 = \left(s_1^{(1)}(u_2), x_1^{(1)}(u_2), s_2^i, \frac{k_2}{k_3}x_1^{(1)}(u_2); u_2\right)$$
$$\text{if } u_3 = u_2 < u_1,$$
$$H_8 = \left(s_1^{(1)}(u_5), x_1^{(1)}(u_5), s_2^{(3)}(u_5), 0; u_5\right) \text{ if } u_5 < u_1.$$

Our goal is to determine the nature of the stability of H_j, $j = 1, \ldots, 8$, for u near the corresponding critical value u_i. To do this, we shall find the reduction of the system (1)–(4) at each nonhyperbolic point to its corresponding center manifold (Carr, 1981; Wiggins, 1990).

Proposition 1. *The system of ODEs (1)–(4) undergoes a transcritical bifurcation at each one of the nonhyperbolic points H_j, $j = 1, 2, \ldots, 8$.*

Proof. We shall consider in detail the point H_8. Write, for simplicity,

$$s_1^* = s_1^{(1)}(u_5), \quad x_1^* = x_1^{(1)}(u_5), \quad s_2^* = s_2^{(3)}(u_5).$$

The following coordinate change

$$\xi_1 = s_1^* - s_1, \quad \eta_1 = x_1^* - x_1, \quad \xi_2 = s_2^* - s_2,$$
$$\eta_2 = x_2, \qquad v = u_5 - u$$

transforms H_8 into zero $(0, 0, 0, 0; 0) =: (0; 0)$. Using Taylor approximations of $\mu_1(s_1^* - \xi_1)$ and $\mu_2(s_2^* - \xi_2)$

Table 2.

Nonhyperbolic points	Eigenvalues	Values of L
$H_1 =$ $\left(s_1^i, 0, s_2^{(1)}(u_1), x_2^*; u_1\right)$ $u_1 < u_3 \leq u_2$ $x_2^* := \frac{s_2^i - s_2^{(1)}(u_1)}{\alpha k_3}$	$\lambda_1 = 0, \ \lambda_2 = -u_1$ $\lambda_{3,4} = \frac{1}{2}(u_1 + k_3 m_2 x_2^*)$ $\pm \frac{1}{2}\sqrt{(u_1 + k_3 m_2 x_2^*)^2 - 4\alpha u_1 k_3 m_2 x_2^*}$ $m_2 := \frac{\mathrm{d}}{\mathrm{d}s_2}\mu_2(s_2^{(1)}(u_1))$	$\frac{k_1}{k_2}\alpha k_3 \frac{\mathrm{d}}{\mathrm{d}s_1}\mu_1(s_1^i)$
$H_2 =$ $\left(s_1^i, 0, s_2^{(2)}(u_1), x_2^*; u_1\right)$ $u_3 < u_1 < u_2$ $x_2^* := \frac{s_2^i - s_2^{(2)}(u_1)}{\alpha k_3}$	$\lambda_1 = 0, \ \lambda_2 = -u_1$ $\lambda_{3,4} = \frac{1}{2}(u_1 + k_3 m_2 x_2^*)$ $\pm \frac{1}{2}\sqrt{(u_1 + k_3 m_2 x_2^*)^2 - 4\alpha u_1 k_3 m_2 x_2^*}$ $m_2 := \frac{\mathrm{d}}{\mathrm{d}s_2}\mu_2(s_2^{(2)}(u_1))$	$\frac{k_1}{k_2}\alpha k_3 \frac{\mathrm{d}}{\mathrm{d}s_1}\mu_1(s_1^i)$
$H_3 = \left(s_1^i, 0, s_2^i, 0; u_1\right)$ $u_1 < u_2, \ \ u_1 \neq u_3$	$\lambda_1 = 0, \ \lambda_2 = \lambda_3 = -u_1, \lambda_4 = \alpha(u_3 - u_1)$	$\alpha k_1 \frac{\mathrm{d}}{\mathrm{d}s_1}\mu_1(s_1^i)$
$H_4 = \left(s_1^i, 0, s_2^i, 0; u_3\right)$ $u_1 < u_3$	$\lambda_1 = 0, \ \lambda_2 = \lambda_3 = -u_3, \lambda_4 = \alpha(u_1 - u_3)$	$\alpha k_3 \frac{\mathrm{d}}{\mathrm{d}s_2}\mu_2(s_2^i)$
$H_5 = \left(s_1^i, 0, \tilde{s}_2, \frac{s_2^i - \tilde{s}_2}{\alpha k_3}; u_2\right)$ $u_1 < u_2, \ \ u_2 \neq u_3$	$\lambda_1 = 0, \ \lambda_2 = \lambda_3 = -u_2, \lambda_4 = \alpha(u_1 - u_2)$	$\frac{1}{2}(s_2^i - \tilde{s}_2)\frac{\mathrm{d}^2}{\mathrm{d}s_2^2}\mu_2(\tilde{s}_2)$
$H_6 =$ $\left(s_1^{(1)}(u_2), x_1^{(1)}(u_2), \tilde{s}_2, x_2^*; u_2\right)$ $u_2 < u_1$ $x_2^* := \frac{s_2^i - \tilde{s}_2 + \alpha k_2 x_1^{(1)}(u_2)}{\alpha k_3}$	$\lambda_1 = 0, \ \lambda_2 = -u_2$ $\lambda_{3,4} = -\frac{1}{2}\left(u_2 + k_1 m_1 x_1^{(1)}(u_2)\right)$ $\pm \frac{1}{2}\sqrt{\left(u_2 + k_1 m_1 x_1^{(1)}(u_2)\right)^2 - 4\alpha u_2 k_1 m_1 x_1^{(1)}(u_2)}$ $m_1 := \frac{\mathrm{d}}{\mathrm{d}s_1}\mu_1(s_1^{(1)}(u_2))$	$\frac{1}{2}\alpha^2 k_3^2 \frac{\mathrm{d}^2}{\mathrm{d}s_2^2}\mu_2(\tilde{s}_2)x_2^*$
$H_7 =$ $\left(s_1^{(1)}(u_2), x_1^{(1)}(u_2), s_2^i, x_2^*; u_2\right)$ $u_3 = u_2 < u_1$ $x_2^* := \frac{k_2}{k_3}x_1^{(1)}(u_2)$	$\lambda_1 = 0, \ \lambda_2 = -u_2$ $\lambda_{3,4} = -\frac{1}{2}\left(u_2 + k_1 m_1 x_1^{(1)}(u_2)\right)$ $\pm \frac{1}{2}\sqrt{\left(u_2 + k_1 m_1 x_1^{(1)}(u_2)\right)^2 - 4\alpha u_2 k_1 m_1 x_1^{(1)}(u_2)}$ $m_1 := \frac{\mathrm{d}}{\mathrm{d}s_1}\mu_1(s_1^{(1)}(u_2))$	$\frac{1}{2}\alpha^2 k_3^2 \frac{\mathrm{d}^2}{\mathrm{d}s_2^2}\mu_2(\tilde{s}_2)x_2^*$
$H_8 =$ $\left(s_1^{(1)}(u_5), x_1^{(1)}(u_5), s_2^{(3)}(u_5), 0; u_5\right)$ $u_5 < u_1$	$\lambda_1 = 0, \ \lambda_2 = -u_5$ $\lambda_{3,4} = -\frac{1}{2}\left(u_2 + k_1 m_1 x_1^{(1)}(u_5)\right)$ $\pm \frac{1}{2}\sqrt{\left(u_5 + k_1 m_1 x_1^{(1)}(u_5)\right)^2 - 4\alpha u_5 k_1 m_1 x_1^{(1)}(u_5)}$ $m_1 := \frac{\mathrm{d}}{\mathrm{d}s_1}\mu_1(s_1^{(1)}(u_5))$	$\alpha k_3 \frac{\mathrm{d}^2}{\mathrm{d}s_2^2}\mu_2(s_2^{(3)}(u_5))$

around $\xi_1 = 0$ and $\xi_2 = 0$, we obtain

$$\mu_1(s_1^* - \xi_1) = \mu_1(s_1^*) - \frac{\mathrm{d}\mu_1}{\mathrm{d}s_1}(s_1^*) \cdot \xi_1 = \alpha u_5 - m_1 \xi_1,$$

$$\mu_2(s_2^* - \xi_2) = \mu_2(s_2^*) - \frac{\mathrm{d}\mu_2}{\mathrm{d}s_2}(s_2^*) \cdot \xi_2 = \alpha u_5 - m_2 \xi_2,$$

where

$$m_1 := \frac{\mathrm{d}\mu_1}{\mathrm{d}s_1}(s_1^*), \quad m_2 := \frac{\mathrm{d}\mu_2}{\mathrm{d}s_2}(s_2^*).$$

Denote by $\widetilde{G}$ the vector of the right hand-side functions of the system (1)–(4) in the new coordinates $(\xi_1, \eta_1, \xi_2, \eta_2)^T$. The latter is then expressed in the following form, where the parameter v has been included as a formal dependent variable:

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\eta}_1 \\ \dot{\xi}_2 \\ \dot{\eta}_2 \end{pmatrix} = D\widetilde{G}(0;0) \cdot \begin{pmatrix} \xi_1 \\ \eta_1 \\ \xi_2 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix},$$

$$\dot{v} = 0$$

within

$$D\widetilde{G}(0;0)$$
$$= \begin{pmatrix} -(u_5 + k_1 m_1 x_1^*) & -\alpha k_1 u_5 & 0 & 0 \\ m_1 x_1^* & 0 & 0 & 0 \\ k_2 m_1 x_1^* & \alpha k_2 u_5 & -u_5 & \alpha k_3 u_5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$g_1 = v\xi_1 + k_1 m_1 \xi_1 \eta_1,$$
$$g_2 = \alpha v \eta_1 - m_1 \xi_1 \eta_1,$$
$$g_3 = v\xi_2 - k_1 m_1 \xi_1 \eta_1 - k_3 m_2 \xi_2 \eta_2,$$
$$g_4 = \alpha v \eta_2 - m_2 \xi_2 \eta_2.$$

The eigenvalues λ_i and the corresponding eigenvectors p_i, $i = 1, 2, 3, 4$, of $D\tilde{G}(0; 0)$ are respectively

$$\lambda_1 = 0, \quad p_1 = (0, 0, \alpha k_3, 1)^T,$$
$$\lambda_2 = -u_5, \quad p_2 = (0, 0, 1, 0)^T,$$
$$\lambda_3 = -\frac{1}{2}(u_5 + k_1 m_1 x_1^*)$$
$$+ \frac{1}{2}\sqrt{(u_5 + k_1 m_1 x_1^*)^2 - 4\alpha k_1 m_1 u_5 x_1^*},$$
$$p_3 = \left(-\frac{k_1}{k_2}, -\frac{\lambda_4}{\alpha k_2 u_5}, 1, 0\right)^T,$$
$$\lambda_4 = -\frac{1}{2}(u_5 + k_1 m_1 x_1^*)$$
$$- \frac{1}{2}\sqrt{(u_5 + k_1 m_1 x_1^*)^2 - 4\alpha k_1 m_1 u_5 x_1^*},$$
$$p_4 = \left(-\frac{k_1}{k_2}, -\frac{\lambda_3}{\alpha k_2 u_5}, 1, 0\right)^T.$$

Obviously, $\lambda_2 < 0$. Since $0 < \alpha < 1$, it can be easily seen that λ_3 and λ_4 are real negative numbers.

Forming the matrix P by taking as columns the eigenvectors p_j, $j = 1, 2, 3, 4$,

$$P = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \end{pmatrix},$$

and finding its inverse P^{-1}, we make the coordinate change

$$\begin{pmatrix} \xi_1 \\ \eta_1 \\ \xi_2 \\ \eta_2 \end{pmatrix} = P \begin{pmatrix} \zeta_1 \\ \varkappa_1 \\ \zeta_2 \\ \varkappa_2 \end{pmatrix}, \quad \begin{pmatrix} \zeta_1 \\ \varkappa_1 \\ \zeta_2 \\ \varkappa_2 \end{pmatrix} = P^{-1} \begin{pmatrix} \xi_1 \\ \eta_1 \\ \xi_2 \\ \eta_2 \end{pmatrix}$$

and obtain

$$\begin{pmatrix} \dot{\zeta}_1 \\ \dot{\varkappa}_1 \\ \dot{\zeta}_2 \\ \dot{\varkappa}_2 \end{pmatrix} = A \cdot \begin{pmatrix} \zeta_1 \\ \varkappa_1 \\ \zeta_2 \\ \varkappa_2 \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}$$
$$\dot{v} = 0,$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$$

and

$$f_1 = \alpha v \zeta_1 - m_2 \zeta_1 (\alpha k_3 \zeta_1 + \varkappa_1 + \zeta_2 \varkappa_2),$$
$$f_2 = v(\zeta_2 + \varkappa_2) - \alpha^2 k_2 v \zeta_1$$
$$+ (\alpha - 1) k_3 m_2 \zeta_1 (\alpha\, k_3 \zeta_1 + \varkappa_1 + \zeta_2 + \varkappa_2),$$
$$f_3 = \frac{1}{\lambda_4 - \lambda_3}[-\lambda_3 v(\zeta_2 + \varkappa_2) + \alpha v(\lambda_4 \zeta_2 + \lambda_3 \varkappa_2)$$
$$+ \frac{k_1}{k_2} m_1 (\zeta_2 + \varkappa_2)(\lambda_4 \zeta_2 + \lambda_3 \varkappa_2)$$
$$+ \frac{k_1 m_1 \lambda_3}{\alpha k_2 u_5}(\zeta_2 + \varkappa_2)(\lambda_4 \zeta_2 + \lambda_3 \varkappa_2)],$$
$$f_4 = \frac{1}{\lambda_4 - \lambda_3}[\lambda_4 v(\zeta_2 + \varkappa_2) - \alpha v(\lambda_4 \zeta_2 + \lambda_3 \varkappa_2)$$
$$- \frac{k_1}{k_2} m_1 (\zeta_2 + \varkappa_2)(\lambda_4 \zeta_2 + \lambda_3 \varkappa_2)$$
$$- \frac{k_1 m_1 \lambda_4}{\alpha k_2 u_5}(\zeta_2 + \varkappa_2)(\lambda_4 \zeta_2 + \lambda_3 \varkappa_2)].$$

Thus, from center manifold theory, the stability of $(\xi_1, \eta_1, \xi_2, \eta_2) = (0, 0, 0, 0)$ near $v = 0$ can be determined by studying a one-parameter family of first-order ODEs on a center manifold. The latter can be represented as a graph over ζ_1 and v by

$$W^c(0)$$
$$= \{(\zeta_1, \varkappa_1, \zeta_2, \varkappa_2; v) \in \mathbb{R}^5 |\ \varkappa_1 = h_1(\zeta_1, v),$$
$$\zeta_2 = h_2(\zeta_1, v), \varkappa_2 = h_3(\zeta_1, v), h_i(0, 0) = 0,$$
$$D_{\zeta_1} h_i(0, 0) = 0,\ i = 1, 2, 3\}$$

for ζ_1 and v sufficiently small. The points $(h_1, h_2, h_3)^T$ of the center manifold should satisfy

$$D_{\zeta_1} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \cdot f_1 - B \cdot \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} - \begin{pmatrix} f_2 \\ f_3 \\ f_4 \end{pmatrix} = 0 \quad (20)$$

within

$$B = \begin{pmatrix} -u_5 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_4 \end{pmatrix}.$$

We now wish to compute the center manifold $W^c(0)$ approximately and to derive the dynamics on $W^c(0)$. We assume that

$$h_1 = h_1(\zeta_1, v) = a_1 \zeta_1^2 + a_2 \zeta_1 v + a_3 v^2 + O(3),$$
$$h_2 = h_2(\zeta_1, v) = b_1 \zeta_1^2 + b_2 \zeta_1 v + b_3 v^2 + O(3),$$
$$h_3 = h_3(\zeta_1, v) = c_1 \zeta_1^2 + c_2 \zeta_1 v + c_3 v^2 + O(3).$$

Substituting this in (20) and equating terms with equal powers to zero, we find

$$h_1(\zeta_1, v) = \frac{\alpha(\alpha - 1)k_3^2 m_2}{u_5}\zeta_1^2 + \frac{\alpha^2 k_2}{u_5}\zeta_1 v + O(3),$$
$$b_1 = b_2 = b_3 = 0, \quad c_1 = c_2 = c_3 = 0.$$

Hence the system of ODEs reduced to the center manifold is given by

$$\dot{\zeta}_1 = \alpha v \zeta_1 - \alpha k_3 m_2 \zeta_1^2, \qquad (21)$$
$$\dot{v} = 0.$$

Obviously, $\zeta_1 = 0$ is always an equilibrium point for the first equation of (21) and it is stable for $v < 0$ and unstable for $v > 0$. A change in stability occurs at the bifurcation value $v = 0$; the type of bifurcation of $\zeta_1 = 0$ is transcritical (Wiggins, 1990). Since the remaining eigenvalues λ_2, λ_3 and λ_4 are negative, it follows (in $(z; u)$-coordinates) that H_8 is a stable equilibrium for the system (1)–(4) if u is sufficiently close to u_5 with $u_5 < u < u_1$.

In a similar way it can be shown that the system of ODEs (1)–(4) undergoes a transcritical bifurcation at H_j, $j = 1, 2, \ldots, 7$. In all cases the reduced dynamics on the center manifold are of the form

$$\dot{\zeta}_1 = \alpha v \zeta_1 - L \zeta_1^2,$$
$$\dot{v} = 0.$$

Table 2 presents the eigenvalues and the constant L for all nonhyperbolic points H_i, $i = 1, \ldots, 8$ (H_8 is included for convenience). It is easy to see that H_3 can have one positive eigenvalue λ_4 if $u_3 > u_1$. The remaining nonhyperbolic points are stable for u sufficiently close to and greater than the corresponding bifurcation value. ∎

At transcritical bifuracion points, two steady states coalesce and exchange stability at the bifurcation value. In the $(z; u)$-coordinates the following pairs of equilibrium points coalesce at the bifurcation points (see Fig. 1):

$$E_1(u_1) \equiv E_4(u_1) \equiv H_1, \quad E_2(u_1) \equiv E_5(u_1) \equiv H_2,$$
$$E_3(u_1) \equiv E_6(u_1) \equiv H_3, \quad E_5(u_3) \equiv E_6(u_3) \equiv H_4,$$
$$E_4(u_2) \equiv E_5(u_2) \equiv H_5, \quad E_1(u_2) \equiv E_2(u_2) \equiv H_6,$$
$$E_1(u_3) \equiv E_2(u_3) \equiv H_7, \quad E_2(u_5) \equiv E_3(u_5) \equiv H_8.$$

Remark 3. If we consider (7)–(8), it is easily seen that $s_1^{(1)}(u)$, $x_1^{(1)}(u)$ are linearly stable (attractors, with two negative eigenvalues of the Jacobian) for any value of u with $u < u_1$. The second (wash-out) equilibrium $s_1^{(2)}(u) = s_1^i$, $x_1^{(2)}(u) = 0$ is a saddle for $u < u_1$ and it is linearly stable if $u > u_1$. It is shown in (Hess and Bernard, 2008) that $(s_1^{(1)}(u), x_1^{(1)}(u))$ is globally asymptotically stable for any $u < u_1$.

Remark 4. The values of $u_1 = (1/\alpha)\mu_1(s_1^i)$ and $u_2 = (1/\alpha)\mu_2(\tilde{s}_2)$ are uncoupled; they depend on the particular coefficient values in the expressions of μ_1 and μ_2. If $u_2 < u_1$, then all bifurcation values u_i will satisfy $u_i < u_1$ (see (18), (19)), i.e., the bifurcations will be related to the s_2- and x_2-components of the steady states. Thus $(s_1^{(1)}(u), x_1^{(1)}(u))$ will be a stable equilibrium point for the acidogenic stage (described by (1)–(2)) whereas the methanogenic phase (Eqns. (3)–(4)) might undergo transcritical bifurcations.

4. Adaptive asymptotic stabilization

As mentioned before, the model (1)–(4) describes a two-stage process where the first two ODEs, (1)–(2), present the acidogenic phase and the second two, (3)–(4), the methanogenic phase. In practice, acidogenesis is much faster than the methanogenic phase. Moreover the latter phase can be inhibited, which on the one hand is limiting and on the other determining for the whole process (Hess and Bernard, 2008).

In what follows we shall assume that the first stage of the process is already stabilized to some previously chosen point (s_1^*, x_1^*) such that $0 < s_1^* < s_1^i$, $x_1^* > 0$ (Simeonov *et al.*, 2004; Simeonov *et al.*, 2007). Write

$$c_1^* = \frac{k_2}{k_1}(s_1^i - s_1^*).$$

In (Antonelli *et al.*, 2003), where a similar approach to asymptotic stabilization of the model is used, the constant c_1^* is assumed to be negligible and thus set to 0. We assume here that $c_1^* > 0$. Then the methanogenic phase of the system is described by the following ODEs:

$$\frac{ds_2}{dt} = u(s_2^i - s_2) + c_1^* u - k_3 \cdot \mu_2(s_2) \cdot x_2, \qquad (22)$$
$$\frac{dx_2}{dt} = (\mu_2(s_2) - \alpha u) \cdot x_2. \qquad (23)$$

Practically, the output methane flow rate Q (see (5)) and the substrate concentration s_2^i, s_2 are the only measurable variables. Our goal is to construct a smooth feedback law for asymptotic stabilization of (22)–(23) to some previously chosen operating (reference) point $\bar{s}_2$. Let us fix such a point $\bar{s}_2 \in (0, s_2^i)$. Then $(\bar{s}_2, \bar{x}_2)$ with

$$\bar{x}_2 = \frac{s_2^i + c_1^* - \bar{s}_2}{\alpha k_3}$$

is an equilibrium point for (22)–(23).

Our main assumption is that α, k_3, k_4, μ_0, k_{s_2} and k_I are unknown but bounded within compact intervals $[\alpha]$, $[k_3]$, $[k_4]$, $[\mu_0]$, $[k_{s_2}]$ and $[k_I]$, respectively.

Let $\omega = (\alpha, k_3, k_4, \mu_0, k_{s_2}, k_I)$ be the vector of the exact (unknown) values of the model parameters. Set

$$\bar{\beta} = \frac{k_3}{k_4(s_2^i + c_1^* - \bar{s}_2)},$$

and let $\beta^- > 0$ and $\beta^+ > 0$ be arbitrary real numbers such that $\bar{\beta} \in (\beta^-, \beta^+)$.

Following (Antonelli *et al.*, 2003), we extend the system (22)–(23) by adding the differential equation

$$\frac{d\beta}{dt} = -C(\beta - \beta^-)(\beta^+ - \beta)k_4\mu_2(s_2)x_2(s_2 - \bar{s}_2), \quad (24)$$

where $C > 0$ is an arbitrary constant.

Define the following map:

$$K(s_2, x_2, \beta)$$
$$= \begin{cases} \beta k_4 \mu_2(s_2) x_2 - \gamma(s_2 - \bar{s}_2) \\ \quad \text{if } \beta k_4 \mu_2(s_2) x_2 - \gamma(s_2 - \bar{s}_2) > 0, \\ \\ \beta k_4 \mu_2(s_2) x_2 \\ \quad \text{if } \beta k_4 \mu_2(s_2) x_2 - \gamma(s_2 - \bar{s}_2) < 0, \quad (25) \\ \\ \{\beta k_4 \mu_2(s_2) x_2 - \varrho\gamma(s_2 - \bar{s}_2) : \varrho \in [0,1]\} \\ \quad \text{if } \beta k_4 \mu_2(s_2) x_2 - \gamma(s_2 - \bar{s}_2) = 0, \end{cases}$$

where γ is a positive constant. This multivalued map $K(s_2, x_2, \beta)$ is upper semicontinuous with compact and convex values.

We have to point out that the quantity $k_4\mu_2(s_2)x_2$ in (24) and (25) is equal to Q and is on-line measurable.

Theorem 1. *For each starting point* $(s_2(0), x_2(0), \beta(0))$ *from the set* $\widetilde{\Omega}_0$ *with*

$$\widetilde{\Omega}_0 = \left\{ (s_2, x_2, \beta) \mid s_2 > 0, \ x_2 > 0, \ \beta \in (\beta^-, \beta^+) \right\},$$

there exists a feedback control law $k(s_2, x_2, \beta) \in K(s_2, x_2, \beta)$ *that stabilizes asymptotically the control system (22)–(23) and (24) to* $(\bar{s}_2, \bar{x}_2, \bar{\beta})$.

Remark 5. We formulate a stabilization result assuming that the control u is unbounded. In practice, some bounds for the control do exist due to physical evidence. Let us assume that $u \in [u_{\min}, u_{\max}]$. In the proof presented below, we construct explicitly a Lyapunov function V. Assuming that the value of the parameter k_4 belongs to the interval $[k_4^-, k_4^+]$, an estimate of the constrained stability region of the above controller can be obtained using the level sets of V, i.e.,
$$\Xi = \left\{ x \in R^n : V(x) \le c^{\max} \right\},$$
where $c^{\max} > 0$ is the largest number for which every element of $\tilde{\Omega}_0 \cap \Xi$ is contained in the set

$$\Theta = \left\{ (s_2, x_2) : \frac{u_{\min} + \gamma(s_2 - \bar{s}_2)}{\beta^+ k_4^+} \right.$$
$$\left. \le \mu_2(s_2)x_2 \le \frac{u_{\max} + \gamma(s_2 - \bar{s}_2)}{\beta^- k_4^-} \right\},$$

consisting of all points that satisfy the control constraints.

Proof. We set

$$\Omega_0 = \left\{ (s_2, x_2, \beta) \mid s_2^i > s_2 > 0, \ x_2 > 0, \right.$$
$$\left. \beta \in (\beta^-, \beta^+) \right\}.$$

Since the multivalued map $K(s_2, x_2, \beta)$ is upper semicontinuous with compact and convex values, there exist a selection $k(s_2, x_2, \beta)$ of the multivalued map $K(s_2, x_2, \beta)$

such that the closed-loop system $\Sigma(\omega)$ with exact but unknown values for the model parameters has at last one local solution starting from an arbitrary point of the set Ω_0 (cf., for example, the monograph (Filippov, 1988)). Let us note that $\Sigma(\omega)$ is obtained from (22)–(23) and (24) by substituting the control variable u by the feedback $k(s_2, x_2, \beta)$. Denote

$$k(s_2, x_2, \beta) = \beta k_4 \cdot \mu_2(s_2) \cdot x_2 - \varrho(s_2, x_2, \beta) \cdot \gamma \cdot (s_2 - \bar{s}_2),$$

where $\varrho(s_2, x_2, \beta)$ is a constant, equal to 1, 0 or ϱ with $\varrho \in [0, 1]$, depending on the conditions in the right-hand side branches of (25).

For convenience, we set

$$\zeta = (s_2, x_2), \quad \overline{\zeta} = (\bar{s}_2, \bar{x}_2).$$

One can directly verify that the set Ω_0 is strongly invariant with respect to (22)–(23) and (24) (Clarke *et al.*, 1998). This means that every trajectory of the closed-loop system $\Sigma(\omega)$ starting from a point $(\zeta, \beta) \in \Omega_0$ remains in Ω_0. In particular, the coordinates of all points of this trajectory will never vanish. Moreover, if the initial point belongs to the larger set $\widetilde{\Omega}_0$, then every trajectory of $\Sigma(\omega)$ starting from a point $(\zeta, \beta) \in \widetilde{\Omega}_0$ enters the set Ω_0 in finite time. For that reason we can assume without loss of generality that the initial point belongs to the set Ω_0.

Using the fact that $s_2^i = \bar{s}_2 + \alpha k_3 \bar{x}_2 - c_1^*$, the equations (22)–(23) of the closed-loop system $\Sigma(\omega)$ can be written as follows:

$$\frac{\mathrm{d}s_2}{\mathrm{d}t} = -k(s_2, x_2, \beta) \cdot (s_2 - \bar{s}_2 + \alpha k_3(x_2 - \bar{x}_2))$$
$$- k_3(\mu_2(s_2) - \alpha k(s_2, x_2, \beta)) \cdot x_2, \quad (26)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = (\mu_2(s_2) - \alpha k(s_2, x_2, \beta)) \cdot x_2. \quad (27)$$

Consider the function

$$V(\zeta, \beta)$$
$$= (s_2 - \bar{s}_2 + k_3(x_2 - \bar{x}_2))^2$$
$$+ \Gamma \left(\int_{\bar{s}_2}^{s_2} \frac{v - \bar{s}_2}{s_2^i - v} \mathrm{d}v + \frac{1}{C} \int_{\bar{\beta}}^{\beta} \frac{w - \bar{\beta}}{(w - \beta^-)(\beta^+ - w)} \mathrm{d}w \right)$$

where the parameter $\Gamma > 0$ will be determined later. Clearly, the values of this function are nonnegative. If we denote by $F(\zeta, \beta)$ the right-hand side of (26)–(27), (24) and take into account the definition of the map $K(s_2, x_2, \beta)$ from (25), then it can be directly checked

that for each point (ζ, β) of Ω_0,

$$
\begin{aligned}
\langle \operatorname{grad} & V(\zeta, \beta), F(\zeta, \beta) \rangle \\
={} & -2k(s_2, x_2, \beta)(s_2 - \bar{s}_2)^2 \\
& - \left(\Gamma \left(\varrho(s_2, x_2, \beta)\gamma + \frac{k_3\mu_2(s_2)x_2}{(s_2^i - s_2)(s_2^i - \bar{s}_2)} \right) \right) \\
& \cdot (s_2 - \bar{s}_2)^2 \\
& - 2(1 + \alpha)k_3 \cdot k(s_2, x_2, \beta)(s_2 - \bar{s}_2)(x_2 - \bar{x}_2) \\
& - 2\alpha k_3^2 \cdot k(s_2, x_2, \beta)(x_2 - \bar{x}_2)^2.
\end{aligned}
$$

The discriminant $D(\zeta, \beta)$ of the last expression, considered as a quadratic function with respect to $s_2 - \bar{s}_2$ and $x_2 - \bar{x}_2$, is

$$
\begin{aligned}
D(\zeta, & \beta) \\
={} & -4k_3^2 \cdot k(s_2, x_2, \beta) \cdot \varrho(s_2, x_2, \beta) \cdot \gamma \\
& \cdot \left(2\alpha \cdot \Gamma + (1 - \alpha)^2(s_2 - \bar{s}_2) \right) \\
& - 4k_3^2 \cdot k(s_2, x_2, \beta) \cdot \mu_2(s_2) \cdot x_2 \\
& \cdot \left(\Gamma \cdot \frac{4\alpha k_3}{(s_2^i - s_2)(s_2^i - \bar{s}_2)} - k_4(1 - \alpha)^2\beta \right).
\end{aligned}
$$

Now we choose the positive parameter Γ in such a way that the following inequalities are satisfied:

$$
2\alpha \cdot \Gamma + (1 - \alpha)^2(s_2 - \bar{s}_2) > 0,
$$

$$
\Gamma \cdot \frac{4\alpha k_3}{(s_2^i - s_2)(s_2^i - \bar{s}_2)} - k_4(1 - \alpha)^2\beta > 0.
$$

Then $D(\zeta, \beta) < 0$ is fulfilled for each point (ζ, β) from the set Ω_0. Hence,

$$
\langle \operatorname{grad} V(\zeta, \beta), F(\zeta, \beta) \rangle < 0
$$

for each point $(\zeta, \beta) \in \Omega_0 \setminus \{(\bar{\zeta}, \bar{\beta})\}$.

Applying LaSalle's invariance principle (cf., for example, (Khalil, 1992)), it follows that every solution of the system (22)–(23), (24) is defined on the interval $[0, +\infty)$ and approaches the largest invariant set of (22)–(23), (24), which is contained in the set $\Omega_0 \cap \{(\zeta, \beta) \in R^3 : s_2 = \bar{s}_2, \beta \in (\beta^-, \beta^+)\}$. Taking into account the definition of the multivalued map $K(\cdot)$, it is easy to see that this invariant set consists of the single point $(\bar{\zeta}, \bar{\beta})$. Hence the feedback $k(\cdot)$ stabilizes asymptotically the control system (22)–(23), (24) to the point $(\bar{\zeta}, \bar{\beta})$ on the set Ω_0. ∎

5. Adaptive stabilization towards the maximum methane flow rate

In practice, only the substrate concentrations s_2 and s_2^i and the effluent methane flow rate Q are measurable on-line (cf., for example, (Antonelli *et al.*, 2003)). Consider

Eqn. (5) and let as before $s_2 \in (0, s_2^i)$ be some reference point. Then the function

$$
Q(s_2) = k_4 \cdot \mu_2(s_2) \cdot \frac{s_2^i + c_1^* - s_2}{\alpha k_3},
$$

which is defined on the set of all steady states, has a maximum at a unique point $s_2^{\max} \in (0, s_2^i)$, that is, $Q_{\max} = Q(s_2^{\max})$. Denote

$$
u_{\max} = \frac{1}{\alpha}\mu_2(s_2^{\max})
$$

and

$$
x_2^{\max} = \frac{s_2^i + c_1^* - s_2^{\max}}{\alpha k_3}.
$$

Our goal now is to stabilize the methanogenic dynamic system (22)–(23) towards the (unknown) maximum methane flow rate $Q_{\max}$. For that purpose we write (24) in the form

$$
\frac{\mathrm{d}\beta}{\mathrm{d}t}(t) = -C(\beta(t) - \beta^-)(\beta^+ - \beta(t))Q(t)(s_2(t) - \bar{s}_2),
\tag{28}
$$

where $Q(t)$ means the methane flow rate measured at the moment of time t. We would like to point out that not only $Q(t)$ but all quantities in (28) are on-line measurable. Thus the values of its solution can also be determined on-line. Since the solution of (28) depends on $\bar{s}_2$, we denote it by $\beta_{\bar{s}_2}(t)$, $t \in [0, +\infty)$. The last fact allows us to apply on-line the feedback control law

$$
\begin{aligned}
(s_2, Q, \beta_{\bar{s}_2}) & \longmapsto k(s_2, Q, \beta_{\bar{s}_2}) \\
& = \beta_{\bar{s}_2} Q - \varrho(s_2, Q, \beta_{\bar{s}_2}) \cdot \gamma \cdot (s_2 - \bar{s}_2).
\end{aligned}
\tag{29}
$$

According to Theorem 1, this feedback will asymptotically stabilize the control system (22)–(23), (28) to the point $(\bar{s}_2, \bar{x}_2, \bar{\beta}_{\bar{s}_2})$ with

$$
(\bar{s}_2, \bar{x}_2, \bar{\beta}_{\bar{s}_2}) = \left(\bar{s}_2, \frac{s_2^i + c_1^* - \bar{s}_2}{\alpha k_3}, \frac{k_3}{k_4(s_2^i + c_1^* - \bar{s}_2)} \right).
$$

To stabilize the dynamics (22)–(23), (28) by means of the feedback (29), we use the fact that Theorem 1 is valid for *any* reference point $\bar{s}_2 \in (0, s_2^i)$. We shall construct a sequence of points $\{\bar{s}_2^n\}$, $n = 1, 2, \ldots$, and use an extremum seeking iterative algorithm to generate a sequence $\{Q_n\}$, which converges to $Q_{\max}$. The algorithm is carried out in two stages: in Stage 1, an interval $[S] = [S^-, S^+]$ is found such that $[S^-, S^+] \subset (0, s_2^i)$ and $s_2^{\max} \in [S^-, S^+]$; in Stage 2, the interval $[S]$ is refined using an elimination procedure based on a Fibonacci search technique (Karmanov, 2000). Stage 2 produces the final interval $[S^{*-}, S^{*+}]$ such that $[S^{*-}, S^{*+}] \subseteq [S^-, S^+] \subset (0, s_2^i)$, $s_2^{\max} \in [S^{*-}, S^{*+}]$ and $S^{*+} - S^{*-} \leq \varepsilon$.

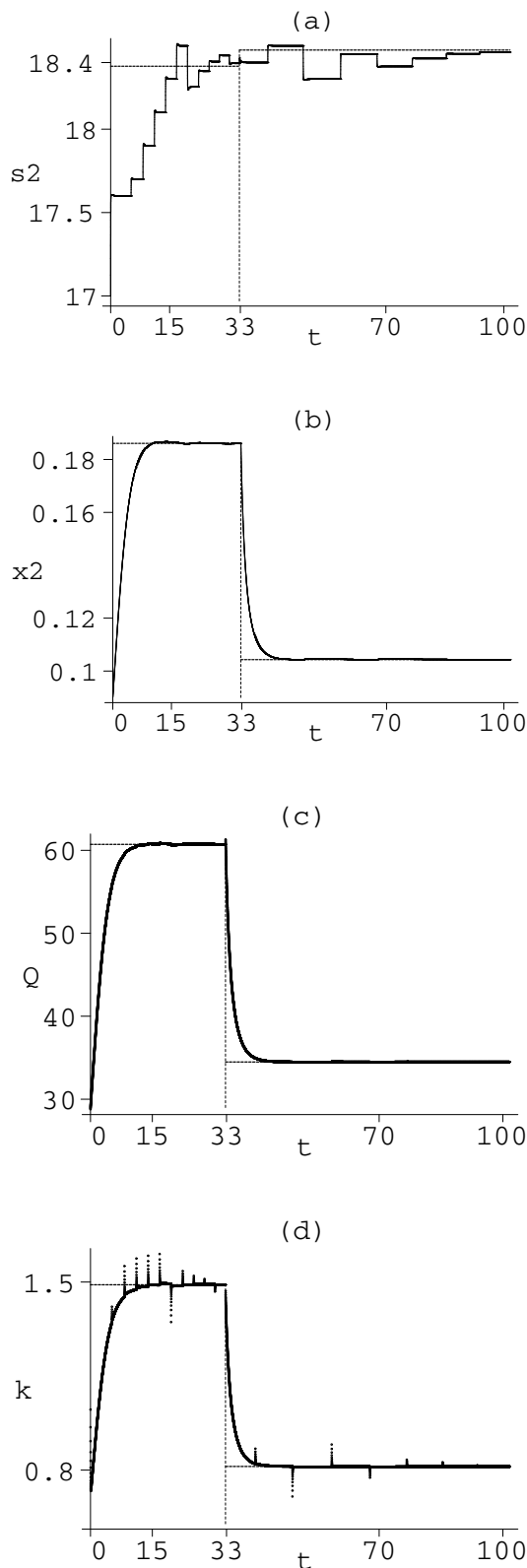The main steps of the numerical extremum seeking algorithm are presented in Appendix.

Fig. 2. Time evolution of (a) $s_2(t)$, (b) $x_2(t)$, (c) $Q(t)$, (d) feedback $k(t)$; horizontal line segments go through $s_2^{\max}$, $x_2^{\max}$, $Q_{\max}$ and $u_{\max}$, respectively.

6. Numerical simulation

As mentioned before, the exact value of the parameter vector

$$\omega = (\alpha, k_1, k_2, k_3, k_4, \mu_{\max}, k_{s_1}, \mu_0, k_{s_2}, k_I)$$

is not known. Practical experiments and parameter estimation results (cf. (Alcaraz-González *et al.*, 2002; Grognard and Bernard, 2006; Mailert *et al.*, 2004; Simeonov, 1994; Simeonov, 1999)) give only bounds for the vector components. For that reason, we assume that the model parameters are bounded within compact real intervals $[\alpha], [k_1], [k_2], [k_3], [k_4], [\mu_{\max}], [k_{s_1}], [\mu_0], [k_{s_2}]$ and $[k_I]$, respectively. Denote by $[\omega]$ the corresponding vector with interval components:

$$[\omega] = ([\alpha], [k_1], [k_2], [k_3], [k_4], [\mu_{\max}], [k_{s_1}],$$
$$[\mu_0], [k_{s_2}], [k_I]).$$

Using the intervals for the parameters, we can find the bounds β^- and β^+ for $\bar{\beta}$ explicitly,

$$\beta^- = \frac{k_3^-}{k_4^+ (s_2^i + c_1^* - \bar{s}_2)}, \quad \beta^+ = \frac{k_3^+}{k_4^- (s_2^i + c_1^* - \bar{s}_2)}$$

for any reference point $\bar{s}_2$. In the extremum seeking algorithm these bounds are computed for every choice of the reference point $\bar{s}_2$.

In the numerical simulations we consider the following intervals for the coefficients

$$
\begin{array}{ll}
[\alpha] = [0.3,\ 0.6], & [k_1] = [9.5,\ 11.5], \\
[k_2] = [27.6,\ 29.6], & [k_3] = [1064,\ 1084], \\
[k_4] = [650,\ 700], & [\mu_{\max}] = [1,\ 1.4], \\
[k_{s_1}] = [6.5,\ 7.9], & [\mu_0] = [0.64,\ 0.84], \\
[k_{s_2}] = [8.28,\ 10.28], & [k_I] = [15,\ 17].
\end{array}
$$

These intervals are chosen to enclose experimentally validated coefficient values, see e.g., (Alcaraz-González *et al.*, 2002; Antonelli *et al.*, 2003).

One can easily see that for any choice of $\omega \in [\omega]$ the relation $u_2 < u_1$ holds true (see Remark 4).

The input concentrations s_1^i and s_2^i are assumed to be constant, $s_1^i = 7$, $s_2^i = 70$. Further, we take $\bar{s}_1 = 1.4$.

To demonstrate the robustness of the feedback with respect to the model uncertainties we proceed as follows: At the initial moment ($t_0 = 0$), we choose random values for the model parameters from the corresponding intervals and consider them as the "exact" vector ω. These values are kept constant until the system stabilizes to $Q_{\max}$. Then, at some time moment $t = t_1 > t_0$, another set of random values for the model parameters is chosen to represent again the "exact" vector ω. The process is repeated. Thereby the last computed values for s_2, x_2 and β are considered as new initial conditions.

During algorithm execution, all intermediate numerical values for s_2, x_2, Q and the feedback k were collected in arrays and then plotted to visualize the results. Figure 2 shows the time profiles of the state variables $s_2(t)$ and $x_2(t)$ (plots (a) and (b), respectively), of $Q(t)$ (plot (c)) and of the feedback $k(t)$ (plot (d)). In the plots the horizontal dash-line segments go through $s_2^{\max}$, $x_2^{\max}$, $Q_{\max}$ and $u_{\max}$, respectively. The vertical line segments mark the moment of time t_1, where the coefficients are randomly changed.

The numerical simulations were carried out in the computer algebra system Maple. Thereby we used symbolic manipulations to find steady states, an ODE solver together with the implemented extremum seeking algorithm and graphic visualization facilities to plot the numerical results.

7. Conclusion

In the paper we studied a four-dimensional nonlinear dynamic system which models a biological two-stage wastewater treatment process. It was shown that the open-loop system undergoes local transcritical bifurcations of the steady states when the control parameter (the dilution rate) u takes different admissible values. Assuming that the acidogenesis (first stage) had been already stabilized to some operating point s_1^*, a nonlinear adaptive feedback was proposed, which stabilizes asymptotically the closed-loop second stage dynamics (the methanogenic phase) towards the maximum methane production rate $Q_{\max}$. For that purpose, we first showed that for *any* previously chosen reference point $\bar{s}_2$ we can asymptotically stabilize the model to an equilibrium point $\bar{\zeta} = (\bar{s}_2, \bar{x}_2)$, whose projection on the s_2-axis is $\bar{s}_2$. It should be pointed out that if the control input u is unbounded, then stabilization is global in the sense that the starting point can be any point from the unbounded set $\widetilde{\Omega}_0$. Further, a numerical extremum seeking algorithm was used to stabilize the closed-loop system into an interval $[S^*]$, containing the equilibrium point $s_2^{\max}$ for which the methane output flow rate Q takes its maximum $Q_{\max}$. The interval $[S^*]$ can be made as tight as desired depending on a constant $\varepsilon > 0$, which has to be celebrated by the user. Assuming that the model parameters are unknown but bounded within compact intervals, numerical experiments were carried out to demonstrate the robustness of the proposed control law. Our further efforts will be directed to designing an adaptive feedback law for asymptotic stabilization of the whole four-dimensional dynamics towards some reference point; this feedback law should depend on on-line measurable model quantities and be robust with respect to model uncertainties.

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Appendix

Below we present the main steps of the numerical extremum seeking algorithm. The steps are executed in the given order except as indicated by branching. We assume tolerances $\varepsilon > 0$, $h > 0$ and $\varepsilon_s > 0$ to be given.

Extremum Seeking Algorithm

Stage 1. Determine an interval $[S] = [S^-, S^+]$ such that $[S] \subset (0, s_2^i)$ and $s_2^{max} \in [S]$.

Step 1.0. Choose $\bar{s}_2^0 \in (0, s_2^i)$. Apply the feedback $k(s_2, Q, \beta_{\bar{s}_2^0})$ to stabilize the system to $\bar{s}_2^0$. According to Theorem 1, there exists a moment of time $t_0 > 0$ such that $|s_2(t_0) - \bar{s}_2^0| < \varepsilon_s$; set $\bar{s}_2^0 := s_2(t_0)$, $Q_0 := Q(t_0)$.

Step 1.1. Set $\sigma := 1$, $\bar{s}_2^1 := \bar{s}_2^0 + \sigma h$. Apply the feedback $k(s_2, Q, \beta_{\bar{s}_2^1})$ to stabilize the system to $\bar{s}_2^1$. According to Theorem 1, there exists a moment of time $t_1 > 0$ such that $|s_2(t_1) - \bar{s}_2^1| < \varepsilon_s$; set $\bar{s}_2^1 := s_2(t_1)$, $Q_1 := Q(t_1)$. If $Q_1 > Q_0$ then go to Step 1.3 else go to Step 1.2.

Step 1.2. Set $\sigma := -1$, $\bar{s}_2^1 := \bar{s}_2^0 + \sigma h$. Apply the feedback $k(s_2, Q, \beta_{\bar{s}_2^1})$ to stabilize the system to $\bar{s}_2^1$. According to Theorem 1, there exists a moment of time $t_1 > 0$ such that $|s_2(t_1) - \bar{s}_2^1| < \varepsilon_s$; set $\bar{s}_2^1 := s_2(t_1)$, $Q_1 := Q(t_1)$.

If $Q_1 > Q_0$ then go to Step 1.3.
If $Q_1 \leq Q_0$ then set $h := h/2$;
   if $h \leq \varepsilon/2$ then set $[S^*] := [\bar{s}_2^0 - \varepsilon, \bar{s}_2^0 + \varepsilon]$ and stop computations without going to Stage 2;
   if $h > \varepsilon/2$ then go to Step 1.1.

Step 1.3. Set $h := 2h$, $\bar{s}_2^2 := \bar{s}_2^1 + \sigma h$. Apply the feedback $k(s_2, Q, \beta_{\bar{s}_2^2})$ to stabilize the system to $\bar{s}_2^2$. According to Theorem 1, there exists a moment of time $t_2 > 0$ such that $|s_2(t_2) - \bar{s}_2^2| < \varepsilon_s$; set $\bar{s}_2^2 := s_2(t_2)$, $Q_2 := Q(t_2)$.

If $Q_2 \leq Q_1$ then set $[S] = [S^-, S^+] := [\bar{s}_2^0, \bar{s}_2^2]$ and go to Stage 2.

If $Q_2 > Q_1$ then set $\bar{s}_2^0 := \bar{s}_2^1$, $\bar{s}_2^1 := \bar{s}_2^2$, $Q_1 := Q_2$; repeat Step 1.3.

Stage 2. Starting with $[S] = [S^-, S^+]$, determine an interval $[S^*] = [S^{*-}, S^{*+}]$ with $s_2^{max} \in [S^*]$ and $S^{*+} - S^{*-} \leq \varepsilon$.

Denote $s_2^{0^-} := S^-$, $s_2^{0^+} := S^+$, $\lambda := (\sqrt{5} - 1)/2$; compute $\Delta_1 := s_2^{0^+} - s_2^{0^-}$.

Step 2.0. Compute $\Delta_2 := (1 - \lambda)\Delta_1$, $p_0 := s_2^{0^-} + \Delta_2$, $q_0 := s_2^{0^+} - \Delta_2$.

Step 2.1. Apply the feedback $k(s_2, Q, \beta_{p_0})$ to stabilize the system to p_0. According to Theorem 1, there exists a moment of time $t_{p_0} > 0$ such that $|s_2(t_{p_0}) - p_0| < \varepsilon_s$; set $p_0 := s_2(t_{p_0})$, $Q_{p_0} := Q(t_{p_0})$.

Apply the feedback $k(s_2, Q, \beta_{q_0})$ to stabilize the system to q_0. According to Theorem 1, there exists a moment of time $t_{q_0} > 0$ such that $|s_2(t_{q_0}) - q_0| < \varepsilon_s$; set $q_0 := s_2(t_{q_0})$, $Q_{q_0} := Q(t_{q_0})$.

Step 2.2. Set $\Delta_3 := q_0 - p_0$.

If $Q_{p_0} > Q_{q_0}$ then set $s_2^{1^-} := s_2^{0^-}$, $s_2^{1^+} := q_0$, $p_1 := s_2^{1^-} + \Delta_3$, $q_1 := p_0$;

If $Q_{p_0} \leq Q_{q_0}$ then set $s_2^{1^-} := p_0$, $s_2^{1^+} := s_0^+$, $p_1 := q_0$, $q_1 := s_2^{1^+} - \Delta_3$.

Compute $\Delta_1 := s_2^{1^+} - s_2^{1^-}$.

Step 2.3. If $\Delta_1 \le \varepsilon$ then set $[S^*] := [s_2^{1^-}, s_2^{1^+}]$ and stop computations.

If $\Delta_1 > \varepsilon$ then

if $p_1 \ge q_1$ then set $s_2^{0^-} := s_2^{1^-}$, $s_2^{0^+} := s_2^{1^+}$ and go to Step 2.0.

if $p_1 < q_1$ then

if $Q_{p_0} > Q_{q_0}$ then apply the feedback $k(s_2, Q, \beta_{p_1})$ to stabilize the system to p_1. According to Theorem 1, there exists a moment of time $t_{p_1} > 0$ such that $|s_2(t_{p_1}) - p_1| < \varepsilon_s$; set $p_1 := s_2(t_{p_1})$, $Q_{p_1} := Q(t_{p_1})$.

if $Q_{p_0} \le Q_{q_0}$ then apply the feedback $k(s_2, Q, \beta_{q_1})$ to stabilize the system to q_1. According to Theorem 1, there exists a moment of time $t_{q_1} > 0$ such that $|s_2(t_{q_1}) - q_1| < \varepsilon_s$; set $q_1 := s_2(t_{q_1})$, $Q_{q_1} := Q(t_{q_1})$.

Set $p_0 := p_1$, $q_0 := q_1$, $s_2^{0^-} := s_2^{1^-}$, $s_2^{0^+} := s_2^{1^+}$, $Q_{p_0} := Q_{p_1}$, $Q_{q_0} := Q_{q_1}$; go to Step 2.2.

Comments on the Algorithm

1. The algorithm works on-line. Having determined the desired interval $[S^*]$, we set $\bar{s}_2 := (S^{*-} + S^{*+})/2$ and stabilize the dynamic system towards it. The computational process continues to work without any changes until the system changes due to parameter perturbations. Then the algorithm starts to work either from Stage 1 or from Stage 2 depending on whether $\bar{s}_2$ belongs to $[S]$ or $[S^*]$.

2. At any step of the algorithm, the last computed values for s_2 and x_2 are used as initial conditions for the next step. For β, the last computed value is checked whether $\beta \in (\beta^-, \beta^+)$; if not, then it is changed to $\beta = (\beta^- + \beta^+)/2$.

3. The algorithm may terminate at Step 1.2 without going to Stage 2 only in the case when the current set point $\bar{s}_2$ is sufficiently close to the maximum point $s_2^{\max}$, that is if $|\bar{s}_2 - s_2^{\max}| \le \varepsilon$.

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