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A SIGN PRESERVING MIXED FINITE ELEMENT APPROXIMATION FOR CONTACT PROBLEMS

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This paper is concerned with the frictionless unilateral contact problem (i.e., a Signorini problem with the elasticity operator). We consider a mixed finite element method in which the unknowns are the displacement field and the contact pressure. The particularity of the method is that it furnishes a normal displacement field and a contact pressure satisfying the sign conditions of the continuous problem. The *a priori* error analysis of the method is closely linked with the study of a specific positivity preserving operator of averaging type which differs from the one of Chen and Nochetto. We show that this method is convergent and satisfies the same *a priori* error estimates as the standard approach in which the approximated contact pressure satisfies only a weak sign condition. Finally we perform some computations to illustrate and compare the sign preserving method with the standard approach.

Keywords: variational inequality, positive operator, averaging operator, contact problem, Signorini problem, mixed finite element method.

1. Introduction

Finite element methods are efficient and widespread tools in computational contact and impact mechanics (see Han and Sofonea, 2002; Haslinger *et al.*, 1996; Kikuchi and Oden, 1988; Laursen, 2002; Wriggers, 2002), and mixed formulations involving a displacement field $\boldsymbol{u}$ in the bodies and the contact pressure $\sigma_n(\boldsymbol{u})$ on the contact zone are commonly used. A particularity of the contact problem lies in the so-called unilateral conditions linking on the contact zone Γ_C, the normal displacement field u_n and the Lagrange multiplier $\lambda = -\sigma_n(\boldsymbol{u})$:

$$u_n \leq 0, \qquad \lambda \geq 0, \qquad \lambda\, u_n = 0 \qquad \text{on } \Gamma_C.$$

The mixed finite element method we consider, introduced by Hild and Nicaise (2007), furnishes an approximated normal displacement field u_{hn} and an approximated multiplier λ_h which satisfy

$$\begin{aligned} u_{hn} \leq 0, \qquad \lambda_h \geq 0 \qquad &\text{on } \Gamma_C,\\ \lambda_h\, u_{hn} = 0 \qquad &\text{at the nodes of } \Gamma_C. \end{aligned}$$

Such a method shows three interesting aspects in comparison with the standard approach in which the multiplier is only nonnegative in a weak sense (see, e.g., Ben Belgacem and Renard, 2003; Coorevits *et al.*, 2002; Hüeber and Wohlmuth, 2005a):

- The nonnegative multiplier is more relevant from a mechanical point of view.
- This multiplier vanishes where the body separates (the multiplier of the standard approach may reveal some artificial oscillations in the separation zone).
- It allows defining a simple *a posteriori* error estimator whose numerical analysis gives better bounds than for the error estimator arising from the standard approach (see Hild and Nicaise, 2007).

Let us mention that there exist other mixed formulations leading to *a priori* error estimates with nonnegative multipliers and normal displacement fields which do not satisfy the nonpositivity condition (see Ben Belgacem and Brenner, 2001; Ben Belgacem and Renard, 2003; Haslinger *et al.*, 1996).

The paper is organized as follows. In Section 2 we introduce the equations modelling the frictionless unilateral contact problem between an elastic body and a rigid foundation. We write the problem using a formulation where the unknowns are the displacement field in the body and the pressure on the contact area. In Section 3, we choose a discretization involving continuous finite elements of degree 1 for the displacements and continuous piecewise affine multipliers on the contact zone. The main particularity of this approach is that both the normal displacement and the multiplier solution to the discrete problem satisfy the same sign conditions as the normal displacement and the multiplier solving the continuous problem.

More precisely, the displacement field of the sign preserving method coincides with the one in the standard approach, and the multipliers are linked by a linear operator which transforms the functions satisfying some "weak" nonnegativity conditions into nonnegative functions. In Section 4, we study and discuss the main basic properties of the positivity preserving averaging operator which requires minimal regularity. Section 5 is concerned with the *a priori* error analysis of the sign preserving method. We prove that the method is convergent when using convenient regularity assumptions on the solution to the continuous problem, and we obtain similar *a priori* error estimates as for the standard approach. In Section 6 we implement both methods and compare them using several examples. As expected, the sign preserving method furnishes more relevant multipliers and no loss of convergence is observed in comparison with the standard approach. Finally, we mention that the results in this paper obviously hold for the simpler Signorini problem with the Laplace operator.

As usual, we denote by $(H^s(\cdot))^d$, $s \in \mathbb{R}, d = 1, 2$ the Sobolev spaces in one and two space dimensions (see Adams, 1975). The usual norm of $(H^s(D))^d$ (dual norm if $s < 0$) is denoted by $\| \cdot \|_{s,D}$, and we keep the same notation when $d = 1$ or $d = 2$.

2. Unilateral contact problem in linear elasticity

We consider an elastic body Ω in $\mathbb{R}^2$ where plane strain assumptions are made. The boundary $\partial\Omega$ of Ω is polygonal, and we suppose that $\partial\Omega$ consists of three nonoverlapping parts Γ_D, Γ_N and Γ_C with $\text{meas}(\Gamma_D) > 0$ and $\text{meas}(\Gamma_C) > 0$. The normal unit outward vector on $\partial\Omega$ is denoted by $\boldsymbol{n} = (n_1, n_2)$, and we choose as the unit tangential vector $\boldsymbol{t} = (-n_2, n_1)$. In its initial stage, the body is in contact on Γ_C which is supposed to be a straight line segment, and we suppose that the unknown final contact zone after deformation will be included in Γ_C. The body is clamped on Γ_D for the sake of simplicity. It is subjected to volume forces $\boldsymbol{f} = (f_1, f_2) \in (L^2(\Omega))^2$ and surface loads $\boldsymbol{g} = (g_1, g_2) \in (L^2(\Gamma_N))^2$.

The unilateral contact problem in linear elasticity consists in finding a displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^2$ satisfying the equations and conditions (1)–(6):

$$\boldsymbol{div}\, \boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \boldsymbol{0} \qquad \text{in } \Omega, \tag{1}$$

where $\boldsymbol{\sigma} = (\sigma_{ij}),\ 1 \le i, j \le 2,$ stands for the stress tensor field and $\boldsymbol{div}$ denotes the divergence operator of tensor valued functions. The stress tensor field is obtained from the displacement field by the constitutive law of linear elasticity:

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in } \Omega, \tag{2}$$

where $\boldsymbol{A}$ is a fourth order symmetric and elliptic tensor whose coefficients lie in $C^1(\overline{\Omega})$, and $\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\nabla \boldsymbol{v} + {}^t \nabla \boldsymbol{v})/2$ represents the linearized strain tensor field. On Γ_D and Γ_N, the conditions are as follows:

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on} \quad \Gamma_D, \tag{3}$$

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{g} \qquad \text{on} \quad \Gamma_N. \tag{4}$$

For any displacement field $\boldsymbol{v}$ and for any density of surface forces $\boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{n}$ defined on $\partial\Omega$, we adopt the following notation:

$$\boldsymbol{v} = v_n \boldsymbol{n} + v_t \boldsymbol{t}, \quad \boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{n} = \sigma_n(\boldsymbol{v})\boldsymbol{n} + \sigma_t(\boldsymbol{v})\boldsymbol{t}.$$

The three conditions describing unilateral contact on Γ_C are (see, e.g., Duvaut and Lions, 1972; Eck *et al.*, 2005; Fichera, 1964; 1972)

$$u_n \le 0, \quad \sigma_n(\boldsymbol{u}) \le 0, \quad \sigma_n(\boldsymbol{u})\, u_n = 0. \tag{5}$$

Finally, the equality

$$\sigma_t(\boldsymbol{u}) = 0 \tag{6}$$

on Γ_C means that friction is omitted.

The mixed variational formulation of (1)–(6) uses the Hilbert space

$$\boldsymbol{V} = \Big\{ \boldsymbol{v} \in \big(H^1(\Omega)\big)^2 : \quad \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D \Big\}.$$

The Lagrange multiplier space M is the dual of the normal trace space N of $\boldsymbol{V}$ restricted to Γ_C. If the end points of Γ_C belong to $\overline{\Gamma}_N$ (resp. $\overline{\Gamma}_D$), then $N = H^{\frac{1}{2}}(\Gamma_C)$ (resp. $H^{\frac{1}{2}}_{00}(\Gamma_C)$). We next define the following convex cone of multipliers on Γ_C:

$$M^+ = \Big\{ \mu \in M : \ \langle \mu, \psi \rangle_{\Gamma_C} \ge 0 \quad \text{for all } \psi \in N, \psi \ge 0 \text{ a.e. on } \Gamma_C \Big\},$$

where the notation $\langle \cdot, \cdot \rangle_{\Gamma_C}$ represents the duality pairing between M and N. Define

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}\Omega, \quad b(\mu, \boldsymbol{v}) = \langle \mu, v_n \rangle_{\Gamma_C},$$

$$L(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\Omega + \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{v}\, \mathrm{d}\Gamma$$

for any $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\boldsymbol{V}$ and μ in M.

The mixed formulation of the unilateral contact problem without friction (1)–(6) consists then in finding $\boldsymbol{u} \in \boldsymbol{V}$ and $\lambda \in M^+$ such that

$$\begin{cases} a(\boldsymbol{u},\boldsymbol{v}) + b(\lambda,\boldsymbol{v}) = L(\boldsymbol{v}), & \forall\, \boldsymbol{v} \in \boldsymbol{V}, \\ b(\mu-\lambda,\boldsymbol{u}) \le 0, & \forall\, \mu \in M^+. \end{cases} \tag{7}$$

An equivalent formulation of (7) consists in finding $(\lambda,\boldsymbol{u}) \in M^+ \times \boldsymbol{V}$ satisfying

$$\mathcal{L}(\mu,\boldsymbol{u}) \le \mathcal{L}(\lambda,\boldsymbol{u}) \le \mathcal{L}(\lambda,\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \boldsymbol{V},\ \forall \mu \in M^+,$$

where $\mathcal{L}(\mu,\boldsymbol{v}) = \frac{1}{2}a(\boldsymbol{v},\boldsymbol{v}) - L(\boldsymbol{v}) + b(\mu,\boldsymbol{v})$. Another classical weak formulation of the problem (1)–(6) is a variational inequality: Find $\boldsymbol{u}$ such that

$$\boldsymbol{u} \in \boldsymbol{K}, \qquad a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) \ge L(\boldsymbol{v}-\boldsymbol{u}), \qquad \forall \boldsymbol{v} \in \boldsymbol{K}, \tag{8}$$

where $\boldsymbol{K}$ denotes the closed convex cone of admissible displacement fields satisfying the non-penetration conditions:

$$\boldsymbol{K} = \big\{\boldsymbol{v} \in \boldsymbol{V} : \quad v_n \le 0 \text{ on } \Gamma_C\big\}.$$

The existence and uniqueness of the solution $(\lambda,\boldsymbol{u})$ to (7) was given by Haslinger *et al.* (1996). Moreover, the first argument $\boldsymbol{u}$ solution to (7) is also the unique solution of the problem (8) and $\lambda = -\sigma_n(\boldsymbol{u})$.

3. Finite element approximation

A regular family of triangulations denoted by T_h is associated with the body Ω (see Brenner and Scott, 2002; Ciarlet, 1991). The closed triangles $K \in T_h$ are of diameter h_K, and we set $h = \max_{K\in T_h} h_K$. In order to use inverse inequalities on the contact area, we suppose that the one-dimensional mesh inherited on Γ_C is uniformly regular, and we denote by h_C a parameter representing the size of the elements on the contact zone (if the entire mesh is uniformly regular, as will be the case in the computations, we can merely choose $h_C = h$).

The finite dimensional space involving continuous affine finite elements is

$$\boldsymbol{V}_h = \Big\{\boldsymbol{v}_h \in (C(\overline{\Omega}))^2 : \forall \kappa \in T_h, \boldsymbol{v}_{h_{|\kappa}} \in (P_1(\kappa))^2, \boldsymbol{v}_{h_{|\Gamma_D}} = \boldsymbol{0}\Big\}.$$

The normal trace space on the contact zone is defined as

$$W_h = \Big\{\mu_h \in C(\overline{\Gamma_C}) : \exists \boldsymbol{v}_h \in \boldsymbol{V}_h \text{ s.t. } \boldsymbol{v}_h \cdot \boldsymbol{n} = \mu_h \text{ on } \Gamma_C\Big\},$$

and the nonnegative functions of W_h become

$$W_h^+ = \Big\{\mu_h \in W_h : \mu_h \ge 0\Big\}.$$

The discrete problem approximating (7) is the following: Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ and $\lambda_h \in W_h^+$ such that

$$\begin{cases} a(\boldsymbol{u}_h,\boldsymbol{v}_h) + \displaystyle\int_{\Gamma_C} I_h(\lambda_h v_{hn})\, \mathrm{d}\Gamma = L(\boldsymbol{v}_h), \\ \hfill \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ \displaystyle\int_{\Gamma_C} I_h((\mu_h-\lambda_h)u_{hn})\, \mathrm{d}\Gamma \le 0, \quad \forall\, \mu_h \in W_h^+, \end{cases} \tag{9}$$

where I_h stands for the standard Lagrange interpolation operator of degree 1 defined at the nodes of $\overline{\Gamma}_C$: $\forall v \in C(\Gamma_C) : I_h v \in C(\Gamma_C), I_h v(\boldsymbol{x}) = v(\boldsymbol{x})$ for any node $\boldsymbol{x}$ in $\overline{\Gamma}_C$, and $I_h v$ is an affine function between two nodes. The following proposition proves the existence of a unique solution to the problem (9). It also gives some elementary properties of the solution and describes links with a standard variational inequality.

Proposition 1.

(i) The problem (9) *admits a unique solution* $(\lambda_h,\boldsymbol{u}_h) \in W_h^+ \times \boldsymbol{V}_h$.

(ii) One has $u_{hn} \le 0, \lambda_h \ge 0$ *on* Γ_C *and* $\lambda_h\, u_{hn} = 0$ *at the nodes of* Γ_C.

(iii) The displacement field $\boldsymbol{u}_h$ *solving* (9) *is the unique solution to the problem: Find* $\boldsymbol{u}_h \in \boldsymbol{K}_h = \{\boldsymbol{v}_h \in \boldsymbol{V}_h : v_{hn} \le 0 \text{ on } \Gamma_C\}$ *such that*

$$a(\boldsymbol{u}_h,\boldsymbol{v}_h-\boldsymbol{u}_h) \ge L(\boldsymbol{v}_h-\boldsymbol{u}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{K}_h. \tag{10}$$

Proof.

(i) Since we deal with the finite dimensional case, we only need to check (see Theorem 3.9 and Example 3.8 of Haslinger *et al.*, 1996) that

$$\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h, \boldsymbol{v}_h\neq\boldsymbol{0}} \frac{\displaystyle\int_{\Gamma_C} I_h(\mu_h v_{hn})\, \mathrm{d}\Gamma}{\|\boldsymbol{v}_h\|_{1,\Omega}}$$

is a norm on W_h. Thus we have to verify that

$$\Big\{\mu_h \in W_h : \int_{\Gamma_C} I_h(\mu_h v_{hn})\, \mathrm{d}\Gamma = 0, \forall \boldsymbol{v}_h \in \boldsymbol{V}_h\Big\} = \{0\},$$

which is satisfied according to the definition of W_h. Hence the problem (9) admits a unique solution $(\lambda_h,\boldsymbol{u}_h) \in W_h^+ \times \boldsymbol{V}_h$.

(ii) Set

$$c(\mu_h,\boldsymbol{v}_h) = \int_{\Gamma_C} I_h(\mu_h v_{hn})\, \mathrm{d}\Gamma, \qquad \forall \mu_h \in W_h, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Taking $\mu_h = 0$ and $\mu_h = 2\lambda_h$ in (9) leads to

$$c(\lambda_h, \boldsymbol{u}_h) = 0 \quad \text{and} \quad c(\mu_h, \boldsymbol{u}_h) \le 0, \quad \forall\, \mu_h \in W_h^+.$$

Taking $\mu_h = \psi_x \in W_h^+$ in the previous inequality where ψ_x is the scalar basis function of W_h (defined on $\overline{\Gamma}_C$) at node $x \in \overline{\Gamma}_C$ satisfying $\psi_x(x') = \delta_{x,x'}$ for any node $x' \in \overline{\Gamma}_C$, we deduce that $u_{hn}(x) \le 0$. Hence $u_{hn} \le 0$ on Γ_C.

From $\lambda_h u_{hn} \le 0$ on Γ_C and since $c(\lambda_h, \boldsymbol{u}_h) = 0$, we come to the conclusion that $I_h(\lambda_h u_{hn}) = 0$ on Γ_C. That proves point (ii).

(iii) From (9) and $c(\lambda_h, \boldsymbol{u}_h) = 0$, we get

$$a(\boldsymbol{u}_h, \boldsymbol{u}_h) = L(\boldsymbol{u}_h) \tag{11}$$

and for any $\boldsymbol{v}_h \in \boldsymbol{K}_h$, we obtain

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) - L(\boldsymbol{v}_h) = -\int_{\Gamma_C} I_h(\lambda_h v_{hn})\, \mathrm{d}\Gamma \ge 0. \tag{12}$$

Putting together (11) and (12) implies that $\boldsymbol{u}_h$ is a solution of the variational inequality (10) which admits a unique solution according to Stampacchia's theorem. ■

The standard approach (see, e.g., Ben Belgacem and Renard, 2003; Coorevits *et al.*, 2002; Hüeber and Wohlmuth, 2005a) consists in solving the following discrete problem (using the same arguments as in the previous proposition, it admits a unique solution): Find $\boldsymbol{w}_h \in \boldsymbol{V}_h$ and $\theta_h \in M_h^+$ such that

$$\left\{ \begin{array}{l} a(\boldsymbol{w}_h, \boldsymbol{v}_h) + b(\theta_h, \boldsymbol{v}_h) = L(\boldsymbol{v}_h), \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b(\mu_h - \theta_h, \boldsymbol{w}_h) \le 0, \quad \forall\, \mu_h \in M_h^+, \end{array} \right. \tag{13}$$

where

$$M_h^+ = \left\{ \mu_h \in W_h : \int_{\Gamma_C} \mu_h \psi_h \, \mathrm{d}\Gamma \ge 0,\ \forall \psi_h \in W_h^+ \right\}. \tag{14}$$

Remark 1. We have $W_h^+ \subset M^+$ and $W_h^+ \subset M_h^+ \not\subset M^+$.

The next proposition establishes a link between the solutions of the problems (9) and (13).

Proposition 2. *The solutions* $(\lambda_h, \boldsymbol{u}_h)$ *and* $(\theta_h, \boldsymbol{w}_h)$ *of the problems* (9) *and* (13) *satisfy what follows:*
(i) $\boldsymbol{u}_h = \boldsymbol{w}_h$,
(ii) $\lambda_h = \pi_h \theta_h$, *where* $\pi_h : L^1(\Gamma_C) \mapsto W_h$ *is the quasi-interpolation operator defined for any function* v *in* $L^1(\Gamma_C)$ *by*

$$\pi_h v = \sum_{x \in N_h} \alpha_x(v) \psi_x,$$

N_h *represents the set of nodes of* $\overline{\Gamma}_C$, ψ_x *is the scalar basis function of* W_h *(defined on* $\overline{\Gamma}_C$*) at the node* x *satisfying* $\psi_x(x') = \delta_{x,x'}$ *for all* $x' \in N_h$ *and*

$$\alpha_x(v) = \left(\int_{\Gamma_C} v \psi_x \, \mathrm{d}\Gamma \right) \left(\int_{\Gamma_C} \psi_x \, \mathrm{d}\Gamma \right)^{-1}.$$

Proof.
(i) The same discussion as in points (ii) and (iii) of Proposition 1 and some polarity arguments (see, e.g., Hild, 2000; Hild and Nicaise, 2007) which we describe hereafter prove that $\boldsymbol{w}_h$ is also the unique solution of the variational inequality (10). Let us briefly summarize the result: Choosing $\mu_h = 0$ and $\mu_h = 2\theta_h$ in (13) implies $b(\theta_h, \boldsymbol{w}_h) = 0$ and $b(\mu_h, \boldsymbol{w}_h) = \int_{\Gamma_C} \mu_h w_{hn} \, \mathrm{d}\Gamma \le 0,\ \forall\, \mu_h \in M_h^+$. Consequently, $w_{hn} \in -(M_h^+)^*$ (the notation X^* stands for the positive polar cone of X for the inner product on W_h induced by $b(\cdot,\cdot)$, (Hiriart-Urruty and Lemaréchal, 1993, p. 119). We have $(M_h^+)^* = ((W_h^+)^*)^* = W_h^+$ since W_h^+ is a closed convex cone. Hence $w_{hn} \in -W_h^+$ and $\boldsymbol{w}_h \in \boldsymbol{K}_h$. Besides, (13) and $b(\theta_h, \boldsymbol{w}_h) = 0$ lead to $a(\boldsymbol{w}_h, \boldsymbol{w}_h) = L(\boldsymbol{w}_h)$, and for any $\boldsymbol{v}_h \in \boldsymbol{K}_h$, we get

$$a(\boldsymbol{w}_h, \boldsymbol{v}_h) - L(\boldsymbol{v}_h) = -\int_{\Gamma_C} \theta_h v_{hn} \, \mathrm{d}\Gamma \ge 0$$

since $\theta_h \in M_h^+ = (W_h^+)^*$ and $v_{hn} \in -W_h^+$. Hence $\boldsymbol{w}_h$ is the unique solution of the variational inequality (10) and point (iii) of Proposition 1 establishes the result.

(ii) From (i) and the equalities in (9) and (13), we deduce that

$$\int_{\Gamma_C} \theta_h v_{hn} \, \mathrm{d}\Gamma = \int_{\Gamma_C} I_h(\lambda_h v_{hn}) \, \mathrm{d}\Gamma, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{15}$$

We choose $\boldsymbol{v}_h$ such that $v_{hn} = \psi_x$ where ψ_x is the scalar basis function of W_h at node $x \in \overline{\Gamma}_C$. As a consequence,

$$\int_{\Gamma_C} \theta_h \psi_x \, \mathrm{d}\Gamma = \lambda_h(x) \int_{\Gamma_C} \psi_x \, \mathrm{d}\Gamma.$$

This proves that $\lambda_h = \pi_h \theta_h$, where π_h is the linear operator defined above. ■

4. Positivity preserving averaging operator: Basic properties

Now, we intend to study the basic properties of the operator π_h defined in Proposition 2. It is obvious that π_h is a linear averaging operator (for other averaging operators, see, e.g., Bernardi and Girault, 1998; Chen and Nochetto, 2000; Clément, 1975; Hilbert, 1973; Scott and Zhang, 1990; Strang, 1972), and that it not only preserves the nonnegative functions, but also satisfies $\pi_h(M_h^+) = W_h^+$, which means that it transforms finite element type functions with a weak nonnegativity condition into nonnegative functions (such a property is also satisfied by the operator in the work of Chen and Nochetto (2000)). For a detailed discussion concerning positivity preserving finite element approximation, we refer the reader to the work of Nochetto and Wahlbin (2002). Obviously, $\pi_h v_h \neq v_h$ in the general case when $v_h \in W_h$. Moreover, it is easy to see that $\pi_h(W_h) = W_h$. Finally, it is straightforward to check that any locally constant function is reproduced locally by π_h (this is not the case for locally affine functions, since the meshes on Γ_C do not have the same length), and that

$$\int_{\Gamma_C} v - \pi_h v \, \mathrm{d}\Gamma = 0 \tag{16}$$

for any $v \in L^1(\Gamma_C)$, which means that the operator preserves globally the average (note that a local average preserving property does not hold). In the following proofs, we denote by C a positive generic constant independent of the discretization parameter h. Now we show the L^2-stability property of π_h.

Lemma 1. *There is a positive constant C independent of h such that for any $v \in L^2(\Gamma_C)$ and any $E \in E_h^C$ (E_h^C denotes the set of closed edges lying in $\overline{\Gamma}_C$)*

$$\|\pi_h v\|_{0,E} \le C\|v\|_{0,\gamma_E},$$

where $\gamma_E = \cup_{\{F\in E_h^C:\, F\cap E\neq\emptyset\}} F$.

Proof. Let γ_x be the support of the basis function ψ_x in Γ_C. Using the definition of $\alpha_x(v)$ in Proposition 2, the Cauchy–Schwarz inequality, and the uniform regularity, we get

$$\begin{aligned}|\alpha_x(v)| &\le \|v\|_{0,\gamma_x}\|\psi_x\|_{0,\gamma_x}\|\psi_x\|^{-1}_{L^1(\gamma_x)}\\ &\le Ch_C^{-\frac{1}{2}}\|v\|_{0,\gamma_x}.\end{aligned}$$

Denoting by N_h the set of nodes of $\overline{\Gamma}_C$, we obtain by a triangular inequality

$$\|\pi_h v\|_{0,E} = \Big\| \sum_{x\in N_h\cap E} \alpha_x(v)\psi_x \Big\|_{0,E} \le C\|v\|_{0,\gamma_E}.$$

∎

The next lemma is concerned with the L^2-approximation properties of π_h.

Lemma 2. *There is a positive constant C independent of h such that for any $v \in H^\eta(\Gamma_C)$, $0 \le \eta \le 1$, and any $E \in E_h^C$ (E_h^C denotes the set of closed edges lying in $\overline{\Gamma}_C$)*

$$\|v - \pi_h v\|_{0,E} \quad\le\quad Ch^\eta\|v\|_{\eta,\gamma_E},$$

where $\gamma_E = \cup_{\{F\in E_h^C:\, F\cap E\neq\emptyset\}} F$.

Proof. When $\eta = 0$, the bound results from the previous lemma. Note that π_h preserves the constant functions on Γ_C. Let there be given an arbitrary constant function $c(x) = c$, $\forall x \in \Gamma_C$. From the definition of π_h, we may write for any $v \in H^\eta(\Gamma_C)$

$$v - \pi_h v = v - c - \pi_h(v - c).$$

Therefore, by Lemma 1 we get

$$\begin{aligned}\|v - \pi_h v\|_{0,E} &\le C\left(\|v - c\|_{0,E} + \|v - c\|_{0,\gamma_E}\right)\\ &\le C\|v - c\|_{0,\gamma_E}, \quad \forall c \in \mathbb{R}.\end{aligned} \tag{17}$$

We then choose $c = \int_{\gamma_E} v(x)\,\mathrm{d}x/|\gamma_E|$ in (17), where $|\gamma_E|$ denotes the length of γ_E. Then, if $x \in \gamma_E$ and $0 < \eta < 1$, we have

$$\begin{aligned}v(x) - c &= |\gamma_E|^{-1}\int_{\gamma_E} v(x) - v(y)\,\mathrm{d}y\\ &= |\gamma_E|^{-1}\int_{\gamma_E} \frac{v(x) - v(y)}{|x-y|^{\frac{1+2\eta}{2}}}|x-y|^{\frac{1+2\eta}{2}}\,\mathrm{d}y.\end{aligned}$$

Using the Cauchy–Schwarz inequality, we deduce that

$$\begin{aligned}&\int_{\gamma_E}(v(x) - c)^2\mathrm{d}x\\ &= |\gamma_E|^{-2}\int_{\gamma_E}\left(\int_{\gamma_E}\frac{v(x)-v(y)}{|x-y|^{\frac{1+2\eta}{2}}}|x-y|^{\frac{1+2\eta}{2}}\mathrm{d}y\right)^2\mathrm{d}x\\ &\le |\gamma_E|^{-2}\\ &\times\int_{\gamma_E}\left(\int_{\gamma_E}\frac{(v(x)-v(y))^2}{|x-y|^{1+2\eta}}\mathrm{d}y\int_{\gamma_E}|x-y|^{1+2\eta}\mathrm{d}y\right)\,\mathrm{d}x\\ &\le |\gamma_E|^{2\eta}\int_{\gamma_E}\int_{\gamma_E}\frac{(v(x)-v(y))^2}{|x-y|^{1+2\eta}}\,\mathrm{d}y\,\mathrm{d}x\\ &\le Ch^{2\eta}\|v\|^2_{\eta,\gamma_E},\end{aligned}$$

which is our claim.

If $x \in \gamma_E$ and $\eta = 1$, we have

$$\begin{aligned}v(x) - c &= |\gamma_E|^{-1}\int_{\gamma_E} v(x) - v(y)\,\mathrm{d}y\\ &= |\gamma_E|^{-1}\int_{\gamma_E}\int_y^x v'(t)\,\mathrm{d}t\,\mathrm{d}y,\end{aligned}$$

where the notation v' stands for the derivative of v. Hence

$$|v(x) - c| \le |\gamma_E|^{\frac{1}{2}}\|v'\|_{0,\gamma_E}.$$

The result is then straightforward. ∎

An open question is concerned with the optimal approximation properties of π_h in dual Sobolev spaces (typically $H^{-\frac{1}{2}}(\Gamma_C)$). It is easily seen that the $L^2(\Gamma_C)$-projection operator onto continuous and piecewise affine functions as well as the $L^2(\Gamma_C)$-projection operator onto piecewise constant functions satisfy such properties. On the contrary, it can be shown that the Lagrange interpolation operator as well as the $L^2(\Gamma_C)$-projection operator applied to nonnegative functions and mapping onto W_h^+ do not fulfil such properties. Unfortunately, the counter examples for the last two operators use the fact that the average of the function is not preserved and this is not the case for π_h (see (16)).

5. *A priori* error estimates

Now we intend to analyze the convergence of the finite element problem (9). In the forthcoming error analysis we suppose that $\boldsymbol{u} \in (H^{\frac{3}{2}+\eta}(\Omega))^2$ with $0 < \eta \leq 1/2$, which implies that u_n is continuous on Γ_C (which is a straight line segment). Set

$$\gamma_c = \Big\{x \in \Gamma_C : u_n(x) = 0\Big\},$$
$$\gamma_s = \Gamma_C \setminus \gamma_c.$$

In order to obtain an optimal convergence rate, we have to use the following assumption:

$$\text{The number of points in } \overset{\circ}{\overline{\gamma_c}} \cap \overline{\gamma_s} \text{ is finite.} \qquad (18)$$

The case where (18) is not valid is considered by Corollary 1. Let us first recall the result established in Hüeber and Wohlmuth (2005a).

Lemma 3. (Hüeber and Wohlmuth, 2005) *Let* $(\lambda, \boldsymbol{u})$ *be the solution of* (7)*, and let* $(\theta_h, \boldsymbol{u}_h)$ *be the solution of* (13)*. Assume that* (18) *holds. Let the regularity assumption* $\boldsymbol{u} \in (H^{\frac{3}{2}+\eta}(\Omega))^2$ *with* $0 < \eta \leq 1/2$ *hold. Then there exists a positive constant* C *independent of* h *and satisfying*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} + \|\lambda - \theta_h\|_{-\frac{1}{2},\Gamma_C} \leq Ch^{\frac{1}{2}+\eta}\|\boldsymbol{u}\|_{\frac{3}{2}+\eta,\Omega}.$$

This result and the triangle inequality imply the bound in the next lemma.

Lemma 4. *Let* $(\lambda, \boldsymbol{u})$ *be the solution of* (7)*, let* $(\lambda_h, \boldsymbol{u}_h)$ *be the solution of* (9)*, and let* $(\theta_h, \boldsymbol{u}_h)$ *be the solution of* (13)*. Assume that* (18) *holds. Let the regularity assumption* $\boldsymbol{u} \in (H^{\frac{3}{2}+\eta}(\Omega))^2$ *with* $0 < \eta \leq 1/2$ *hold. Then there exists a positive constant* C *independent of* h *satisfying*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-\frac{1}{2},\Gamma_C} \leq Ch^{\frac{1}{2}+\eta}\|\boldsymbol{u}\|_{\frac{3}{2}+\eta,\Omega} + \|\lambda_h - \theta_h\|_{-\frac{1}{2},\Gamma_C}.$$

Now we have to estimate the term $\|\lambda_h - \theta_h\|_{-\frac{1}{2},\Gamma_C}$. A first bound is given hereafter.

Lemma 5. *Assume that the hypotheses of Lemma 4 hold. Then there exists a positive constant* C *independent of* h *and satisfying*

$$\|\lambda_h - \theta_h\|_{-\frac{1}{2},\Gamma_C} \leq C\left(h^{\frac{1}{2}+\eta}\|\boldsymbol{u}\|_{\frac{3}{2}+\eta,\Omega} + h_C^{\frac{1}{2}}\|\lambda_h - \theta_h\|_{0,\Gamma_C}\right).$$

Proof. From the discrete inf-sup condition (see, e.g., Coorevits *et al.*, 2002)

$$0 < C \leq \inf_{\mu_h \in W_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\mu_h, \boldsymbol{v}_h)}{\|\mu_h\|_{-\frac{1}{2},\Gamma_C}\|\boldsymbol{v}_h\|_{1,\Omega}}$$

and (15), we get

$$\begin{aligned}\|\lambda_h - \theta_h\|_{-\frac{1}{2},\Gamma_C} &\leq C \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\lambda_h - \theta_h, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{1,\Omega}} \\ &= C \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{\displaystyle\int_{\Gamma_C} \lambda_h v_{hn} - I_h(\lambda_h v_{hn})\, \mathrm{d}\Gamma}{\|\boldsymbol{v}_h\|_{1,\Omega}}. \qquad (19)\end{aligned}$$

Besides, we have

$$\int_{\Gamma_C} \lambda_h v_{hn} - I_h(\lambda_h v_{hn})\, \mathrm{d}\Gamma = \sum_{E \in E_h^C} \int_E \lambda_h v_{hn} - I_h(\lambda_h v_{hn})\, \mathrm{d}\Gamma,$$

where E_h^C denotes the set of closed edges (of triangles) lying in $\overline{\Gamma}_C$. From numerical integration (trapezoidal formula) and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\int_E \lambda_h v_{hn} - I_h(\lambda_h v_{hn})\, \mathrm{d}\Gamma &\leq Ch_E^3|(\lambda_h v_{hn})''_{|_E}| \\ &\leq Ch_E^3|(\lambda'_h v'_{hn})_{|_E}| \\ &\leq Ch_E^2\|\lambda'_h\|_{0,E}\|v'_{hn}\|_{0,E} \\ &= Ch_E^2\|(\lambda_h - \bar{\lambda})'\|_{0,E}\|v'_{hn}\|_{0,E},\end{aligned}$$

where h_E denotes the length of the edge E, $\bar{\lambda} = (\int_E \lambda\ d\Gamma)/h_E$, and v', v'' denote the derivatives of the first and second orders of v. An inverse inequality implies

$$\int_E \lambda_h v_{hn} - I_h(\lambda_h v_{hn})\, \mathrm{d}\Gamma \leq Ch_E\|\lambda_h - \bar{\lambda}\|_{0,E}\|v'_{hn}\|_{0,E}.$$

Writing $\boldsymbol{v}_h = (v_{hx}, v_{hy})$, we can suppose without loss of generality that Γ_C is parallel to the horizontal x-axis (the y-axis is vertical). Using the scaled trace theorem (see, e.g., Grisvard, 1985),

$$\|v\|_{0,E} \leq C\left(h_E^{-\frac{1}{2}}\|v\|_{0,K} + h_E^{\frac{1}{2}}\|\nabla v\|_{0,K}\right), \quad \forall E \in E_K, \forall v \in H^1(K),$$

(E_K represents the set of the three edges belonging to the triangle K), we deduce that

$$\begin{aligned}\|v'_{hn}\|_{0,E} &= \left\|\frac{\partial v_{hy}}{\partial x}\right\|_{0,E} \\ &\leq Ch_E^{-\frac{1}{2}}\left\|\frac{\partial v_{hy}}{\partial x}\right\|_{0,K} \\ &\leq Ch_E^{-\frac{1}{2}}\|v_{hy}\|_{1,K} \leq Ch_E^{-\frac{1}{2}}\|\boldsymbol{v}_h\|_{1,K}.\end{aligned}$$

Hence

$$\int_E \lambda_h v_{hn} - I_h(\lambda_h v_{hn})\,\mathrm{d}\Gamma \le C h_E^{\frac{1}{2}} \|\lambda_h - \bar{\lambda}\|_{0,E} \|\boldsymbol{v}_h\|_{1,K}.$$

Therefore, denoting again by $\bar{\lambda}$ the piecewise constant function defined on Γ_C such that $\bar{\lambda}_{|_E} = (\int_E \lambda\,\mathrm{d}\Gamma)/h_E$, we obtain by addition

$$\int_{\Gamma_C} \lambda_h v_{hn} - I_h(\lambda_h v_{hn})\,\mathrm{d}\Gamma \le C h_C^{\frac{1}{2}} \|\bar{\lambda} - \lambda_h\|_{0,\Gamma_C} \|\boldsymbol{v}_h\|_{1,\Omega}.$$

According to (19), we deduce that

$$\begin{aligned} &\|\lambda_h - \theta_h\|_{-\frac{1}{2},\Gamma_C} \\ &\le C h_C^{\frac{1}{2}} \|\bar{\lambda} - \lambda_h\|_{0,\Gamma_C} \\ &\le C h_C^{\frac{1}{2}} \big(\|\lambda - \theta_h\|_{0,\Gamma_C} \\ &\quad + \|\theta_h - \lambda_h\|_{0,\Gamma_C} + \|\lambda - \bar{\lambda}\|_{0,\Gamma_C} \big). \end{aligned}$$

Then we use the standard estimate $\|\lambda - \bar{\lambda}\|_{0,\Gamma_C} \le C h^{\eta} \|\lambda\|_{\eta,\Gamma_C}$ (the latter result is obtained in the proof of Lemma 2) together with the trace theorem (the coefficients in the elasticity operator are supposed to lie in $C^1(\overline{\Omega})$).

The term $\|\lambda - \theta_h\|_{0,\Gamma_C}$ is estimated by using an inverse inequality, Lemma 3, and the optimal approximation properties in $H^{-\frac{1}{2}}(\Gamma_C)$ of the $L^2(\Gamma_C)$-projection operator p_h mapping onto W_h. We recall that p_h is defined for any $v \in L^2(\Gamma_C)$ by

$$p_h v \in W_h, \quad \int_{\Gamma_C} (v - p_h v)\psi_h\,\mathrm{d}\Gamma = 0, \quad \forall \psi_h \in W_h. \tag{20}$$

More precisely, we have

$$\begin{aligned} &\|\lambda - \theta_h\|_{0,\Gamma_C} \\ &\le \|\lambda - p_h\lambda\|_{0,\Gamma_C} + \|p_h\lambda - \theta_h\|_{0,\Gamma_C} \\ &\le C\left(h^{\eta}\|\lambda\|_{\eta,\Gamma_C} + h_C^{-\frac{1}{2}} \|p_h\lambda - \theta_h\|_{-\frac{1}{2},\Gamma_C} \right) \\ &\le C\Big(h^{\eta}\|\boldsymbol{u}\|_{\frac{3}{2}+\eta,\Omega} + h_C^{-\frac{1}{2}} \|p_h\lambda - \lambda\|_{-\frac{1}{2},\Gamma_C} \\ &\quad + h_C^{-\frac{1}{2}} \|\lambda - \theta_h\|_{-\frac{1}{2},\Gamma_C} \Big) \end{aligned}$$

and

$$h_C^{\frac{1}{2}} \|\lambda - \theta_h\|_{0,\Gamma_C} \le C h^{\frac{1}{2}+\eta} \|\boldsymbol{u}\|_{\frac{3}{2}+\eta,\Omega}. \tag{21}$$

Finally,

$$\begin{aligned} &\|\lambda_h - \theta_h\|_{-\frac{1}{2},\Gamma_C} \\ &\le C\left(h^{\frac{1}{2}+\eta} \|\boldsymbol{u}\|_{\frac{3}{2}+\eta,\Omega} + h_C^{\frac{1}{2}} \|\lambda_h - \theta_h\|_{0,\Gamma_C} \right). \end{aligned}$$

■

Lemma 6. *Assume that the hypotheses of Lemma 4 hold. Then there exists a positive constant C independent of h and satisfying*

$$h_C^{\frac{1}{2}} \|\lambda_h - \theta_h\|_{0,\Gamma_C} \le C h^{\frac{1}{2}+\eta} \|\boldsymbol{u}\|_{\frac{3}{2}+\eta,\Omega}.$$

Proof. We write

$$\begin{aligned} \|\lambda_h - \theta_h\|_{0,\Gamma_C} &= \|\theta_h - \pi_h\theta_h\|_{0,\Gamma_C} \\ &\le \|(\theta_h - \lambda) - \pi_h(\theta_h - \lambda)\|_{0,\Gamma_C} \\ &\quad + \|\lambda - \pi_h\lambda\|_{0,\Gamma_C}. \end{aligned}$$

Using Lemma 2 when adding the local estimates gives

$$\|\lambda_h - \theta_h\|_{0,\Gamma_C} \le C\left(\|\lambda - \theta_h\|_{0,\Gamma_C} + h^{\eta}\|\lambda\|_{\eta,\Gamma_C} \right),$$

and the bound (21) yields

$$h_C^{\frac{1}{2}} \|\lambda_h - \theta_h\|_{0,\Gamma_C} \le C h^{\frac{1}{2}+\eta} \|\boldsymbol{u}\|_{\frac{3}{2}+\eta,\Omega}.$$

■

We finally obtain the optimal *a priori* error estimate for the sign preserving method.

Theorem 1. *Let $(\lambda, \boldsymbol{u})$ be the solution of* (7) *and let $(\lambda_h, \boldsymbol{u}_h)$ be the solution of* (9)*. Assume that* (18) *holds. Let the regularity assumption $\boldsymbol{u} \in (H^{\frac{3}{2}+\eta}(\Omega))^2$ with $0 < \eta \le 1/2$ hold. Then, there exists a positive constant C independent of h satisfying*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-\frac{1}{2},\Gamma_C} \le C h^{\frac{1}{2}+\eta} \|\boldsymbol{u}\|_{\frac{3}{2}+\eta,\Omega}.$$

Proof. It suffices to put together the results of Lemmas 4–6. ■

Remark 2. If the operator π_h satisfied optimal approximation properties in dual Sobolev spaces (as $H^{-\frac{1}{2}}(\Gamma_C)$), then the proof of Theorem 1 would be straightforward (in this case one could avoid Lemma 5) since it suffices to write $\|\lambda_h - \theta_h\|_{-\frac{1}{2},\Gamma_C} = \|\theta_h - \pi_h\theta_h\|_{-\frac{1}{2},\Gamma_C} \le \|(\lambda - \theta_h) - \pi_h(\lambda - \theta_h)\|_{-\frac{1}{2},\Gamma_C} + \|\lambda - \pi_h\lambda\|_{-\frac{1}{2},\Gamma_C}$, and these properties (together with some inverse estimates) would end the proof. Unfortunately, such properties are not available (see also the discussion at the end of Section 4).

Remark 3. A deeper insight into the estimates shows that the direct error analysis of the finite element method (9) by circumventing the standard approximation (13) would be nontrivial (at least not shorter than the present analysis).

The assumption (18) is concerned with the finite number of transition points between contact and separation zones. Actually, we cannot prove that such an assumption is satisfied in practice. Without this hypothesis we can obtain a convergence result for the finite element method (9). This is achieved in the next corollary.

Corollary 1. *Let $(\lambda, \boldsymbol{u})$ be the solution of* (7)*, and let $(\lambda_h, \boldsymbol{u}_h)$ be the solution of* (9)*. Assume that $\boldsymbol{u} \in (H^{\frac{3}{2}+\eta}(\Omega))^2$ with $0 < \eta \le 1/2$. Then there exists a positive constant C independent of h and satisfying*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-\frac{1}{2},\Gamma_C} \le Ch^{\frac{1+\eta}{2}} \|\boldsymbol{u}\|_{\frac{3}{2}+\eta,\Omega}.$$

Proof. The result is straightforward by noting that the solution $(\theta_h, \boldsymbol{u}_h)$ of (13) satisfies, under the $(H^{\frac{3}{2}+\eta}(\Omega))^2$ regularity hypothesis (see, e.g., Ben Belgacem *et al.*, 1999)

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} + \|\lambda - \theta_h\|_{-\frac{1}{2},\Gamma_C} \le Ch^{\frac{1+\eta}{2}} \|\boldsymbol{u}\|_{\frac{3}{2}+\eta,\Omega},$$

and that the proofs of Lemmas 4–6 remain the same when dropping the assumption (18). ■

Remark 4. Using the same techniques as in (21), it becomes possible to obtain the same bounds as in Theorem 1 and Corollary 1 for the error with a weighted L^2-norm on the multipliers: $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} + h_C^{\frac{1}{2}}\|\lambda - \lambda_h\|_{0,\Gamma_C}$.

6. Numerical experiments

This section is concerned with the numerical implementation of the finite element method (9) and its comparison with the standard approach (13). We suppose that the contacting bodies are homogeneous isotropic so that Hooke's law (2) becomes

$$\boldsymbol{\sigma}(\boldsymbol{v}) = \frac{E\nu}{(1-2\nu)(1+\nu)} \operatorname{tr}(\boldsymbol{\varepsilon}(\boldsymbol{v}))\boldsymbol{I} + \frac{E}{1+\nu}\boldsymbol{\varepsilon}(\boldsymbol{v}),$$

where $\boldsymbol{I}$ represents the identity matrix, 'tr' is the trace operator, E and ν denote Young's modulus and Poisson's ratio, respectively, with $E > 0$ and $0 \le \nu < 1/2$. Hereafter we denote by N_C the number of elements on the contact area Γ_C.

In the first test we compute the values of the standard and nonstandard multipliers θ_h and λ_h, and we discuss the convergence rate of $\|\lambda_h - \theta_h\|_{0,\Gamma_C}$. The second example deals with Hertzian contact where the exact multiplier λ is known. This allows us to compare the accuracy of both discrete multipliers. A case with two contacting bodies and nonmatching meshes on the contact area is considered in the third example. We show how the sign preserving approach can be extended to this framework, at least numerically.

6.1. First example with slow variation in the contact pressure.

We study a realistic physical example also considered by Hild and Nicaise (2007) (see Fig. 1). We choose the domain $\Omega =]0,1[\times]0,1[$, and we suppose that the body is an iron square of $1\,\mathrm{m}^2$ whose material characteristics are $E = 2.1\,10^{11}\,\mathrm{Pa}, \nu = 0.3$ and $\rho = 7800\,\mathrm{kg}\cdot\mathrm{m}^{-3}$. The body is clamped on its right side; it is initially in contact on its left side and no forces are applied on the upper and lower boundary parts of Ω. Moreover, the body is acted on by its own weight only (with $g = 9.81\,\mathrm{m}\cdot\mathrm{s}^{-2}$). We consider quasi-uniform un-

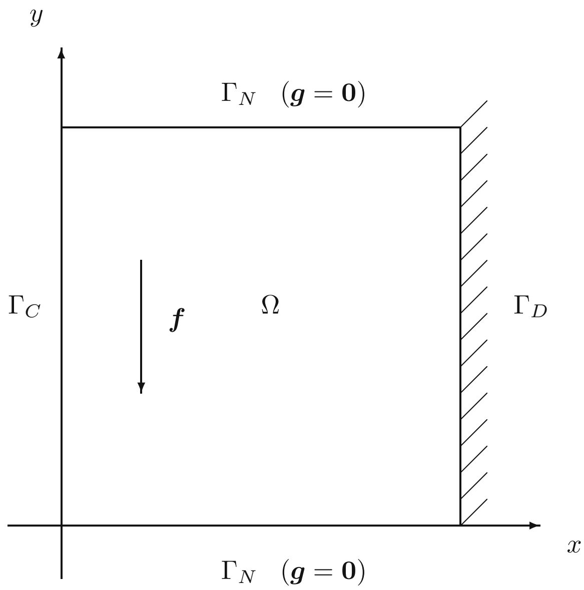

Fig. 1. Geometry of the body Ω.

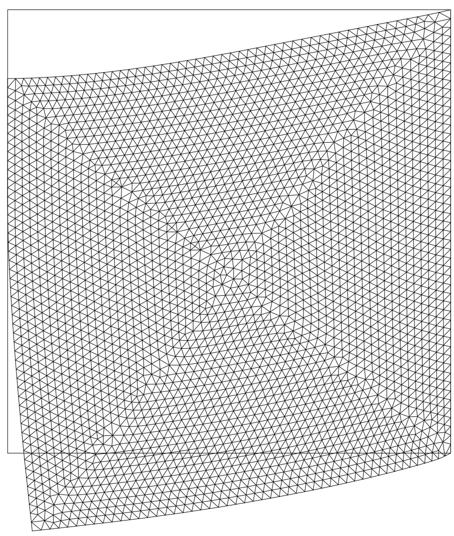

Fig. 2. Initial and deformed configuration with $N_C = 50$ (deformation is amplified by a factor $2 \cdot 10^5$).

structured meshes. A first configuration with 51 nodes on the contact area is depicted in Fig. 2. We see that Γ_C is divided into two parts: an upper part where the body remains in contact with the axis $x = 0$, and the lower part of Γ_C where it separates from this axis.

The nodes on Γ_C are numbered from 1 (up) to 51 (bottom), and $u_{hn} = 0$ at nodes 1 to 16 whereas $u_{hn} < 0$ at the other nodes. The corresponding standard (resp. nonstandard) multipliers θ_h (resp. λ_h) are reported in Table 1. As expected, we observe that θ_h is sometimes negative and that it shows some artificial (from a mechanical point of view) oscillations on the separation part (nodes 16 to 51). These oscillations weaken when moving away from the transition point (node 16).

Table 1. Nonstandard and standard multipliers λ_h and θ_h.

Node	λ_h	θ_h
1	1.20010E+05	1.26478E+05
2	1.09245E+05	1.07075E+05
3	1.00097E+05	1.00693E+05
4	9.10860E+04	9.07382E+04
5	8.29212E+04	8.28702E+04
6	7.53913E+04	7.53082E+04
7	6.83133E+04	6.82448E+04
8	6.16557E+04	6.15925E+04
9	5.53683E+04	5.53196E+04
10	4.93761E+04	4.93389E+04
11	4.36050E+04	4.35813E+04
12	3.79678E+04	3.79658E+04
13	3.23468E+04	3.23622E+04
14	2.65571E+04	2.66665E+04
15	2.02133E+04	2.03146E+04
16	1.16927E+04	1.33550E+04
17	0	- 3.57846E+03
18	0	9.58847E+02
19	0	- 2.56922E+02
20	0	6.88421E+01
21	0	- 1.84462E+01
22	0	4.94264E+00
23	0	- 1.32438E+00
24	0	3.54866E-01
25	0	- 9.50859E-02
26	0	2.54782E-02
27	0	- 6.82686E-03
28	0	1.82925E-03
29	0	- 4.90147E-04
30	0	1.31334E-04
31	0	- 3.51910E-05
32	0	9.42939E-06
33	0	- 2.52660E-06
34	0	6.77000E-07
35	0	- 1.81401E-07
36	0	4.86064E-08
37	0	- 1.30240E-08
38	0	3.48978E-09
39	0	- 9.35084E-10
40	0	2.50555E-10
41	0	- 6.71360E-11
42	0	1.79890E-11
43	0	- 4.82015E-12
44	0	1.29155E-12
45	0	- 3.46071E-13
46	0	9.27297E-14
47	0	- 2.48474E-14
48	0	6.66014E-15
49	0	- 1.79311E-15
50	0	5.12319E-16
51	0	- 2.56159E-16

We then compute the convergence rate of $\|\lambda_h - \theta_h\|_{0,\Gamma_C}$ in order to illustrate Lemma 6. The results are reported in Table 2, where this expression is computed from $N_C = 1$ to $N_C = 128$. The average convergence rate (between $N_C = 8$ and $N_C = 128$) is 1.25, and a limit rate near 1.24 is observed. In this example, we avoid computing the convergence rates of $\|\lambda - \lambda_h\|_{0,\Gamma_C}$ and $\|\lambda - \theta_h\|_{0,\Gamma_C}$ since the problem (7) does not admit an explicit solution $(\lambda, \boldsymbol{u})$ in this case and the choice of a reference multiplier would require to choose one of the methods (9) or (13). This study will be performed in the next example, where the exact expression of the multiplier λ is known. Of course, such a phenomenon does not occur for the reference displacement since they coincide for both finite element methods (9) and (13) according to Proposition 2. Thus we compute a reference displacement denoted by $\boldsymbol{u}_{\text{ref}}$ corresponding to a mesh which is as fine as possible. The most refined mesh corresponds to 129 nodes on the contact area and it furnishes the reference solution $\boldsymbol{u}_{\text{ref}}$ which is the chosen approximation for $\boldsymbol{u}$.

Table 2. Multipliers difference and error on the displacements.

	$\|\lambda_h - \theta_h\|_{0,\Gamma_C}$	$(a(\boldsymbol{e},\boldsymbol{e}))^{1/2}$
N_C =1	17063	0.12778
N_C =2	17299	$9.72400 \ 10^{-2}$
N_C = 4	13355	$7.01423 \ 10^{-2}$
N_C =8	6181.4	$4.40570 \ 10^{-2}$
N_C =16	2789	$2.54805 \ 10^{-2}$
N_C =32	1121	$1.40710 \ 10^{-2}$
N_C =64	453.27	–
N_C =128	191.29	–
Limit rate	1.24	0.86

We set $\boldsymbol{e} = \boldsymbol{u}_{\text{ref}} - \boldsymbol{u}_h$. Since the limit convergence rate of $(a(\boldsymbol{e},\boldsymbol{e}))^{1/2}$ (which is a norm equivalent to $\|\boldsymbol{u}_{\text{ref}} - \boldsymbol{u}_h\|_{1,\Omega}$) is near 0.86, one could merely believe that the convergence rate of $\|\lambda_h - \theta_h\|_{0,\Gamma_C}$ would be around 0.36. In fact, the computed rate (of 1.25) is much higher, a phenomenon that we cannot explain.

From this example we conclude, as expected, that the multiplier λ_h is more relevant from a mechanical point of view than θ_h.

6.2. Example of Hertzian contact. The next example is concerned with the Hertzian contact problem of an elastic ball with an infinite half plane. The material characteristics of the ball of radius $r = 1\,\text{mm}$ are chosen as in the work of Hüeber and Wohlmuth (2005a): $\nu = 0.3$, $E = 7000\,\text{MPa}$, and a force of $(0, -f)$ with $f = 100\ N$ is applied at the top of the ball. Since the analytical expression of the contact pressure is

$$\lambda(x) = \frac{2f}{\pi b^2}\sqrt{b^2 - x^2}, \quad -b \le x \le b,$$

$$b = 2\sqrt{\frac{fr(1-\nu^2)}{E\pi}}, \qquad (22)$$

we have at our disposal a useful analytical solution for a comparison of λ_h and θ_h. Here $b \approx 0.1286\,\mathrm{mm}$ and $\lambda(x) \approx 494.8\sqrt{1-(x/b)^2}\,\mathrm{N/mm}, -b \le x \le b$. In our computations we choose quasi-uniform unstructured meshes (we do not symmetrize the problem and the mesh is not symmetric). The results are reported in Tables 3 and 4.

Table 3. Errors on the multipliers.

Nodes on $\partial\Omega$	$\|\lambda-\lambda_h\|_{0,\Gamma_C}$	$\|\lambda-\theta_h\|_{0,\Gamma_C}$
24	47.080	100.23
48	54.651	80.604
96	23.704	30.822
192	8.9620	15.223
384	1.8805	9.8732
768	1.2057	4.4893
Average rate	1.057	0.896

Table 4. Comparison of the multipliers.

Nodes on $\partial\Omega$	$\max_{\Gamma_C}\lambda_h$; $\min_{\Gamma_C}\lambda_h$	$\max_{\Gamma_C}\theta_h$; $\min_{\Gamma_C}\theta_h$
24	381.97; 0	663.98; −178.76
48	511.47; 0	769.27; −3.0305
96	503.82; 0	535.93; −19.902
192	498.20; 0	501.49; −40.091
384	496.93; 0	498.27; −59.165
768	496.43; 0	496.77; −45.345

We first observe that the convergence rates of $\|\lambda-\lambda_h\|_{0,\Gamma_C}$ and $\|\lambda-\theta_h\|_{0,\Gamma_C}$ are not constant when h decreases: the average rates are 1.057 and 0.896, respectively, so that the terms $\|\lambda-\lambda_h\|_{0,\Gamma_C}$ remain smaller than $\|\lambda-\theta_h\|_{0,\Gamma_C}$ as h vanishes. From the expression (22), we see that $(\max_{\Gamma_C}\lambda, \min_{\Gamma_C}\lambda)$ is approximately $(494.8, 0)$. The first argument is reached by the two approaches, but the value 0 is not obtained in a satisfactory way by θ_h.

From this example we conclude that the sign preserving approach involving the nonnegative multiplier λ_h is more accurate than the standard method.

6.3. Example with two contacting bodies and nonmatching meshes. As the third example we choose a problem of two contacting bodies Ω^1 and Ω^2 with nonmatching meshes on the common contact zone $\Gamma_C = \overline{\Omega^1} \cap \overline{\Omega^2}$. The dimensions of Ω^1 and Ω^2 are $1\,\mathrm{mm} \times 0.05\,\mathrm{mm}$. Poisson's ratio $\nu = 0.2$ for both solids, Young's modulus $E_1 = 13000\,\mathrm{MPa}$ for the upper body, and $E_2 = 30000\,\mathrm{MPa}$ for the lower body are assumed. There are two applied boundary loads on Ω^1, of a value of $100\ \mathrm{N/mm}$ (see Fig. 3): $\boldsymbol{g}_1$ (on the upper half of the left side) and $\boldsymbol{g}_2$ (on the right half of the upper side). Symmetry conditions are applied on the lower and right parts of the structure. The mesh of Ω^1 (resp. Ω^2) divides Γ_C into 119 (resp. 120) identical segments.

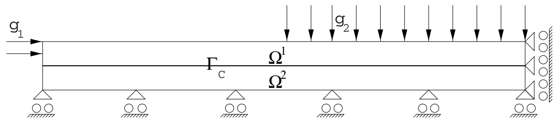

Fig. 3. Setting of the problem.

In order to handle nonmatching meshes, we consider a global contact condition of a mortar type. For error estimates dealing with mortar methods for contact problems, we refer the reader to, e.g., the works of Ben Belgacem *et al.* (1999), Coorevits *et al.* (2002), Hild (2000), Hüeber and Wohlmuth (2005a; 2005b), and Wohlmuth and Krause (2003). Such a contact condition furnishes a multiplier denoted θ_h^1 which does not satisfy the nonnegativity condition. Our aim is to extend, at least numerically, the range of applicability of the sign preserving method to a configuration with nonmatching meshes.

We denote by $\boldsymbol{V}_h^1$ and $\boldsymbol{V}_h^2$ the finite element spaces associated with Ω^1 and Ω^2, and by M_h^{1+} the positive polar cone of W_h^{1+} (see the definition in (14)). Note that the set W_h^{1+} involves functions defined on Γ_C which are continuous, nonnegative and piecewise of degree one on the mesh of Ω^1. Of course, one could also choose a symmetrical definition (e.g., M_h^{2+}) using the mesh of Ω^2. The standard approach is to find $\boldsymbol{u}_h = (\boldsymbol{u}_h^1, \boldsymbol{u}_h^2) \in \boldsymbol{V}_h^1 \times \boldsymbol{V}_h^2$ and $\theta_h^1 \in M_h^{1+}$ satisfying (see Coorevits *et al.*, 2002; Hüeber and Wohlmuth, 2005a)

$$\begin{cases} a(\boldsymbol{u}_h, \boldsymbol{v}_h) + \displaystyle\int_{\Gamma_C} \theta_h^1 (v_{hn}^1 + v_{hn}^2)\,\mathrm{d}\Gamma = L(\boldsymbol{v}_h), \\ \qquad\qquad\qquad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h^1 \times \boldsymbol{V}_h^2, \\ \displaystyle\int_{\Gamma_C} (\mu_h^1 - \theta_h^1)(u_{hn}^1 + u_{hn}^2)\,\mathrm{d}\Gamma \le 0,\ \forall\, \mu_h^1 \in M_h^{1+}, \end{cases}$$

where $a(\cdot,\cdot)$ and $L(\cdot)$ denote respectively the bilinear and linear forms involving both bodies Ω^1 and Ω^2. The sign preserving approach is to find $\boldsymbol{u}_h = (\boldsymbol{u}_h^1, \boldsymbol{u}_h^2) \in \boldsymbol{V}_h^1 \times \boldsymbol{V}_h^2$ and $\lambda_h^1 \in W_h^{1+}$ satisfying

$$\begin{cases} a(\boldsymbol{u}_h, \boldsymbol{v}_h) + \displaystyle\int_{\Gamma_C} I_h^1(\lambda_h^1 (v_{hn}^1 + p_h^1(v_{hn}^2)))\,\mathrm{d}\Gamma = L(\boldsymbol{v}_h), \\ \qquad\qquad\qquad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h^1 \times \boldsymbol{V}_h^2, \\ \displaystyle\int_{\Gamma_C} I_h^1((\mu_h^1 - \lambda_h^1)(u_{hn}^1 + p_h^1(u_{hn}^2)))\,\mathrm{d}\Gamma \le 0, \\ \qquad\qquad\qquad \forall\, \mu_h^1 \in W_h^{1+}, \end{cases}$$

where I_h^1 denotes the Lagrange interpolation operator of degree 1 at the nodes of Ω^1 on Γ_C, and p_h^1 stands for the $L^2(\Gamma_C)$-projection operator onto W_h^1 (see (20)).

As expected, the deformed configuration shows a separation area on the left part of Γ_C and a contact area on the right part of Γ_C (see Fig. 4). The multiplier θ_h^1, representing the contact pressure is depicted in Fig. 5. As already noticed, the multiplier is not always nonnegative and it shows some artificial oscillations near the transition point from contact to separation. Besides, the multiplier λ_h^1 is represented in Fig. 6 and we observe that it is more relevant from a mechanical point of view. We observe that the multiplier value is close to 100 on the contact zone which corresponds to the value of $\boldsymbol{g}_2$. Finally, the difference $\theta_h^1 - \lambda_h^1$ is depicted in Fig. 7 and we see that θ_h^1 and λ_h^1 differ in a significant way near the transition point. Again, we conclude that the new approach involving λ_h^1 seems to be more accurate than the standard one when handling nonmatching meshes.

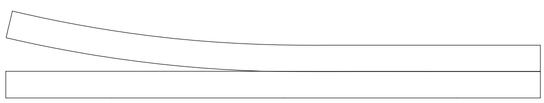

Fig. 4. Deformed configuration.

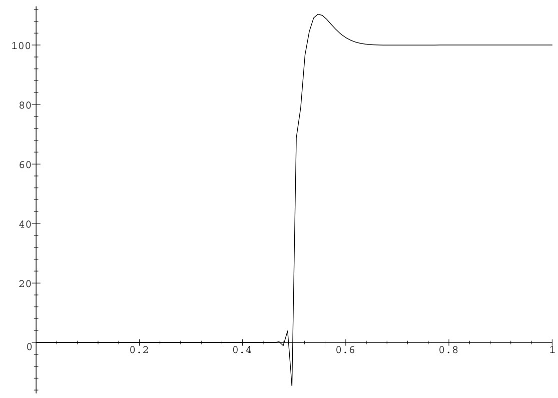

Fig. 5. Multiplier θ_h^1.

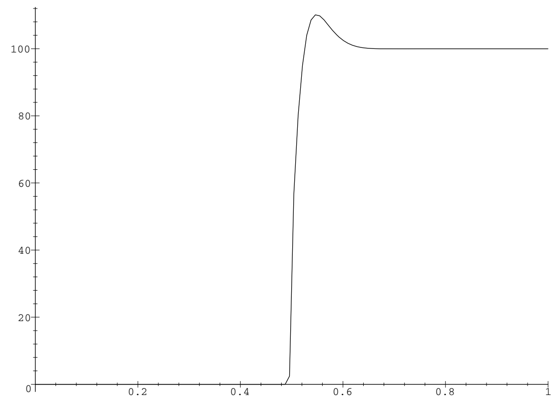

Fig. 6. Multiplier λ_h^1.

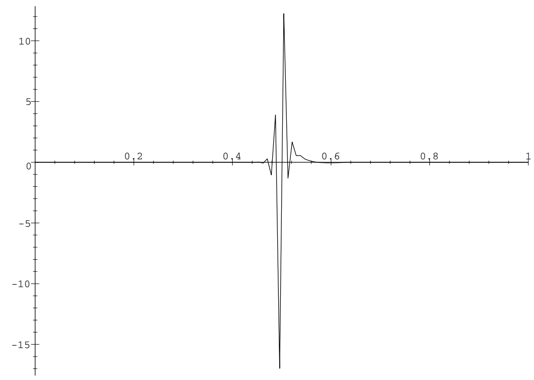

Fig. 7. Difference between the multipliers: $\theta_h^1 - \lambda_h^1$.

7. Conclusion

In this work we consider a mixed finite element method which furnishes primal and dual variables with a good sign in opposition to the already known mixed methods for contact problems (in particular, the classical approach). The study of the method uses an averaging positivity preserving operator, which is analyzed and discussed. The convergence analysis in this paper leads to the same error estimates as the standard approach. The numerical experiments obtained with the new method seem to be more relevant and efficient in comparison with the standard method. Finally, the friction (see, e.g., Hild, 2002) or the crack problems (see, e.g., Belhachmi *et al.*, 2005; Khludnev and Sokolowski, 2004) are some possible applications of the method.

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