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ESTIMATING THE COUNTERPARTY RISK EXPOSURE BY USING THE BROWNIAN MOTION LOCAL TIME

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In recent years, the counterparty credit risk measure, namely the default risk in over-the-counter (OTC) derivatives contracts, has received great attention by banking regulators, specifically within the frameworks of Basel II and Basel III. More explicitly, to obtain the related risk figures, one is first obliged to compute intermediate output functionals related to the mark-to-market position at a given time no exceeding a positive and finite time horizon. The latter implies an enormous amount of computational effort is needed, with related highly time consuming procedures to be carried out, turning out into significant costs. To overcome the latter issue, we propose a smart exploitation of the properties of the (local) time spent by the Brownian motion close to a given value.

Keywords: counterparty credit risk, exposure at default, local times Brownian motion, over-the-counter derivatives, Basel financial framework.

1. Introduction

For some years now, due to the occurrence of events leading to the financial crisis between 2007 and 2008, regulators have forced financial institutions to adopt *ad-hoc* procedures to predict, and therefore prevent, defaults. In other words, banks have to be able to measure and manage their default risk. As for both the credit and the counterparty risk, in 2006 the Basel Committee for Banking Supervision has inserted in the well-known Basel II reform, two rather general methodologies for calculating banks capital requirements, namely the *standardized approach* and the *internal approach*. While the former one is based on the use of ratings from external credit rating agencies, the latter envisages the evaluation of certain risk parameters, such as the exposure at default (EAD) (cf. BCBS, 2006).

An interesting perspective concerns the so-called *counterparty credit risk* (CCR), which represents the default risk linked to *over-the-counter* (OTC) derivatives contracts. The latter case implies the computation, as intermediate outputs, of a large set of different functionals related to the *mark-to-market* (MtM) of the position over a future time horizon, at a given time $t \in [0, T]$, where $T < +\infty$ is the *time horizon*. Standard techniques for the evaluation of such an exposure are based on classical Monte Carlo methods, which are characterized by a strong dependence on the number of assets considered

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and related high computational time costs (see, e.g., Liu, 2015). Other approaches have also been given, considering, e.g., a geometric point of view (Sinkala and Nkalashe, 2015), or general ambit stochastic processes (Di Persio and Perin, 2015), or some optimal investment control problems (Chevalier *et al.*, 2013), even if, as a general benchmark, the Monte Carlo set of methods are the most widely used. Nevertheless, as mentioned, Monte Carlo techniques are far from being computationally satisfactory, even in simple cases. For example, a medium bank requires $D = \mathcal{O}(10^4)$ derivative deals and $U = \mathcal{O}(10^3)$ risk factors, evaluated in $K = 20$ time steps with $N = 2000$ simulations, which allow for $K \times N \times U = 4 \times 10^7$ grid points for the risk factor simulation and $K \times N \times D = 4 \times 10^8$ tasks for deals evaluation.

To overcome these drawbacks, the literature has recently proposed new techniques, e.g., vector quantization (Bonollo *et al.*, 2015; Callegaro *et al.*, 2015; 2017; Callegaro and Sagna, 2013), or more enhanced hardware technologies, such as in the case of grid computing and graphical processing units (GPUs) (see, e.g., the works of Castagna (2013) or Pagés and Wilbertz (2011) and the references therein). In the context of American option pricing, other methods recently investigated are the martingale-based approach *à la Rogers* see, e.g., Lelong (2016), and the simple least-squares approach (see Antonov *et al.*, 2015; Glasserman, 2012). A different solution can be achieved exploiting the so-called *polynomial chaos expansion* approach (see, e.g., Bernis and Scotti, 2017; Di Persio *et al.*, 2015) and the references therein. Another possibility consists in exploiting the properties of suitable mathematical tools, as for the case of derivatives pricing via *Brownian local time.* Given a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, we consider a standard Brownian motion, $\{W_t\}_{t\geq 0}$ defined on it. Then, for $\omega \in \Omega$ and a level a, an interesting point is to determine how much time the sample path $W_t(\omega)$ spends close to a. A possible answer dates back to the works written by Paul Lévy in 1948, where the author introduced the concept of *Brownian local time* (see Lévy, 1965).

The right approach consists in defining the Brownian local time (BLT from now on) as the following density:

$$L_t(a) := \frac{1}{2\epsilon} \lim_{\epsilon \longrightarrow 0} \mu\{x : |x - a| \leq \epsilon\} , \qquad (1)$$

where μ represents the Lebesgue measure on the real line.

Remark 1. It is worth mentioning that there does not exist a standard notation to define the BLT, since some authors prefer to multiply the limit in (1) by $1/4\epsilon$, instead of $1/2\epsilon$ (see, e.g., Karatzas and Shreve, 1991).

More formally, the local time can be defined through the so-called *occupation formula* (see Karatzas and Shreve, 1991), namely by the following equation:

$$\int_0^t f(W_s)\,\mathrm{d}s = 2\int_{\mathbb{R}} f(x)L_t(x)\,\mathrm{d}x , \qquad (2)$$

where the left-hand side is a random measure, called *occupation measure* or *sojourn measure,* at fixed time t and level $x \in \mathbb{R}$, while f is an L^1 function, $f : \mathbb{R} \to \mathbb{R}$. We refer to Section 3.2 for a more detailed discussion of the BLT properties.

To what concerns the fine properties of the local time, e.g., the identification of both its distribution function and related density function and moments, we refer to the works of Doney and Yor (1998), Karatzas and Shreve (1991) or Takacs (1995), and references therein. It is also worth mentioning that there exist many works dealing with the theoretical applications of the BLT such as an extension of Itô's formula to convex functions, the definition of the density of the occupation measure for a Brownian motion with respect to the Lebesgue measure (Bonollo *et al.*, 2015), etc.

On the other hand, relatively limited literature has been devoted to concrete applications of the BLT and its properties. The latter lack can be easily recognized in frameworks related to economy and finance. Nevertheless, theoretical aspects of the BLT can be fruitfully exploited to analyze a wide range of financial tools, particularly with respect to the pricing of some kinds of exotic path-dependent options as in the case, e.g., of *range accrual options* and *accumulators,* where the payoff depends on the time spent by the underlying below or above a given level, between two boundaries, or outside of them (see, e.g., Mijatovic, 2010). Moreover, the use of the BLT is almost absent in the risk management field. The present work aims at filling this gap by showing that the numerical integration of the BLT density function can be used to evaluate the risk exposure, hence obtaining results that are very compelling when compared with classical Monte Carlo benchmark algorithms.

The paper is organized as follows. In Section 2 we introduce the financial framework, focusing on the regulatory viewpoint, and with emphasis to the instructions for calculating the EAD and the credit value adjustment (CVA). Then, in Section 3, the mathematical setting is introduced also recalling the main properties of the BLT, while in Section 4, we provide the *local time approach* to the aforementioned type of financial problems, also analyzing its performances compared with more standard techniques with respect to an EAD application. Finally, in Section 5, we state our main conclusions and outline future research directions.

2. Counterparty risk: The financial framework

2.1. Credit counterparty risk in the Basel approach. In the Basel II framework, the *counterparty credit risk* (CCR from now on) is a specific class of the broader credit risk category. Let us recall the definition of the Basel committee, shortly BCBS, as it is written by BCBS (2006).

Definition 1. (*Counterparty credit risk (CCR)*) is the risk that the counterparty in a transaction could default before the final settlement of the transaction's cash flows. An economic loss would occur if the transactions or portfolio of transactions with the counterparty has a positive economic value at the time of default.

Unlike a firm's exposure to credit risk through a loan, the CCR creates a bilateral risk of loss: the market value of the transaction is uncertain, it can be positive or negative to either counterparty and can vary over time with the movement of the underlying market factors. A typical example is given by IRS. Several classes of financial transactions are considered in the regulatory perimeter, but most of the CCR arise from OTC derivatives, in the peer-to-peer relationships with a defaultable counterparty. From a practical perspective, the buyer of any option, or the holder of a derivative with positive MtM, both are facing a CCR. If the two counterparties agree upon a *netting set,* e.g., a running compensation process in their deals, the current exposure will be given by the positive part of the algebraic sum of all deals.

As in the whole Basel setting, the risk must be dealt with by setting apart an amount regulatory capital of the bank which is linked to the risk measure called *capital requirement* (BCBS, 2011):

$$K = \mathrm{EAD} \cdot 1.06 \cdot \mathrm{LGD} \left\{ \Phi \left[\left(\frac{1}{1-\rho} \right)^{0.5} \Phi^{-1}(\mathrm{PD}) + \left(\frac{\rho}{1-\rho} \right)^{0.5} \Phi^{-1}(0.999) \right] - \mathrm{PD} \right\} \cdot c \,, \qquad (3)$$

where EAD is the *exposure at default,* namely an estimate of the extent to which a bank may be exposed to a counterparty in the case of a default; LGD is the *loss given default,* namely an estimate of the percentage of the credit not recoverable in the case of insolvency; PD is the *probability of default,* namely an estimate of the likelihood that a default will occur; ρ is the *asset return correlation coefficient*; c is a constant which takes into account some maturity adjustment and may vary with respect to different regulatory *portfolios,* such as enterprise or retail loans; 1.06 is a coefficient depending on the calibration procedure made by the Basel committee; Φ is the cumulative distribution function of a standard Gaussian random variable; Φ^{-1} is simply the inverse of Φ, also referred to as the *quantile function*.

As well highlighted in the BCBS definition (see Definition 1), the EAD estimate makes the counterparty risk very different from the normal credit risk for loans and mortgages. In fact, the Basel formula (3) requires a one-year measurement process, and the default time τ could be, or it could not be, *AT* any future time t.

For a mortgage, we know the future exposure profile, since it can be computed using the amortizing plan. Differently, in the CCR, the EAD estimation is fairly difficult, because of two different reasons: the future exposure is stochastic and, further, it depends on the market parameters via its specific evolution pricing model.

In other words, the CCR depends in its magnitude both on the credit parameters (PD, LGD) and on the market influenced EAD parameter; that is why it is also referred to as the *boundary risk.* To summarize, the CCR has to be determined according to (3) for the credit risk, but its EAD input estimation is itself a hard challenge, to which the Basel committee and the financial operators pay most of their attention.

2.2. Exposure and CVA calculation in the Basel II–III setting. In order to calculate the EAD quantity in the CCR context in a robust and conservative way, the Basel II framework (BCBS, 2006) defines two important different approaches: the *standard model* and the *internal model*, also called *EPE-based* approach. In the standard model, we have $\mathrm{EAD} = \mathrm{MtM} + \mathrm{Add-On}$, where the Add-On is computed exploiting a table which depends on both the underlying asset class and on the time to maturity. In this case, the idea is that such an Add-On takes into account the future volatility by additive coefficients. As an example, for an equity option with maturity M years and such that $1 \le M \le 5$, we have that the Add-On is 8% of the notional amount, while for an interest rate derivative it is just 0.5%. In the *EPE-based* approach, to which the present work refers, some notation has to be pointed out. Given a derivative maturity time $0 < T < +\infty$, we consider $K \in \mathbb{N}^+$ time steps $0 < t_1 < t_2 < \cdots < t_K$, which constitute the so-called *buckets array*, denoted by $\mathbf{B}^{T,K}$, where usually, but not mandatory, $t_K = T$. For every $t_k \in \mathbf{B}^{T,K}$, we denote by $\mathrm{MtM}\,(t_k, S_k) := \mathrm{MtM}\,(t_k, S_{t_k})$ the fair value, mark-to-market, of a derivative at time bucket t_k, with respect to the *underlying value* S_k considered at time t_k.

For every $t_k \in \mathbf{B}^{T,K}$, we denote by $\mathrm{MtM}\left(t_k, S^k\right) := \mathrm{MtM}\left(t_k, S^{t_k}\right)$ the fair value (mark-to-market) of a derivative at time bucket t_k, with respect to the whole sample path $S^k := \{S_t : 0 \le t \le t_k\}$, and with initial time $t_0 = 0$.

Taking into account previous definitions, we indicate by $\varphi = \varphi\,(T - t_k, S_k, \Theta)$ the pricing function for the given derivative, where Θ represents the set of parameters from which such a pricing function may depend, e.g., the free risk rate r or the volatility σ.

We give an account of the main amounts, as they are defined in Basel III (BCBS, 2006), that will be used later on to estimate the EAD. We introduce the *expected exposure* of the derivative at time $t_k \in \mathbf{B}^{T,K}$ (EE_k), as

$$\mathrm{EE}_k := \frac{1}{N}\sum_{n=1}^{N} \mathrm{MtM}\,(t_k, S_{k,n})^+\,, \quad N \in \mathbb{N}^+, \tag{4}$$

which is the arithmetic mean of the positive part of N Monte Carlo simulated MtM values, computed at the k-th time bucket t_k, with respect to the underlying S.

Remark 2. The *positive part* operator is effective if we are managing a symmetric derivative, such as an interest rate swap or a portfolio of derivatives. Nevertheless, it is redundant if we consider a single option, as the fair value of the option is always positive from the buy side situation. We want to stress that the sell side does not imply counterparty risk, hence it is out of context.

We evaluate the *expected positive exposure* (EPE) as

$$\mathrm{EPE} := \frac{1}{T}\sum_{k=1}^{K} \mathrm{EE}_k \cdot \Delta_k, \tag{5}$$

where $\Delta_k = t_k - t_{k-1}$ indicates the time space between two consecutive time buckets at the k-th level. If the time buckets t_k are equally spaced, then the formula reduces to $\mathrm{EPE} = \frac{1}{K}\sum_{k=1}^{K} \mathrm{EE}_k$. Therefore, the EPE value gives the time average of EE_k and reflects the hypothesis that the default could happen, as a first approximation, at any time with the same probability. We define the *effected expected exposure* as follows $\mathrm{EEE}_1 := \mathrm{EE}_1$; $\mathrm{EEE}_k := \max\{\mathrm{EE}_k, \mathrm{EEE}_{k-1}\}\,,\; k = 1,\dots,K$, observing that, due to its non decreasing property, EEE_k takes into account the fact that, once the time decay effect reduces the MtM as well as the counterparty risk exposure, the bank applies a roll out with some new deals. We also define the *effected expected positive exposure* (EEPE) by

$$\mathrm{EEPE} := \frac{1}{T}\sum_{k=1}^{K} \mathrm{EEE}_k \cdot \Delta_k.$$

Remark 3. In order to avoid too many inessential regulatory details, we will work on EE_k and the EPE, the others quantities being just arithmetic modifications of them.

In what follows we shall rewrite previously defined quantities in continuous time, and we add the index A to indicate the *adjusted* definitions. Moreover we consider the dynamics of the underlying $S_t := \{S_t\}_{t\in[0,T]}$, $T \in \mathbb{R}^+$ being some expiration date, as an Itô process, defined on some filtered probability space $\left(\Omega, \mathfrak{F}, \mathfrak{F}_{t\in[0,T]}, \mathbb{P}\right)$. As an example, S_t is the solution of the stochastic differential equation defining the geometric Brownian motion, $\mathfrak{F}_{t\in[0,T]}$ being the natural filtration generated by a standard Brownian motion $W_t = (W_t)_{t\in[0,T]}$ and with respect to a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, where $\mathbb{P}$ is often referred to as the so-called *real world* probability measure, or an equivalent risk neutral measure under the martingale approach to option pricing (see, e.g., Karatzas and Shreve, 1991).

The *adjusted expected exposure* EE^A is given by

$$\begin{aligned}\mathrm{EE}^A_k &:= \mathbb{E}_{\mathbb{P}}\left[\mathrm{MtM}\,(t_k, S_k)^+\right]\\ &= \int \varphi\,(T - t_k, S_k, \Theta)\;\mathrm{d}\mathbb{P}\\ &\cong \frac{1}{N}\sum_{n=1}^{N}\mathrm{MtM}\,(t_k, S_{k,n})^+ = \widehat{\mathrm{EE}^A_k}.\end{aligned} \tag{6}$$

Similarly, we define the *adjusted expected positive exposure* EPE^A as

$$\mathrm{EPE}^A := \int \mathrm{EE}^A_t \mathrm{d}t = \iint \varphi\,(t, S_k, \Theta)\,\mathrm{d}\mathbb{P}\,\mathrm{d}t\,. \tag{7}$$

With respect to the latter formulation, the Basel definition is simply one of many methods that can be used to estimate the expected fair value of the derivative in the future.

Remark 4. We skip any comment about the choice of the most suitable probability measure $\mathbb{P}$ to be used in the calculation of EE_k, the latter being beyond the aim of the present paper. For a detailed discussion on the role played by the *risk neutral* probability, or by the *historical* real world probability (see, e.g., Brigo *et al*., 2013).

Remark 5. Let us underline that the component usually indicated as *a discount factor*, or a *numeraire*, is missing in the EPE definition, the latter being a byproduct of the conservative approach used in the risk regulation.

Besides the EAD, understood as a CCR measure, also the *credit value adjustment* (CVA) may be specified. According to Basel guidelines (BCBS, 2011), the CVA represents the capital charge for potential MtM losses associated with a deterioration in the credit worthiness of a counterparty. Moreover, by introducing the CVA, the expression of the derivative payoff provides a new term, related to the value of the security emerging in the case of a default. In particular, we have

$$\begin{aligned}\mathrm{Payoff} = \phi(\mathbf{m}^T, \mathbf{c}) \cdot \mathbb{1}_{\{\tau > T\}}&\\ + \mathrm{RR}\phi(\mathbf{m}^{\mathbf{T}}, \mathbf{c}) \cdot \mathbb{1}_{\{\tau \le T\}}&,\end{aligned}$$

where τ is the counterparty default time, $\phi(\mathbf{m}^T, \mathbf{c})$ is the terminal payoff at maturity T, where $\mathbf{m}^T$, resp. $\mathbf{c}$, stands for the path of the market parameters in $[0, T]$, resp. for the contract clauses on which the payoff depends, while $\mathrm{RR} := 1 - \mathrm{LGD}$ is the so called *recovery rate,* that is, the extent to which principal and accrued interest on a debt instrument that is in default can be recovered, expressed as a percentage of the instrument's face value. Hence, the CVA metrics performs an average reduction of the MtM value and involves another form of risk, the CVA *risk,* characterizing the uncertainty of the future CVA evolution.

Remark 6. Let us note that one of the major credit rating agencies, namely Moody's, estimates defaulted debt recovery rates using market bid prices observed roughly 30 days after the date of default. Recovery rates are measured as the ratio of price to par value (see the report for Moody's (2009) for further details).

2.3. Computational challenges. An extremely interesting and challenging problem consists in the concrete implementation of both the EPE and the CVA. Because of the EPE (EAD) volatility, the counterparty risk must be monitored frequently, hence the standard requirement for an internal model validation is a daily frequency. To have an idea of the magnitude of the computational efforts for such a procedure, let us consider that, in a medium size banking group that aims to satisfy the regulators indications, we could observe $D = 10000$ deals in the book, $N = 2000$ simulations and $K = 20$ time steps. If we denote by PT the number of pricing tasks for each CCR run, we easily get

$$PT = D \cdot N \cdot K = 4 \cdot 10^8 \ . \tag{8}$$

This example easily shows how great the required computational effort is even though a big part of the pricing algorithms is still represented by specific *à la* Monte Carlo techniques. Hence, although the pricing software and CPU features are adequate for front office purposes, they become unsatisfactory for CCR evaluation constraints. As for the storage requirements, we define the new parameter α, i.e., the number of execution cases that have to be stored to allow traceability and auditability of the output results. We can fix $\alpha = 13$, if we suppose an *end-of-month* backup with one-year memory. Of course, the storage is run on different record types, e.g., deal information, payoff information, simulation information, etc. For the sake of simplicity, we can think of the storage as a unique large record type, denoted by RT, which takes into account all the relevant information. Hence

$$\mathrm{RT} = D \cdot N \cdot K \cdot \alpha = 5.2 \cdot 10^9 \ . \tag{9}$$

As each record could easily require 1000 bytes, we raise to 5.2 terabytes of storage. In other words, the CCR involves the computational hard challenges related to the credit and market risk fields. In particular, the high frequency of monitoring implies a number of concrete practical implementations of efficient and robust CCR calculation. In order to address the previous challenges, important results have been achieved exploiting techniques related to the so-called *BigData* analysis as well as using graphical processing units (GPU); see, e.g., the numerical investigations provided by Castagna (2013) or Pagés and Wilbertz (2011). Nevertheless, the solution to the computational challenges posed by the CCR evaluation are neither completely, nor satisfactory solved by the aforementioned software improvements. That is why there is a growing and wide interest in finding more effective theoretical techniques, and related applied algorithmic procedures.

Remark 7. We would like to underline that while the Basel Committee generally defines frameworks and principles, it does not prescribe a mandatory model or some numerical technique that one has to apply. Hence, starting from the next section, we propose a novel method to perform the EPE calculation, in the broad CCR setting, by exploiting a BLT approach.

3. Mathematical setting

3.1. Black–Scholes market model. In what follows we will refer to the celebrated Black and Scholes diffusion process (see Black and Scholes, 1973), as a theoretical benchmark for our proposal's verification. Let us consider a financial market, composed of a risk-less security B, with constant return r, and a risky asset S, defined by means of a geometric Brownian motion, namely

$$\begin{cases} \mathrm{d}B_t = rB_t \,\mathrm{d}t, \\ \mathrm{d}S_t = S_t\mu \,\mathrm{d}t + S_t\sigma \,\mathrm{d}W_t, \end{cases} \tag{10}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and $\{W_t\}_{t\geq 0}$ represents a standard Brownian motion.

The SDE representing the geometric Brownian motion in (10) admits the following unique solution:

$$S_t = S_0 \exp\left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\} , \tag{11}$$

which characterizes the dynamics of the underlying of a *derivative,* namely a financial instrument that gives to its owner a terminal *payoff* $\phi = \phi(\mathbf{m}^T, \mathbf{c})$ evaluated at the maturity T. To give an example, in the simple case represented by considering a *European call option,* we have $\phi := (S_T - K)^+$, where the *level* K is called the *strike price* of the option, since it provides a positive profit if and only if $S_T > K$. Let us recall that the parameters r and σ represent the *risk-free rate* and the *volatility* of the underlying, respectively. The risk-free rate plays a key role in the evaluation process, that is, the definition of the fair value (FV from now on) at time 0. In other words, by an application of the Itô–Döblin lemma, it is possible to show that, in the fair value evaluation, the actual drift μ, with $\mu > r$, and the unknown risk aversion of the market, or *utility function,* both disappear, while the fair value can be simply calculated as the discounted expected payoff, where the risk-neutral drift r can straightly replace the expected drift μ in (11); see, e.g., the work of Hull (1999) for further details. In the basic Black–Scholes simplified model, where the risk-free rate r is deterministic and constant over time, this principle leads to a general evaluation strategy given by

$$\mathrm{FV}_t = \mathrm{E}[e^{-r(T-t)}\phi(\mathbf{m}^T, \mathbf{c}_t)] \, .$$

The Black–Scholes model gained several extensions and criticism, e.g., sophistication in the *payoff algebra,* due to the natural innovation process in the financial markets. They allow to cover the effective requirements or to get new profits by issuing new appealing products. Generally speaking, we can have several clauses, e.g., the bundling of different strikes, barriers, memory effects, occupation time clauses, etc., or the dependence of ϕ on the whole sample path of S_t, as it happens when dealing with the so-called *Asian, look-back* options; new models for the *underlying,* that arise from the different dynamics among the asset classes, e.g., considering interest rates *versus* equity *versus* forex, or from the need of a better calibration of the empirical data, e.g., volatility surface *versus* flat volatility.

As a benchmark model for an interest rate underlying, the *Vasicek model* (Vasicek, 1977) and the *Hull–White model* (Hull and White, 1990) usually replace the Black–Scholes one; an increase in the number of *risk sources,* e.g., by taking into account the stochastic behavior of volatility, as it happens in the *Heston model* (Heston, 1993). For a complete review of models, resp. of pricing formulas, see the work of Hull (1999), resp. Haug (1983). Nevertheless, let us recall that, if dealing with a whole portfolio of financial instruments, independently from their features, the Mark-to-Market dynamics can be adequately fitted by a log-normal process because of the compensation or aggregation effect among several single position returns.

This is a common practice in the asset management sector, often referred to as the *normal portfolio* approach (see, e.g., Saita, 2007). Moreover, also in the risk management approach, the lognormal Black–Scholes model is quite satisfactory, as pointed out, e.g., by Glasserman (2012), who proposes a particular type of *incremental risk charge* (IRC) model. We recall that, in the real world, one buys or sells a derivative for a given quantity, or *notional*, namely one *takes a position.* Hence, in the following, we will often replace the fair value by its related *mark-to-market* expression (MtM), hence by the fair value equipped with a quantity and a sign.

3.2. Local time and occupation time. Let $\{W_t\}_{t\geq 0}$ be a standard Brownian motion, defined over the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. The local time for the Brownian motion W_t, or equivalently, the *Brownian local time* (BLT), first introduced by Lévy (1965), can be seen as a stochastic process indicating the amount of time spent by the Brownian motion process close to a given *level* $a \in \mathbb{R}$. To quantify such a random time, Lévy (1965) introduced the following random field:

$$L_t(a) = \frac{1}{2\varepsilon} \lim_{\varepsilon \to 0} \mu\big\{0 \leq s \leq t, : |W_s - a| \leq \varepsilon\big\},$$

where $t \in [0,T]$, $a \in \mathbb{R}$ and μ is the Lebesgue measure. $L_t(a)$ was defined as the *mesure de voisinage*, and Lévy proved its existence, its finiteness and its continuity, (Lévy, 1965). More rigorously, let us recall the following useful definition:

Definition 2. The random field $\{L_t(x,\omega) : (t,x) \in [0,T] \times \mathbb{R}, \omega \in \Omega\}$ is called a *Brownian local time* if the random variable $L_t(x)$ is $\mathfrak{F}$-measurable, the function $(t,x) \longmapsto L_t(x,\omega)$ results to be continuous and

$$\Gamma_t(B,\omega) := \int_0^t \mathbb{1}_B(W_s)\,\mathrm{d}s = \int_B L_t(x,\omega)\,\mathrm{d}x, \quad (12)$$

with $0 \leq t \leq \infty$ and $B \in \mathcal{B}(\mathbb{R})$.

Let us also recall that the quantity on the left-hand side of (12) is known as the *occupation time* of the Brownian motion up to time t. A crucial theoretical point consists in establishing the BLT existence. This is ensured by Karatzas and Shreve (1991, Thm 6.1.1, Ch. 3). The Brownian local time satisfies several useful properties. For the sake of convenience, we report only the ones that we are going to use for our computational purposes, while we refer the interested reader to Karatzas and Shreve (1991, Section 3.6) for a more comprehensive treatment of the subject as well as for the proofs of the results which we will exploit in what follows.

Proposition 1. *For every Borel-measurable function* $f : \mathbb{R} \to [0,T]$, *we have*

$$\begin{aligned}\int_0^t f(W_s(\omega))\,\mathrm{d}s \\ = \int_{\mathbb{R}} f(x) L_t(x,\omega)\,\mathrm{d}x, \quad 0 \leq t \leq T \, .\end{aligned} \quad (13)$$

As a consequence of Proposition (1), we have

$$\int_0^t \mathbb{1}_{\mathbb{R}}(W_s)\,\mathrm{d}s = \int_{\mathbb{R}} L_t(x,\omega)\,\mathrm{d}x = t. \quad (14)$$

The following result is known in the literature as the *Tanaka–Meyer decomposition,* (see the work of Karatzas and Shreve (1991) for further details).

Proposition 2. *Let us assume that the BLT exists and let $a \in \mathbb{R}$ be a given number. Then the process $\{L_t(a)\}_{0\leq t\leq T}$ is a nonnegative, continuous, additive functional which satisfies*

$$L_t(a) = (W_t - a)^+ - (z-a)^+ - \int_0^t \mathbb{1}_{(a,+\infty)}(W_s)\,\mathrm{d}W_s, \tag{15}$$

for $0 \leq t \leq T$ and for every $z \in \mathbb{R}$.

Remark 8. Note that the representation given in Proposition 2 can be generalized to a semimartingale.

The Brownian motion spends a random time over any set A. Hence it is important to be able to derive its density, namely, the probability that the BLT stands close to a given *level* a, for a time $\mathrm{d}y$. Such a density is given by

$$g(y;t,a) = \sqrt{\frac{2}{\pi t}} e^{-\frac{(y+|a|)^2}{2t}}, \tag{16}$$

(see Borodin and Salminen, 2002, Eqn. (1.3.4), p. 155).

4. Local time proposal for the CCR

4.1. Application of Brownian local time in finance: Accumulator derivatives. In what follows we focus our attention on a particular type of derivatives, namely the *Accumulator*, which is a path-dependent forward enhancement without a guaranteed worst case. More precisely, an *Accumulator* is characterized by a contract, agreed upon two parties, which provides that the investor purchases/sells a pre-determined quantity of stock at a settled strike price K, on specified observation days $t_1,\dots,t_n$, $t_n \leq T$, T being the expiry of the contract.

Usually, an *Accumulator* is linked to an underlying which is an exchange rate, but we have similar payoffs with different names, *range accrual*, in the broad interest rate derivatives frameworks. An example is given by the *FTSE Income Accumulator,* identified through the ISIN code *XS1000869211,* over the *FTSE 100 Index,* with the plan start date on February 14th, 2014, the plan end date on August 14th, 2020, and the maturity date on August 28th, 2020. The plan is expected to pay every three months, the level depending on how the FTSE 100 Index has performed over the quarter. The maximum income is 6.75% every year, paid if the underlying closes between 5000 and 8000 points on each weekly observation date. Otherwise, the income will proportionally be reduced, according to the time spent out of the range. Although such a kind of derivative product exhibits some benefits, e.g., a noticeable improvement in the exchange rate, the lack of product costs and the existence of several tailor-made features. On the other hand, there are some drawbacks. The latter allowed the accumulator derivatives to earn the nickname of *"I will kill you later"* products.

In order to permit more flexibility and to reduce hedging costs, the accumulator contracts may include one or two *knock-out barriers* in order to restrict the maximum profit and/or the maximum loss by the investor. Basically, if at the end of the i-th observation day, the closing price S_i of the underlying hits the barrier H, for all $i = 1,\dots,n$, then the option stops. We distinguish among *accumulator-out one-sided knock-out, accumulator-in one-sided knock-out, accumulator-out range knock-out, accumulator-in range knock-out,* depending on whether the investor purchases (resp. sells) a one-sided or range knock-out call (resp. put) and sells (resp. purchases) a one-sided or range knock-out put (resp./ call), with the same strike price, fixing dates and expiry date. Hence, the payoff $\mathscr{P}_i$ of an accumulator derivative at the observation day t_i, $i = 1,\dots,n$, is given by

$$\mathscr{P}_i = \begin{cases} 0 & \text{if } \max\limits_{0\leq\tau\leq t_i} S_\tau \geq H, \\ Q(S_{t_i} - K) & \text{if } \max\limits_{0\leq\tau\leq t_i} S_\tau < H,\ S_\tau \geq K, \\ gQ(S_{t_i} - K) & \text{if } \max\limits_{0\leq\tau\leq t_i} S_\tau < H,\ S_\tau < K, \end{cases} \tag{17}$$

where Q is the purchase quantity and g is the gearing ratio, both fixed by contract; see, e.g., the work of Lam *et al.* (2009) for further details. For our purposes, we set $Q = 1$ and $g = 2$, hence implying that the fair value FV is given by

$$FV_i = \sum_{j=1}^{N} \left[C_{t_j} - P_{t_j}\right] \cdot e^{-r(T-t_i)}, \tag{18}$$

where $C_{t_j} := C(S_0, K, T - t_j, \sigma, H)$, resp. $P_{t_j} := P(S_0, K, T - t_j, \sigma, H)$, represents the fair price of a knock-out call option, resp. of knock-out put one. We recall that, by assuming that the underlying evolves according to the Black–Scholes, model, the call price and the put price appearing in (18) have a closed form (see, e.g., Lam *et al.*, 2009).

4.2. Proposal for EE evaluation. In what follows we show how the local time may be used as a handy tool in the evaluation of the counterparty credit risk (CCR) for accumulator derivatives. In the setting described by (10) and (11), it is still possible to determine how long the geometric Brownian motion, remains in the neighborhood of any point a, for any given set. In other words, we could attain the density of local time with respect to a geometric Brownian motion; see, e.g., the work of Borodin and Salminen (2002) for further details. In particular, we have

$$\mathrm{P}(L(t,a) \in \mathrm{d}y) = f(y;t,a,\sigma,\nu,S)$$

$$= \sqrt{\frac{2}{\pi t}}\sigma a \left(\frac{a}{S}\right)^{\nu} e^{-\nu^2\sigma^2\frac{t}{2} - \frac{(\sigma^2 ay + |\log(a/S)|)^2}{2\sigma^2 t}}$$
$$+ |\nu|\sigma^2 a \left(\frac{a}{S}\right)^{\nu} \Big[e^{-|\nu|(\sigma^2 ay + |\log(a/S)|)}$$
$$\times \mathrm{Erfc}\left(\frac{\sigma^2 ay + |\log(a/S)|}{\sigma\sqrt{2t}} - |\nu|\sigma\sqrt{\frac{t}{2}}\right)$$
$$- e^{|\nu|(\sigma^2 ay + |\log(a/S)|)}$$
$$\times \mathrm{Erfc}\left(\frac{\sigma^2 ay + |\log(a/S)|}{\sigma\sqrt{2t}} + |\nu|\sigma\sqrt{\frac{t}{2}}\right)\Big] , \quad (19)$$

where t represents the time up to which the local time is evaluated, a is the underlying, σ is the volatility parameter,

$$\nu := -\frac{1}{2} + \frac{r}{\sigma^2},$$

r being the risk-free rate, S represents the spot price, and $\mathrm{Erfc}(z)$ is the complementary error function, namely

$$\mathrm{Erfc}(z) = 1 - \mathrm{Erf}(z),$$
$$\mathrm{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2}\,\mathrm{d}x.$$

For convenience, from now on we will not consider the presence of a knock-out barrier.

By recalling the expressions of the payoff and the fair value stated in (17) and (18), and supposing a high fixing frequency, we obtain

$$\mathcal{P}^{(\mathrm{LT})} = \sum_{j=1}^{N} [(S_{t_j} - K)^+ - 2(K - S_{t_j})^+]$$
$$\approx \int_0^T [(S_t - K)^+ - 2(K - S_t)^+]\,\mathrm{d}t$$
$$= \int_{\mathbb{R}} L(T,x)[(x-K)^+ - 2(K-x)^+]\,\mathrm{d}x, \quad (20)$$

where the last equality in (20) follows exploiting (13), while $L(t,x)$ is the BLT up to maturity T. As a consequence, we are able to evaluate the corresponding fair value for every observation day t_i, $i = 1,\dots,n$,

$$FV_{t_i}^{(\mathrm{LT})}$$
$$= e^{-r(T-t_i)}\mathbb{E}\Big(\int_{\mathbb{R}} L(T,x)[(x-K)^+ - 2(K-x)^+]\mathrm{d}x\Big)$$
$$= e^{-r(T-t_i)} \int_{\mathbb{R}} \mathbb{E}[L(T,x)]\,\big[(x-K)^+ - 2(K-x)^+\big]\,\mathrm{d}x$$
$$= e^{-r(T-t_i)} \int_{\mathbb{R}}\int_0^{\infty} y f(y;T,x,\sigma,\nu,S) \times \big[(x-K)^+ - 2(K-x)^+\big]\,\mathrm{d}y\,\mathrm{d}x, \quad (21)$$

basically as an application of the Fubini theorem, in the second equality, and by the very definition of the BLT density given in (19).

Hence, as an intermediate first application, we use the above pricing formula for our *Accumulator*, and we compare three different pricing techniques for the *Accumulator* defined by $(C - 2P)$, where C and P are respectively the Call option price and the Put option price, namely: BSD, the straight BS evaluation, i.e., Eqn. (18); BSC, the continuous time version of BSD, described in Section 4.3; LT: the time proposal given by the formula (21). For a more detailed discussion of the aforementioned quantities, i.e., concerning BSD, BSC and LT, see Section 4.3. The results have been reported in Table 1 and they have been obtained setting $S_0 = 1$, with $N = 250$ fixing dates. We can see that the accuracy is very good, with just a small decay when the volatility parameter increases. We are interested in evaluating the EE and EPE introduced in Section 2.2. Hence, for all t_i, $i = 1,\dots,n$, we have

$$\mathrm{EE}_{t_i}^{(\mathrm{LT})} = \mathbb{E}\left(FV_{t_i}^{(\mathrm{LT})}\right) \quad (22)$$
$$\mathrm{EPE}_{t_i}^{(\mathrm{LT})} = \frac{1}{T}\int_0^T \int_{\mathbb{R}} e^{-r(T-t)}\mathbb{E}(L(T,x)) \times [(x-K)^+ - 2(K-x)^+]\,\mathrm{d}x\,\mathrm{d}t. \quad (23)$$

Remark 9. By recalling that the expectation functional $\mathbb{E}$ involves integration, it follows that the EPE requires the evaluation of a triple integral. In consequence, we have two further integration steps with respect to the usual MtM current evaluation of the deal, against the expectation with respect to the market parameters scenarios and the time average, respectively.

Remark 10. We wonder which probability measure is better to use when the expectation functional is evaluated. In other terms, we are interested in choosing the most appropriate distribution at any time t and for all market parameters, which represent the input data for the pricing function. As is well known in the literature, there are two alternatives, namely the risk neutral distribution, or the historical one. Since we mainly focus on computation issues, we believe that the latter is not a relevant point. Anyway, in agreement with the majority of the authors, we follow the convention of adopting the historical distribution. In the Black–Scholes framework, the latter implies that there is a real world drift μ different from the risk-free rate r, and such that $\mu > r$.

4.3. Application and numerical results. To the extent of testing the goodness of our local time proposal to estimate the EE as well as the EPE, we compare the algorithm described in the previous subsection with a benchmark *à la* Black and Scholes (BSD). First of all,

Table 1. Comparison between fair values obtained with the three methods.

r	K	σ	FV BSD	FV BSC	FV LT	$\Delta(LT, BSD)$
0.01	0.9	15%	0.0961	0.0961	0.0961	0.00%
0.01	0.9	25%	0.0783	0.0784	0.0781	– 0.26%
0.01	1	15%	– 0.0323	– 0.0322	– 0.0322	– 0.31%
0.01	1	25%	– 0.0587	– 0.0585	– 0.0576	– 1.87%
0.02	0.9	15%	0.1008	0.1008	0.1008	0,00%
0.02	0.9	25%	0.0837	0.0837	0.0839	0,24%
0.02	1	15%	– 0.0248	– 0.0247	– 0.0247	– 0.40%
0.02	1	25%	– 0.0509	– 0.0508	– 0.0501	– 1.57%

let us fix the number of simulations, indicating them by Nsim. Then, for every simulation,

- we consider $\mathrm{Nday} = 250$ business days, indicated by $t_i,\ i = 1, \dots, \mathrm{Nday}$, for each of which t_i, we simulate the price of the underlying by using the following *discretization procedure:*

$$S_{t_i} = S_{t_{i-1}} e^{\left\{\left(r-\frac{\sigma^2}{2}\right)\Delta t_i + \sigma\sqrt{\Delta t_i}\cdot\mathcal{N}(0,1)\right\}},$$

where

$$\Delta t_i = t_i - t_{i-1} = \frac{1}{\mathrm{Nday}}, \forall i = 1, \dots, \quad \mathrm{Nday}.$$

- then we compute the *Accumulator* price by means of the following formula:

$$\begin{aligned}\mathrm{FV}_{t_i}^{(\mathrm{BSD})} &= \Delta\left(\sum_{j=1}^{i}[(S_j-K)^+ - 2(K-S_j)^+]\right.\\ &\left.+\sum_{k=i+1}^{\mathrm{Nday}}[C_{t_k} - 2\cdot P_{t_k}]\right)\cdot e^{-r(T-t_i)}, \qquad (24)\end{aligned}$$

where

$$\Delta := \frac{T}{\mathrm{Nday}}$$

and

$$C_{t_k} = C(S_0, K, r, \sigma, T-t_k),$$
$$P_{t_k} = P(S_0, K, r, \sigma, T-t_k)$$

are the call and put prices, respectively.

In order to evaluate the counterparty credit risk, we choose 10 time steps, one every 25 business days. Then we determine the expected exposure $\mathrm{EE}^{(\mathrm{BSD})}$ and the expected positive exposure $\mathrm{EPE}^{(\mathrm{BSD})}$ by using (4), and (5). Since an *Accumulator* derivative is characterized by a daily, at least, fixing frequency, we could take into account a continuous version of the derivative fair value. Hence we consider

$$\mathrm{FV}_t^{(\mathrm{BSC})} \cong \int_0^T e^{-r(T-t)}(C_t - 2P_t)\,\mathrm{d}t, \qquad (25)$$

where

$$C_t = C(S_0, K, r, \sigma, T-t),$$
$$P_t = P(S_0, K, r, \sigma, T-t)$$

are the call and put prices computed as before. Equation (25) allows us to consider a continuous version of the benchmark, denoted by BSC. In order to compare the LT and BSC approaches, we carry out a time discretization approximating the BSC by retracing the steps of the BSD algorithm and by considering 10^4 fixing dates, namely 40 observations per day, instead of one. Finally, we are able to appraise the expected exposure $\mathrm{EE}^{(\mathrm{BSC})}$, resp. the expected positive exposure $\mathrm{EPE}^{(\mathrm{BSC})}$, again by exploiting (4), resp., (5). As regards the local time algorithm, we use a numerical integration, and, in order to have such an integration as efficient as possible, we fixed convenient lower and upper bounds.

Numerical results. To show how the local time techniques behave compared with classical approaches, we provide the results reported in Table 2 which contains the EPE values obtained with methods introduced in the previous sections. More precisely, we run all the algorithms for several strike, volatility and risk-free parameters, according to the following choices: spot price $S_0 = 5.7$; strike price: $K = [4.78,\ 3.75,\ 2.98]$; volatility: $\sigma = [0.15,\ 0.2,\ 0.3]$ risk-free rate: $r = [0.01,\ 0.02]$ We have analyzed the aforementioned three methods, whose values are described in columns 2–4, focusing on the changes (Δ) in the EPE values in columns 5–7. Every row is characterized by a triplet $(K_i, \sigma_j, r_h),\ i = 1, \dots, 3,\ j = 1, \dots, 3,\ h = 1, 2$, to specify which values of strike price, volatility and risk-free rate we refer.

Remark 11. Let us underline the meaning of the three Δ comparisons in the right part of Table 2. $\Delta(\mathrm{BSC}, \mathrm{BSD})$ does not take into account our proposal, but it measures the difference between the real world (BSD), where the fixing is discrete over time, and its continuous version, namely the BSC one. $\Delta(\mathrm{BSC}, \mathrm{LT})$ has a double role. On one hand, it measures the rightness of our algorithm implementation, as the two methods are theoretically equivalent. Once we verify that the difference is small,

Table 2. Expected positive exposure of an accumulator derivative.

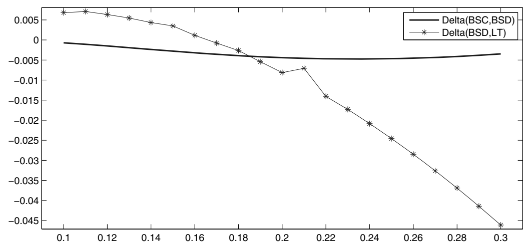
(K, σ, r)	BSD	BSC	LT	$\Delta(BSD, BSC)$	$\Delta(\mathrm{BSC}, \mathrm{LT})$	$\Delta(\mathrm{BSD}, \mathrm{LT})$
$(4.78, 0.15, 0.01)$	0.9303454395	0.9303714275	0.9303781163	– 0.00279%	0.00072%	0.00351%
$(4.78, 0.2, 0.01)$	0.9049015095	0.9049413280	0.9048279190	– 0.00440%	– 0.01253%	– 0.00813%
$(4.78, 0.3, 0.01)$	0.8251642939	0.8251928714	0.8247838254	– 0.00346%	– 0.04957%	– 0.04611%
$(3.75, 0.15, 0.01)$	1.9686102762	1.9686111645	1.9686848833	– 0.00005%	0.00374%	0.00379%
$(3.75, 0.2, 0.01)$	1.9675941521	1.9676004514	1.9676336107	– 0.00032%	0.00169%	0.00201%
$(3.75, 0.3, 0.01)$	1.9547899122	1.9548275248	1.9547735333	– 0.00192%	– 0.00276%	– 0.00084%
$(2.98, 0.15, 0.01)$	2.7348505526	2.7348507540	2.7349168463	– 0.00001%	0.00242%	0.00242%
$(2.98, 0.2, 0.01)$	2.7348375084	2.7348378657	2.7348585689	– 0.00001%	0.00076%	0.00077%
$(2.98, 0.3, 0.01)$	2.7336498018	2.7336570672	2.7336545073	– 0.00027%	– 0.00009%	0.00017%
$(4.78, 0.15, 0.02)$	0.9556220367	0.9556450742	0.9558308683	– 0.00241%	0.01944%	0.02185%
$(4.78, 0.2, 0.02)$	0.9318141129	0.9318494415	0.9318254905	– 0.00379%	– 0.00257%	0.00122%
$(4.78, 0.3, 0.02)$	0.8548444479	0.8548671968	0.8542861489	– 0.00266%	– 0.06797%	– 0.06531%
$(3.75, 0.15, 0.02)$	1.9871484085	1.9871500602	1.9873803563	– 0.00008%	0.01159%	0.01167%
$(3.75, 0.2, 0.02)$	1.9862637277	1.9862700504	1.9864841609	– 0.00032%	0.01078%	0.01110%
$(3.75, 0.3, 0.02)$	1.9743416072	1.9743766907	1.9744396571	– 0.00178%	0.00319%	0.00497%
$(2.98, 0.15, 0.02)$	2.7496039372	2.7496047313	2.7497784936	– 0.00003%	0.00632%	0.00635%
$(2.98, 0.2, 0.02)$	2.7495932160	2.7495941376	2.7497625846	– 0.00003%	0.00613%	0.00616%
$(2.98, 0.3, 0.02)$	2.7485174039	2.7485245347	2.748661624	– 0.00026%	0.00499%	0.00525%

Table 3. Average elapsed time of the three algorithms, measured in seconds.

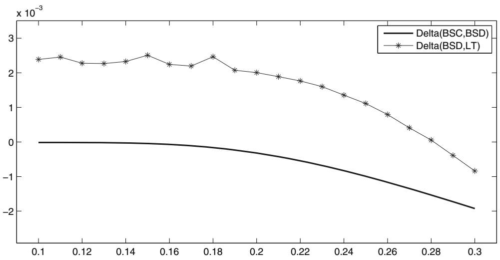
BSC	BSD	LT
18.314768	4.98831	2.12311

with a more practical perspective it allows us to monitor the numerical accuracy of the tools we used to perform the various numerical integration involved in both the techniques. Finally, $\Delta(\mathrm{BSD}, \mathrm{LT})$ considers both the previous effects and measures the global accuracy of our BLT proposal, where we proxy, by continuous time, the real world problem by a new, local time based, technique.

In order to complete the comparison between the different methods proposed, we draw a parallel between the execution times of the individual methods, which is reported in Table 3. In particular, we invite the reader to dwell on the last two columns, for which the computational effort is comparable. We observe that the elapsed time of the local time algorithm is less than the elapsed time of the BSD approach and, on the average, the former is about half the latter. Finally, we exhibit a couple of graphs comparing the errors of the algorithm LT and BSC, with respect to the *exact* case BSD, once the strike price and the free risk rate have been set, while the volatility σ changes. We observe that the relative error in very good for small volatilities. To this extent, we will investigate further the software implementation details. Anyway, referring to the computational time in the Table 3 above, we think that in the usual trade-off (accuracy, time) the LT approach undoubtedly dominates the BSC approximation, and it can compete with the true BSD model.



(a)



(b)

Fig. 1. Comparison between EPE changes for fixed risk-free rates: $\Delta(\mathrm{BSD}, \mathrm{BSC})$ vs. $\Delta(\mathrm{BSD}, \mathrm{LT})$ with $r = 1\%$ and $K = 3.75$ (a), $\Delta(\mathrm{BSD}, \mathrm{BSC})$ vs. $\Delta(\mathrm{BSD}, \mathrm{LT})$ with $r = 1\%$ and $K = 4.78$ (b). The volatility parameter varies between 10% and 30%.

4.4. Some remarks about computational complexity. Once a new methodology or algorithm has been proposed, one would like to make a general analysis of the computation complexity of the new method compared with its more traditional competitors; the some concerns the accuracy and the convergence rate. In the simplest and naive case, one has just one parameter, N, say, e.g., the number of simulations, the number of deals in the portfolio, the number of time steps, etc., and the computational complexity could be stylized by a single "order" such as $\mathcal{O}(N)$, $\mathcal{O}(N^2)$ and so on. Despite this elegant theoretical approach, concrete applications are characterised by extra difficulties. First, the proposal, or the set of competitors, could depend on some different parameters, and N could not be a proper summary of the technique set up. Second, for each atomic algorithmic task, namely for any simulation of a loop of N simulations, the different competitors could contain calculations with very different levels of complexity and elapsed time, t_1 and t_2, say. Hence it may happen that for small or medium values of the parameter N the actual computational time of the two algorithms does not match the asymptotic order ranking, e.g., it may happen that $t_1 \cdot N > t_2 \cdot N^{3/2}$. Third, finally, the observed computational times depend on many implementation details: the numerical integration method, bounded or unbounded integration, the efficiency of the libraries embedded in the exploited programming languages.

Coming back to the above tables of execution times for BSD, BSC and LT, also focusing on the evaluation of the EE and EPE values, loops behave similarly with the cross-method in increasing the number of calculations, and we observe that the BSC involves a time integration of the rather complicated BS formula, while the BSD has a complexity given by $\mathrm{BS} \cdot t_n$, the second term being the number of fixing times, and eventually the LT has a complexity given by time-space integration of a quite simple function which is the payoff itself. Moreover, we optimized the latter by bounding both the infimum and the supremum of the space integral. Therefore, even without an exhaustive comparison, also for various implementations, we can conclude that the LT proposal allows for a good *Accuracy versus Effort* trade-off. We also underline that extensions to other market parameters, clauses and payoffs are needed.

5. Conclusions and further research

We have addressed the issue of the CCR assessment for the so-called accumulator derivatives, within the Black–Scholes, financial framework with one risky asset. Since the corresponding payoff depends on the time spent by a geometric Brownian motion *near* a given value, we have exploited the notion of the BLT which turns to play a crucial role in the derivative pricing step for CCR evaluation. However, it is possible to involve the BLT also in the risk factors simulation step: roughly speaking, for each time bucket t_k, we could employ the BLT to build up the grid $(t_k, S_{t_k,n})$ and the corresponding probabilities, and evaluate the k-th expected exposure EE_k as the sum of weighted probability masses. We have proposed an original approach founded on the possibility of expressing the BLT in terms of its probability density.

The associated implementation with regard to EPE evaluation leads to numerical results that significantly improve those obtained by standard procedures *à la* Black–Scholes. A smaller execution time and a better EE appraisal accuracy make our method a competitive tool, suggesting an extension of the local time approach to more general derivatives, such as barrier options or Asian options.

The next step consists in comparing our results with those derived by Cordoni and Di Persio (2014; 2016). Moreover, we also plan to use the results presented by Takacs (1995), namely a generalization of the well-known Lévy's *arc-sine law*; see also Lévy (1939), who provides the distribution of the *occupation time* given in (12). In fact, we intend to use the related Takacs formula as an alternative expression for the probability density stated in (19) which has been extensively used in this paper.

Finally, we are aware that the one-dimensional case turns out to be unrealistic, though relatively easy to implement, albeit to work in the one-dimensional framework is a very acceptable proxy for derivatives of banks with corporate customers, i.e., small and medium size enterprises; in these cases the i-th customer has a very small number of deals, with the main dependence on a single risk factor, e.g., the EUR interest rate curve. After all, a large number of risk factors entails a very hard estimation of correlations.

To overcome such drawbacks, financial institutions resort to some heuristic and easy-to-extend methods. For example, in the two-dimensional case it is common practice to consider

$$\left\langle \mathrm{d}W_t^{(1)}, \mathrm{d}W_t^{(2)} \right\rangle = 0$$

between the asset classes, e.g., interest rate, forex or equity, and

$$\left\langle \mathrm{d}W_t^{(1)}, \mathrm{d}W_t^{(2)} \right\rangle = \mathrm{d}t$$

within the asset class. Such a procedure could be easily extended to the N-dimensional case, with $N \gg 1$. This is clearly a complicated issue. From a theoretical point of view, the literature provides contributions related to the study of the multidimensional BLT; see, e.g., the work of Brydges *et al*. (2007) and the references therein.

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