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# A SPECTRAL METHOD OF THE ANALYSIS OF LINEAR CONTROL SYSTEMS

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A spectral method of the analysis of linear control systems is considered. Within the framework of this approach the $\sigma$-entropy of input signals and the $\sigma$-entropy norm of systems are introduced. The invariance of the introduced norm makes it possible to get invariant results of $\sigma$-entropy analysis.

**Keywords:** anisotropy-based control theory, anisotropic norm, $\mathcal{H}_2$-norm, $\mathcal{H}_\infty$-norm.

## 1. Introduction

The well-known $\mathcal{H}_2$ and $\mathcal{H}_\infty$ theories of designing optimal controllers minimizing the impact of the external perturbations on the output of a time invariant linear system rely on using the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the matrix-valued transfer functions of closed-loop systems as the performance criteria of the transfer functions. The $\mathcal{H}_2$ theory assumes that the system input receives a random signal which is Gaussian white noise. The $\mathcal{H}_\infty$ theory assumes that the input perturbation is a quadratically summable signal.

Despite the successes of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ theories, the application to practical problems was limited. On the one hand, when the input disturbance is strongly correlated noise, the $\mathcal{H}_2$ controller is unable to satisfy the desired performance. On the other hand, if the input disturbance is slightly correlated or Gaussian white noise, a robust $\mathcal{H}_\infty$ controller leads to unnecessary energy losses. This leads to a new problem: minimize the $\mathcal{H}_2$ norm under an $\mathcal{H}_\infty$ norm bound. It is hoped that the $\mathcal{H}_\infty$ norm bound yields the desired level of robustness while the performance is optimized simultaneously via the minimization of the $\mathcal{H}_2$ norm. To overcome drawbacks of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control theories, the entropy functional was established (Mustafa and Glover, 1990). Similar results for discrete-time systems were obtained by Iglesias *et al*. (1990) as well as Iglesias and Mustafa (1993).

Another approach to overcome drawbacks of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control theories is to use an additional performance index that describes a set of input disturbances with bounded spectral density. This approach was first introduced by Semyonov *et al*. (1994). The basic concepts of the anisotropy-based control theory are the anisotropy of the random vector, the mean anisotropy of the random sequence, and the anisotropic norm of the system (Diamond *et al*., 2001; Vladimirov *et al*., 2005). Briefly, the anisotropy of the random vector is defined as the minimal relative entropy (Kullback–Leibler information divergence) between the probability density functions of the random vector and the Gaussian signal with zero mean and a scalar covariance matrix. Mean anisotropy is defined as the limit of the ratio of the anisotropy of the vector composed of $n$ random vectors, to the number $n$, as $n$ tends to infinity. Mean anisotropy characterizes "spectral color" of the input sequence, or its difference from the Gaussian white noise that has zero "spectral color." The induced $\mathcal{H}_2$ norm of the system with random input signals with limited mean anisotropy is called the anisotropic norm of the time invariant system.

Within the framework of the anisotropy-based control theory the following common control problems were solved: optimal and suboptimal anisotropy-based control with uncertainties (Kurdyukov and Maximov, 2005; Kurdyukov *et al*., 2006), the generalization of the anisotropy-based control theory to descriptor systems (Belov *et al*., 2018; Belov and Andrianova, 2016), the anisotropy-based control theory in the case of a nonzero mathematical expectation of the input signal (Kurdyukov

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*et al.*, 2013; Kustov, 2014), the anisotropy-based theory of filtering and control for time variance control systems (Timin and Kurdyukov, 2015; 2016; Tchaikovsky and Timin, 2017), anisotropy-based analysis in the case of multiplicative disturbances (Kustov *et al.*, 2016; Yurchenkov *et al.*, 2016).

Unfortunately, anisotropy-based control theory found its application only for discrete-time linear systems with stationary stochastic input signals. The main purpose of this paper is to suggest a control theory of continuous-time systems with norm-bounded deterministic or stochastic perturbations on the analogy of anisotropic control theory.

## 2. Basic concepts of anisotropic analysis

Let $\mathbf{L}_2^m$ denote the set of square integrable $\mathbb{R}^m$-valued random vectors distributed absolutely continuously with respect to the $m$-dimensional Lebesgue measure $\mathrm{mes}_m$. For any $w \in \mathbf{L}_2^m$ with the probability density function $f\colon \mathbb{R}^m \to \mathbb{R}_+$, the anisotropy $\mathbf{A}(w)$ is defined by Vladimirov *et al.* (2005) as the minimal informational Kullback-Leibler divergence $\mathbf{D}(f\|p_{m,\lambda})$ with respect to the Gaussian distributions $p_{m,\lambda}$ in $\mathbb{R}^m$ with zero mean and scalar covariance matrices $\lambda I_m$:

$$\begin{aligned}\mathbf{A}(w) &= \min_{\lambda>0} \mathbf{D}(f\|p_{m,\lambda})\\ &= \frac{m}{2}\ln\left(\frac{2\pi e}{m}\mathbf{E}[|w|^2]\right) - \mathbf{h}(w),\end{aligned}$$

where $\mathbf{E}[\cdot]$ is expectation and $\mathbf{h}(w)$ denotes the differential entropy of $w$ with respect to $\mathrm{mes}_m$ (Cover and Tomas, 1991).

Let $\mathbf{G}^m(\Sigma)$ denote the class of $\mathbb{R}^m$-valued Gaussian random vectors with zero mean ($\mathbf{E}[w] = 0$) and a nonsingular covariance matrix $\mathrm{cov}(w) = \mathbf{E}[w\,w^{\mathrm{T}}] = \Sigma$.

**Lemma 1.** (Vladimirov *et al.*, 2005)

*(a) The anisotropy $\mathbf{A}(w)$ is invariant with respect to rotations and homotheties, i.e., $\mathbf{A}(\lambda U w) = \mathbf{A}(w)$ for any $\lambda \in \mathbb{R}\setminus\{0\}$ and any orthogonal matrix $U \in \mathbb{R}^{m\times m}$.*

*(b) For any positive definite matrix $\Sigma \in \mathbb{R}^{m\times m}$*

$$\begin{aligned}\min\left\{\mathbf{A}(w) \,:\, w \in \mathbf{L}_2^m,\ \mathbf{E}[ww^{\mathrm{T}}] = \Sigma\right\}\\ = -\frac{1}{2}\ln\det\frac{m\Sigma}{\mathrm{tr}\,\Sigma};\end{aligned}$$

*furthermore, the minimum is attained only at $w \in \mathbf{G}^m(\Sigma)$.*

*(c) $\mathbf{A}(w) \geq 0$ is satisfied for any $w \in \mathbf{L}_2^m$ and $\mathbf{A}(w) = 0$ if and only if $w \in \mathbf{G}^m(\lambda I_m)$ for some $\lambda > 0$.*

Let $W = \{w_k\}_{-\infty<k<+\infty}$ be a stationary sequence of vectors $w_k \in \mathbf{L}_2^m$ interpreted as a discrete-time random signal. Assemble the elements of $W$ associated with a time interval $[s,t]$ into a random vector:

$$W_{s:t} = \begin{bmatrix} w_s \\ \vdots \\ w_t \end{bmatrix}.$$

The mean anisotropy of the sequence $W$ is defined as the anisotropy production rate per time step:

$$\overline{\mathbf{A}}(W) = \lim_{N\to+\infty}\frac{\mathbf{A}(W_{0:N-1})}{N}. \tag{1}$$

Suppose $W = GV$ is generated from the Gaussian white noise sequence $V$ by a stable shaping filter with the transfer function $G(z) \in \mathcal{H}_2^{m\times m}$. Then the spectral density of $W$ is given by

$$S(\omega) = \widehat{G}(\omega)\widehat{G}^*(\omega), \quad -\pi \leq \omega < \pi, \tag{2}$$

where $\widehat{G}(\omega) = \lim_{r\to 1-} G(re^{i\omega})$ is the boundary value of the transfer function $G(z)$. As shown by Vladimirov *et al.* (2005), the mean anisotropy (1) can be calculated in terms of spectral density (2) and the $\mathcal{H}_2$-norm of the shaping filter $G$:

$$\overline{\mathbf{A}}(W) = -\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\det\frac{mS(\omega)}{\|G\|_2^2}\,\mathrm{d}\omega. \tag{3}$$

The mean anisotropy functional (3) is always nonnegative. It takes a finite value if the shaping filter $G$ has full rank, otherwise $\overline{\mathbf{A}}(G) = +\infty$. The equality $\overline{\mathbf{A}}(G) = 0$ holds true if and only if $G$ is an all-pass system up to a nonzero constant factor. In this case, the spectral density (2) is equal to $S(\omega) = \lambda I_m$ for some $\lambda > 0$, so that $W$ is a Gaussian white noise sequence with zero mean and a scalar covariance matrix.

Let $F \in \mathcal{H}_\infty^{p\times m}$ be a linear discrete time invariant system with an $m$-dimensional input $W$ and a $p$-dimensional output $Z = FW$. Consider the random input sequence $W = GV$ where $V$ is the $m$-dimensional Gaussian white noise sequence. Define

$$\mathcal{G}_a = \left\{G \in \mathcal{H}_2^{m\times m} : \overline{\mathbf{A}}(G) \leq a\right\}$$

as the set of shaping filters $G$ that produce Gaussian random sequences $W$ with mean anisotropy (3) bounded by a given parameter $a \geq 0$.

The anisotropic norm of the system $F$ is defined (Vladimirov *et al.*, 2005) as

$$|\!|\!| F |\!|\!|_a = \sup_{G\in\mathcal{G}_a}\frac{\|FG\|_2}{\|G\|_2}. \tag{4}$$

The anisotropic norm of a given system $F \in \mathcal{H}_\infty^{p\times m}$ is a nondecreasing continuous function of the mean anisotropy level $a$, which satisfies (Vladimirov *et al.*, 2005):

$$\frac{1}{\sqrt{m}}\|F\|_2 = |||F|||_0 \leq |||F|||_a \leq \lim_{a\to+\infty} |||F|||_a = \|F\|_\infty.$$

Therefore, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms are the limiting cases of the anisotropic norm as $a \to 0$ and $a \to +\infty$, respectively.

## 3. Discrete systems with uncorrelated input

**3.1. Stochastic input with bounded $l_2$ norm.** Consider a linear discrete time invariant system:

$$\begin{cases} x_{k+1} = Ax_k + Bw_k, & x_0 = 0, \\ z_k = Cx_k + Dw_k, & k = 0, 1, 2, \dots, \end{cases} \tag{5}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, $A$ is a stable matrix, $x_k \in \mathbb{R}^n$ is the state, $z_k \in \mathbb{R}^p$ is the output vector, $w_k \in \mathbb{R}^m$ is the input stochastic vector. Let the sequence $W = \{w_k\}_{0\leqslant k<+\infty}$ satisfy the conditions

$$\mathbf{E}\,[w_k] = 0, \tag{6}$$

$$\sum_{k=0}^{+\infty} \mathbf{E}\big[|w_k|^2\big] < \infty, \tag{7}$$

$$\begin{aligned} \mathbf{E}\big[w_j^{\mathrm{T}} w_i\big] &= \operatorname{tr}\mathbf{E}\big[w_i w_j^{\mathrm{T}}\big] \\ &= \operatorname{tr} S_{ij} = \begin{cases} \operatorname{tr} S_i & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases} \end{aligned} \tag{8}$$

For the system (5) with the input sequence $W$ that satisfies (7), define the root mean square (RMS) gain $\Theta$:

$$\Theta = \frac{\|z_k\|_2}{\|w_k\|_2} = \frac{\sqrt{\sum_{k=0}^{+\infty}\mathbf{E}\big[|z_k|^2\big]}}{\sqrt{\sum_{k=0}^{+\infty}\mathbf{E}\big[|w_k|^2\big]}}, \tag{9}$$

where $\|\cdot\|_2$ is the $l_2$ norm of a sequence and $|\cdot|$ is the Euclidean norm of a vector.

**Lemma 2.** *The gain $\Theta$ of the system* (5) *with the input sequence $W$ that satisfies* (6)–(8) *is given by*

$$\Theta^2 = \frac{\operatorname{tr}\big[\big(B^{\mathrm{T}}\Gamma B + D^{\mathrm{T}}D\big)\,S\big]}{\operatorname{tr} S},$$

*where $\Gamma$ is the observability Gramian, which is the solution of the Lyapunov equation*

$$A^{\mathrm{T}}\Gamma A - \Gamma + C^{\mathrm{T}}C = 0, \tag{10}$$

*and*

$$S = \sum_{k=0}^{+\infty} S_k.$$

Introduce the property of the input sequence, which is called the $\sigma$-entropy:

$$\mathfrak{S}(S) = -\frac{1}{2}\ln\det\frac{mS}{\operatorname{tr} S}\cdot \tag{11}$$

By means of the $\sigma$-entropy we define the $\sigma$-entropy gain $\Theta_s$ of the system (5) with the input sequence (6)–(8) as the maximum of the gain $\Theta$ wherein the $\sigma$-entropy (11) of input sequences does not exceed a given value $s$:

$$\Theta_s^2 = \sup_{\mathfrak{S}(S)\leq s} \Theta^2 = \sup_{\mathfrak{S}(S)\leq s} \frac{\operatorname{tr}\big[\big(B^{\mathrm{T}}\Gamma B + D^{\mathrm{T}}D\big)\,S\big]}{\operatorname{tr} S}\cdot \tag{12}$$

**Theorem 1.** *For any $s \geq 0$ the $\sigma$-entropy gain* (12) *is*

$$\Theta_s^2 = \frac{\sum_{i=1}^{m}\dfrac{\lambda_i}{1-q\lambda_i}}{\sum_{i=1}^{m}\dfrac{1}{1-q\lambda_i}},$$

*where $\lambda_i$ are eigenvalues of the matrix $\Lambda = B^{\mathrm{T}}\Gamma B + D^{\mathrm{T}}D$, $\lambda_{\max}$ is the largest eigenvalue of $\Lambda$ and $q \in \big[0,\, \lambda_{\max}^{-1}\big)$ is the unique solution of the equation*

$$-\frac{1}{2}\ln\det\frac{m(I_m - q\Lambda)^{-1}}{\operatorname{tr}[(I_m - q\Lambda)^{-1}]} = s.$$

The proofs of Lemma 2 and Theorem 1 are given by Boichenko and Kurdyukov (2016; 2017).

**3.2. Stochastic input with bounded power norm.** Consider the system (5) with an input stochastic sequence $W = \{w_k\}_{0\leq k<+\infty}$ which satisfies the conditions

$$\mathbf{E}\,[w_k] = 0, \tag{13}$$

$$\lim_{N\to\infty}\left\{\frac{1}{N}\sum_{k=0}^{N-1}\mathbf{E}\big[|w_k|^2\big]\right\} < \infty, \tag{14}$$

$$\mathbf{E}\big[w_j^{\mathrm{T}} w_i\big] = \operatorname{tr} S_{ij} = \begin{cases} \operatorname{tr} S_i & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases} \tag{15}$$

For the system (5) and the input sequence with the power norm $\|w\|_{\mathcal{P}}$ define the RMS gain

$$\Theta = \frac{\|z\|_{\mathcal{P}}}{\|w\|_{\mathcal{P}}} = \frac{\sqrt{\lim_{N\to\infty}\left\{\frac{1}{N}\sum_{k=0}^{N-1}\mathbf{E}\big[|z_k|^2\big]\right\}}}{\sqrt{\lim_{N\to\infty}\left\{\frac{1}{N}\sum_{k=0}^{N-1}\mathbf{E}\big[|w_k|^2\big]\right\}}}\cdot \tag{16}$$

**Lemma 3.** *The gain $\Theta$ of the system* (5) *with the input sequence* (13)–(15) *is given by*

$$\Theta^2 = \frac{\operatorname{tr}\big[\big(B^{\mathrm{T}}\Gamma B + D^{\mathrm{T}}D\big)\,S\big]}{\operatorname{tr} S},$$

*where $\mathit{\Gamma}$ signifies the observability Gramian* (10) *and*

$$S = \lim_{N\to\infty} \left\{ \frac{1}{N} \sum_{k=0}^{N-1} S_k \right\}. \tag{17}$$

Note that the limit in (17) exists as a consequence of the definitions (14) and (15).

Define the $\sigma$-entropy of the sequence (13)–(15) as

$$\mathfrak{S}(S) = -\frac{1}{2} \ln\det \frac{mS}{\operatorname{tr} S}\cdot \tag{18}$$

Similar to (12), the $\sigma$-entropy gain $\Theta_s$ of the system (5) with the input sequence (13)–(15) is equal to

$$\Theta_s^2 = \sup_{\mathfrak{S}(S)\le s} \Theta^2 = \sup_{\mathfrak{S}(S)\le s} \frac{\operatorname{tr}\left[\left(B^{\mathrm{T}}\mathit{\Gamma} B + D^{\mathrm{T}}D\right) S\right]}{\operatorname{tr} S}\cdot \tag{19}$$

**Theorem 2.** *For any $s \ge 0$ the $\sigma$-entropy gain* (19) *is*

$$\Theta_s^2 = \frac{\displaystyle\sum_{i=1}^{m} \frac{\lambda_i}{1-q\lambda_i}}{\displaystyle\sum_{i=1}^{m} \frac{1}{1-q\lambda_i}},$$

*where $\lambda_i$ are eigenvalues of the matrix $\Lambda = B^{\mathrm{T}}\mathit{\Gamma} B + D^{\mathrm{T}}D$ and $q \in \left[0,\, \lambda_{\max}^{-1}\right)$ is the unique solution of the equation*

$$-\frac{1}{2}\ln\det\frac{m(I_m - q\Lambda)^{-1}}{\operatorname{tr}[(I_m - q\Lambda)^{-1}]} = s.$$

The proofs of Lemma 3 and Theorem 2 are similar the proofs of Lemma 2 and Theorem 1.

### 3.3. Stochastic system with a nonzero initial condition.

Consider a linear discrete time invariant system with a nonzero initial condition:

$$\begin{cases} x_{k+1} = A\,x_k + B\,w_k, & x_0 \ne 0, \\ z_k = C\,x_k + D\,w_k, & k = 0,1,2\dots, \end{cases} \tag{20}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times l}$, $x_k \in \mathbb{R}^n$ is the state and $x_0 \in \mathbf{L}_2^n$ is a stochastic vector of the nonzero initial condition, $z_k \in \mathbb{R}^p$ is the output vector, $w_k \in \mathbb{R}^l$ is an input stochastic vector. Let the vector $x_0$ and the sequence $W = \{w_k\}_{0\le k<+\infty}$ satisfy the conditions

$$\mathbf{E}\,[w_k] = 0, \tag{21}$$

$$\mathbf{E}\left[x_0 w_k^{\mathrm{T}}\right] = 0, \tag{22}$$

$$\sum_{k=0}^{+\infty} \mathbf{E}\left[|w_k|^2\right] < \infty, \tag{23}$$

$$\mathbf{E}\left[w_j^{\mathrm{T}} w_i\right] = \operatorname{tr} S_{ij} = \begin{cases} \operatorname{tr} S_i & \text{for } i = j, \\ 0 & \text{for } i \ne j. \end{cases} \tag{24}$$

For the system (20) with the nonzero initial condition $x_0$ define the generalized RMS gain

$$\Theta = \frac{\sqrt{\displaystyle\sum_{k=0}^{+\infty} \mathbf{E}\left[|z_k|^2\right]}}{\sqrt{\displaystyle\sum_{k=0}^{+\infty} \mathbf{E}\left[|w_k|^2\right] + \mathbf{E}\left[|x_0|^2\right]}}\cdot \tag{25}$$

**Lemma 4.** *The generalized gain $\Theta$ of the system* (20) *with the nonzero initial condition $x_0$ and the input sequence* (21)–(24) *is given by*

$$\Theta^2 = \frac{\operatorname{tr}\left(\Lambda S\right)}{\operatorname{tr} S},$$

*where $\Lambda$ and $S$ are the following square block-diagonal matrices of order $m = n + l$:*

$$\Lambda = \begin{bmatrix} \mathit{\Gamma} & 0 \\ 0 & B^{\mathrm{T}}\mathit{\Gamma} B + D^{\mathrm{T}}D \end{bmatrix},$$

$$S = \begin{bmatrix} \mathbf{E}\left[x_0 x_0^{\mathrm{T}}\right] & 0 \\ 0 & \sum_{k=0}^{+\infty} S_k \end{bmatrix}.$$

Define the $\sigma$-entropy of the sequence $\{w_k\}$ and the nonzero initial condition (21)–(24) as follows:

$$\mathfrak{S}(S) = -\frac{1}{2}\ln\det\frac{mS}{\operatorname{tr} S}\cdot \tag{26}$$

Then the generalized $\sigma$-entropy gain $\Theta_s$ of the system (20) with the nonzero initial condition and the input sequence (21)–(24) may be defined as the supremum over all the positive definite matrices $S$ subject to the constraint $\mathfrak{S}(S) \le s$:

$$\Theta_s^2 = \sup_{\mathfrak{S}(S)\le s} \Theta^2 = \sup_{\mathfrak{S}(S)\le s} \frac{\operatorname{tr}(\Lambda S)}{\operatorname{tr} S}\cdot \tag{27}$$

**Theorem 3.** *For any $s \ge 0$ the generalized $\sigma$-entropy gain* (27) *is equal to*

$$\Theta_s^2 = \frac{\displaystyle\sum_{i=1}^{m} \frac{\lambda_i}{1-q\lambda_i}}{\displaystyle\sum_{i=1}^{m} \frac{1}{1-q\lambda_i}},$$

*where $\lambda_i$ are the eigenvalues of the matrix $\Lambda$ and $q \in \left[0,\, \lambda_{\max}^{-1}\right)$ is the unique solution of the equation:*

$$-\frac{1}{2}\ln\det\frac{m(I_m - q\Lambda)^{-1}}{\operatorname{tr}[(I_m - q\Lambda)^{-1}]} = s.$$

The proofs of Lemma 4 and Theorem 3 are given by Boichenko (2017) as well as Boichenko and Belov (2017).

## 4. Continuous systems

**4.1. Stochastic signal with bounded $\mathcal{L}_2$ norm.** Consider a linear time invariant system with the zero initial condition (Kwakernaak and Sivan, 1972; Zhou *et al.*, 1996):

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,w(t), & x(0) = 0, \\ z(t) = C\,x(t) + D\,w(t), \end{cases} \tag{28}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, $A$ is a stable matrix, $x(t) \in \mathbb{R}^n$ is a system state, $z(t) \in \mathbb{R}^p$ is an output signal, $w(t) \in \mathbb{R}^m$ is an input stochastic signal. Let $w(t)$ satisfy the condition

$$\|w(t)\|_2^2 = \int_{-\infty}^{+\infty} \mathbf{E}\big[|w(t)|^2\big]\,\mathrm{d}t < \infty, \tag{29}$$

where $\|\cdot\|_2$ is the $\mathcal{L}_2$ norm of a stochastic signal.

For the system (28) with the input signal (29), define the gain $\Theta$ as the ratio of the $\mathcal{L}_2$ norm of the system output $z(t)$ to the $\mathcal{L}_2$ norm of the input signal $w(t)$:

$$\Theta = \frac{\|z(t)\|_2}{\|w(t)\|_2} = \frac{\sqrt{\displaystyle\int_{-\infty}^{+\infty} \mathbf{E}\big[|z(t)|^2\big]\,\mathrm{d}t}}{\sqrt{\displaystyle\int_{-\infty}^{+\infty} \mathbf{E}\big[|w(t)|^2\big]\,\mathrm{d}t}}\cdot \tag{30}$$

Given a signal $w(t)$, define its autocorrelation matrix $K(\tau)$ as

$$K(\tau) = \int_{-\infty}^{+\infty} \mathbf{E}\big[w(t+\tau)w^{\mathrm{T}}(t)\big]\,\mathrm{d}t.$$

For the purpose of this paper, we further assume that the Fourier transform of the signal's autocorrelation matrix function exists. This Fourier transform is called the spectral density of $w(t)$ and is defined as

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} K(\tau)\,\mathrm{e}^{-i\omega\tau}\,\mathrm{d}\tau.$$

The autocorrelation matrix $K(\tau)$ can be obtained from $S(\omega)$ by the inverse Fourier transform,

$$K(\tau) = \int_{-\infty}^{+\infty} S(\omega)\,\mathrm{e}^{i\omega\tau}\,\mathrm{d}\omega.$$

Then the $\mathcal{L}_2$ norm of a signal can be computed from its spectral density matrix

$$\|w(t)\|_2^2 = \operatorname{tr} K(0) = \int_{-\infty}^{+\infty} \operatorname{tr} S(\omega)\,\mathrm{d}\omega.$$

Similarly, the $\mathcal{L}_2$ norm of an output signal $z(t)$ is

$$\|z(t)\|_2^2 = \int_{-\infty}^{+\infty} \operatorname{tr} S_z(\omega)\,\mathrm{d}\omega,$$

where $S_z(\omega)$ is the spectral density matrix of $z(t)$ and for the system (28) with the transfer matrix $G(s) = C(sI - A)^{-1}B + D$ this spectral density matrix is (Zhou *et al.*, 1994)

$$S_z(\omega) = G(i\omega)S(\omega)G^*(i\omega).$$

Then the gain (30) can be written as

$$\Theta^2 = \frac{\displaystyle\int_{-\infty}^{+\infty} \operatorname{tr}\big[\Lambda(\omega)S(\omega)\big]\,\mathrm{d}\omega}{\displaystyle\int_{-\infty}^{+\infty} \operatorname{tr} S(\omega)\,\mathrm{d}\omega}, \tag{31}$$

where $\Lambda(\omega) = G^*(i\omega)G(i\omega)$.

Define the $\sigma$-entropy of the input signal $w(t)$:

$$\mathfrak{S}(S) = -\frac{1}{2}\int_{-\infty}^{+\infty} \varphi(\omega_0,\omega)\ln\det\frac{S(\omega)}{\displaystyle\int_{-\infty}^{+\infty}\operatorname{tr} S(\omega)\,\mathrm{d}\omega}\,\mathrm{d}\omega, \tag{32}$$

where $\varphi(\omega_0,\omega)$ is the function which yields the convergence of the integral

$$\int_{-\infty}^{+\infty} \ln\det S(\omega)\,\mathrm{d}\omega. \tag{33}$$

For example, it could be

$$\varphi(\omega_0,\omega) = \frac{\omega_0^2}{\omega_0^2+\omega^2}$$

or in the form of the Fermi–Dirac distribution

$$\varphi(\omega_0,\omega) = \frac{1}{\exp\dfrac{\omega^2-\omega_0^2}{(\triangle\omega)^2}+1}\cdot$$

As a matter of fact, the condition

$$\int_{-\infty}^{+\infty} \operatorname{tr} S(\omega)\,\mathrm{d}\omega < \infty \tag{34}$$

prevents the convergence of the integral (33). Indeed, suppose that $S(\omega)$ is a rational function. Then, in accordance with (34), the asymptotics of $S(\omega)$ is equal to

$$S(\omega) \sim \frac{1}{\omega^n}S_\infty, \quad \omega\to\infty,$$

where $S_\infty$ is a positive definite matrix and $n \geq 2$. Consequently, the integral (33) diverges asymptotically.

For the system (28), define the $\sigma$-entropy norm $|||F|||_s$ as the maximum of the gain (31) wherein the $\sigma$-entropy (32) of input signals does not exceed a given value $s$:

$$|||F|||_s^2 = \sup_{\mathfrak{S}(S)\leq s} \Theta^2 = \sup_{\mathfrak{S}(S)\leq s} \frac{\int_{-\infty}^{+\infty} \operatorname{tr}\big[\Lambda(\omega)S(\omega)\big]\,\mathrm{d}\omega}{\int_{-\infty}^{+\infty} \operatorname{tr} S(\omega)\,\mathrm{d}\omega}. \quad (35)$$

**Theorem 4.** *For any $s \geq 0$ the $\sigma$-entropy norm $|||F|||_s$ of the system* (28) *with the input signal* (29) *is*

$$|||F|||_s^2 = \int_{-\infty}^{+\infty} \operatorname{tr} \frac{\varphi(\omega_0,\omega)\,\Lambda(\omega)\big[I - q\Lambda(\omega)\big]^{-1}}{\int_{-\infty}^{+\infty} \varphi(\omega_0,\omega)\operatorname{tr}\big[I - q\Lambda(\omega)\big]^{-1}\,\mathrm{d}\omega}\,\mathrm{d}\omega,$$

*where $q \in \big[0, \max_\omega \lambda_{\max}^{-1}(\omega)\big)$ is the unique solution of the equation*

$$-\frac{1}{2}\int_{-\infty}^{+\infty} \varphi(\omega_0,\omega) \ln\det \frac{\varphi(\omega_0,\omega)\big[I-q\Lambda(\omega)\big]^{-1}}{\int_{-\infty}^{+\infty}\varphi(\omega_0,\omega)\operatorname{tr}\big[I-q\Lambda(\omega)\big]^{-1}\,\mathrm{d}\omega}\,\mathrm{d}\omega = s. \quad (36)$$

*Proof.* Equation (35) may be rewritten in the equivalent form

$$|||F|||_s^2 = \sup_{S(\omega)} \left\{ \int_{-\infty}^{+\infty} \operatorname{tr}[\Lambda(\omega)S(\omega)]\,\mathrm{d}\omega \;: -\frac{1}{2}\int_{-\infty}^{+\infty} \varphi(\omega_0,\omega)\ln\det S(\omega)\,\mathrm{d}\omega \leq s, \quad (37) \quad \int_{-\infty}^{+\infty} \operatorname{tr} S(\omega)\,\mathrm{d}\omega = 1 \right\}.$$

The solution of the constrained optimization problem (37) will be obtained by using the method of Lagrange multipliers. The Lagrange function in this case is equal to

$$\begin{aligned} L[S] = & \int_{-\infty}^{+\infty} \operatorname{tr}[\Lambda(\omega)S(\omega)]\,\mathrm{d}\omega \\ & + \lambda_1 \left\{ s + \frac{1}{2}\int_{-\infty}^{+\infty} \varphi(\omega_0,\omega)\ln\det S(\omega)\,\mathrm{d}\omega \right\} \\ & + \lambda_2 \left\{ 1 - \int_{-\infty}^{+\infty} \operatorname{tr} S(\omega)\,\mathrm{d}\omega \right\}, \end{aligned}$$

where $\lambda_1, \lambda_2$ are the Lagrange multipliers. The necessary optimality condition (Poznyak, 2008; Rockafellar, 1970; Bertsekas, 2003; Boyd and Vandenberghe, 2004) requires that the variation $\delta L[S]$ be zero,

$$\begin{aligned} \delta L[S] = & \int_{-\infty}^{+\infty} \operatorname{tr}\big[\Lambda(\omega)\,\delta S(\omega)\big]\,\mathrm{d}\omega \\ & + \frac{\lambda_1}{2}\int_{-\infty}^{+\infty} \varphi(\omega_0,\omega)\operatorname{tr}\big[S(\omega)^{-1}\,\delta S(\omega)\big]\,\mathrm{d}\omega \\ & - \lambda_2 \int_{-\infty}^{+\infty} \operatorname{tr}\big[\delta S(\omega)\big]\,\mathrm{d}\omega = 0. \end{aligned}$$

Therefore, the spectral density $S(\omega)$ on which the Lagrangian attains the maximum is

$$S(\omega) = p\,\varphi(\omega_0,\omega)\big(I - q\Lambda(\omega)\big)^{-1}, \quad (38)$$

where parameters

$$p = \frac{\lambda_1}{2\lambda_2}, \quad q = \frac{1}{\lambda_2}$$

are determined from

$$\int_{-\infty}^{+\infty} \operatorname{tr} S(\omega)\,\mathrm{d}\omega = 1, \quad (39)$$

$$-\frac{1}{2}\int_{-\infty}^{+\infty} \varphi(\omega_0,\omega)\ln\det S(\omega)\,\mathrm{d}\omega = s. \quad (40)$$

From (38) it follows that the matrices $\Lambda(\omega)$ and $S(\omega)$ commute, i.e.,

$$\Lambda(\omega)S(\omega) - S(\omega)\Lambda(\omega) = 0.$$

In addition, they are Hermitian and positive definite:

$$\Lambda^*(\omega) = \Lambda(\omega) > 0,$$
$$S^*(\omega) = S(\omega) > 0.$$

Therefore, there is a unitary matrix $\big(U(\omega)\big)^* = \big(U(\omega)\big)^{-1}$ that simultaneously transforms these matrices into diagonal forms (Gantmacher, 2000) and hence the eigenvalues $s_i(\omega)$ of the matrix $S(\omega)$ are equal to

$$s_i(\omega) = \varphi(\omega_0,\omega)\frac{p}{1-q\lambda_i(\omega)},$$

where $\lambda_i(\omega)$ are positive eigenvalues of the matrix $\Lambda(\omega)$.

In order for the eigenvalues $s_i(\omega)$ to be positive, it is necessary that the parameter $q$ should be limited:

$$0 \le q < \max_{\omega} \lambda_{\max}^{-1}(\omega),$$

here $\lambda_{\max}(\omega)$ is the largest eigenvalue of $\Lambda(\omega)$.

From (38)–(39) it follows that

$$p = \frac{1}{\displaystyle\int_{-\infty}^{+\infty}\varphi(\omega_0,\omega)\,\mathrm{tr}\big[I-q\Lambda(\omega)\big]^{-1}\,\mathrm{d}\omega}$$

and hence the spectral density $S(q,\omega)$ on which the gain (37) attains the maximum is equal to

$$S(q,\omega) = \frac{\varphi(\omega_0,\omega)\big[I-q\Lambda(\omega)\big]^{-1}}{\displaystyle\int_{-\infty}^{+\infty}\varphi(\omega_0,\omega)\,\mathrm{tr}\big[I-q\Lambda(\omega)\big]^{-1}d\omega}\cdot \tag{41}$$

Since the spectral density $S(q,\omega)$ is proportional to the resolvent $R(\mu) = (\Lambda - \mu I)^{-1}$ of the operator $\Lambda$,

$$S(q,\omega) \sim \big[\Lambda(\omega) - \frac{1}{q}\, I\big]^{-1},$$

$S(q,\omega)$ is an analytic function on the half-open interval $q \in \big[0, \max_\omega \lambda_{\max}^{-1}(\omega)\big)$ and the variable $q$ parameterizes the spectral density set.

From (32) and (41) it follows that the $\sigma$-entropy $\mathfrak{S}(q)$ of the input signal, at which the gain (37) attains the maximum, is

$$\mathfrak{S}(q) = -\frac{1}{2}\int_{-\infty}^{+\infty}\varphi(\omega_0,\omega)\ln\det\frac{\varphi(\omega_0,\omega)\big[I-q\Lambda(\omega)\big]^{-1}}{\displaystyle\int_{-\infty}^{+\infty}\varphi(\omega_0,\omega)\,\mathrm{tr}\big[I-q\Lambda(\omega)\big]^{-1}\,\mathrm{d}\omega}\,\mathrm{d}\omega.$$

Using (41), we define the function

$$\mathcal{F}^2(q) = \int_{-\infty}^{+\infty}\mathrm{tr}\,\frac{\varphi(\omega_0,\omega)\,\Lambda(\omega)\big[I-q\Lambda(\omega)\big]^{-1}}{\displaystyle\int_{-\infty}^{+\infty}\varphi(\omega_0,\omega)\,\mathrm{tr}\big[I-q\Lambda(\omega)\big]^{-1}\,\mathrm{d}\omega}\,\mathrm{d}\omega.$$

The functions $\mathfrak{S}(q)$ and $\mathcal{F}(q)$ are analytic and strictly increasing in $q$. This allows the $\sigma$-entropy norm (37) to be calculated as $|\!|\!|F|\!|\!|_s^2 = \mathcal{F}^2(\mathfrak{S}^{-1}(s))$, where $\mathfrak{S}^{-1}$ is the functional inverse of $\mathfrak{S}$:

$$|\!|\!|F|\!|\!|_s^2 = \int_{-\infty}^{+\infty}\mathrm{tr}\,\frac{\varphi(\omega_0,\omega)\,\Lambda(\omega)\big[I-q\Lambda(\omega)\big]^{-1}}{\displaystyle\int_{-\infty}^{+\infty}\varphi(\omega_0,\omega)\,\mathrm{tr}\big[I-q\Lambda(\omega)\big]^{-1}\,\mathrm{d}\omega}\,\mathrm{d}\omega$$

and $q \in \big[0, \max_\omega \lambda_{\max}^{-1}(\omega)\big)$ is the unique solution of the equation

$$-\frac{1}{2}\int_{-\infty}^{+\infty}\varphi(\omega_0,\omega)\ln\det\frac{\varphi(\omega_0,\omega)\big[I-q\Lambda(\omega)\big]^{-1}}{\displaystyle\int_{-\infty}^{+\infty}\varphi(\omega_0,\omega)\mathrm{tr}\big[I-q\Lambda(\omega)\big]^{-1}\,\mathrm{d}\omega}\,\mathrm{d}\omega = s.$$

■

**4.2. Stochastic signal with bounded power norm.** Consider the system (28) with the input stochastic signal $w(t)$ which satisfies the condition:

$$\|w(t)\|_{\mathcal{P}} = \sqrt{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathbf{E}\Big[|w(t)|^2\Big]\,\mathrm{d}t} < \infty, \tag{42}$$

where $\|\cdot\|_{\mathcal{P}}$ is the power norm of a stochastic signal.

For the system (28) with the input signal (42), define the gain $\Theta$ as the ratio of the power norm of the system output $z(t)$ to the power norm of the input signal $w(t)$:

$$\Theta = \frac{\|z(t)\|_{\mathcal{P}}}{\|w(t)\|_{\mathcal{P}}} = \frac{\sqrt{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathbf{E}\Big[|z(t)|^2\Big]\,\mathrm{d}t}}{\sqrt{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathbf{E}\Big[|w(t)|^2\Big]\,\mathrm{d}t}}. \tag{43}$$

Define the autocorrelation matrix

$$K(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathbf{E}\left[w(t+\tau)w^{\mathrm{T}}(t)\right]\,\mathrm{d}t.$$

Then the spectral density matrix is

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}K(\tau)\,\mathrm{e}^{-i\omega\tau}\,\mathrm{d}\tau,$$

and the spectral density matrix of $z(t)$ is (Zhou *et al.*, 1994)

$$S_z(\omega) = G(i\omega)S(\omega)G^*(i\omega).$$

Thus the gain (43) can be written as

$$\Theta^2 = \frac{\int_{-\infty}^{+\infty} \operatorname{tr}\left[\Lambda(\omega)S(\omega)\right] \mathrm{d}\omega}{\int_{-\infty}^{+\infty} \operatorname{tr} S(\omega)\, \mathrm{d}\omega}. \tag{44}$$

In much the same way as in Section 4.1, define the $\sigma$-entropy of the input signal $w(t)$:

$$\mathfrak{S}(S) = -\frac{1}{2}\int_{-\infty}^{+\infty} \varphi(\omega_0, \omega) \ln\det \frac{S(\omega)}{\int_{-\infty}^{+\infty} \operatorname{tr} S(\omega)\, \mathrm{d}\omega}\, \mathrm{d}\omega \tag{45}$$

and the $\sigma$-entropy norm $|||F|||_s$ of the system (28) as the maximum of the gain (44) where the $\sigma$-entropy (45) of input signals does not exceed a given value $s$:

$$|||F|||_s^2 = \sup_{\mathfrak{S}(S)\leq s} \frac{\int_{-\infty}^{+\infty} \operatorname{tr}\left[\Lambda(\omega)S(\omega)\right] \mathrm{d}\omega}{\int_{-\infty}^{+\infty} \operatorname{tr} S(\omega)\, \mathrm{d}\omega}. \tag{46}$$

**Theorem 5.** *For any $s \geq 0$ the $\sigma$-entropy norm $|||F|||_s$ of the system* (28) *with the input signal* (42) *is*

$$|||F|||_s^2 = \int_{-\infty}^{+\infty} \operatorname{tr} \frac{\varphi(\omega_0, \omega)\, \Lambda(\omega)\left[I - q\Lambda(\omega)\right]^{-1}}{\int_{-\infty}^{+\infty} \varphi(\omega_0, \omega) \operatorname{tr}\left[I - q\Lambda(\omega)\right]^{-1} \mathrm{d}\omega}\, \mathrm{d}\omega,$$

*where $q \in \left[0, \max_\omega \lambda_{\max}^{-1}(\omega)\right)$ is the unique solution of the equation*

$$-\frac{1}{2}\int_{-\infty}^{+\infty} \varphi(\omega_0, \omega) \ln\det \frac{\varphi(\omega_0, \omega)\left[I - q\Lambda(\omega)\right]^{-1}}{\int_{-\infty}^{+\infty} \varphi(\omega_0, \omega) \operatorname{tr}\left[I - q\Lambda(\omega)\right]^{-1} \mathrm{d}\omega}\, \mathrm{d}\omega = s.$$

The proof of Theorem 5 is similar to that of Theorem 4.

**4.3. Deterministic signal with bounded $\mathcal{L}_2$ norm.** Consider the system (28) with a deterministic input signal $w(t)$ which satisfies the condition

$$\left\|w(t)\right\|_2^2 = \int_{-\infty}^{+\infty} \left|w(t)\right|^2 \mathrm{d}t < \infty, \tag{47}$$

where $\|\cdot\|_2$ is the $\mathcal{L}_2$ norm of a deterministic signal.

For the system (28) with the input signal (47), define the gain $\Theta$ as the ratio of the $\mathcal{L}_2$ norm of the system output $z(t)$ to the $\mathcal{L}_2$ norm of the input signal $w(t)$:

$$\Theta = \frac{\left\|z(t)\right\|_2}{\left\|w(t)\right\|_2} = \frac{\sqrt{\int_{-\infty}^{+\infty} \left|z(t)\right|^2 \mathrm{d}t}}{\sqrt{\int_{-\infty}^{+\infty} \left|w(t)\right|^2 \mathrm{d}t}}. \tag{48}$$

Define the autocorrelation matrix

$$K(\tau) = \int_{-\infty}^{+\infty} w(t+\tau)w^{\mathrm{T}}(t)\, \mathrm{d}t.$$

Then the spectral density matrix is

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} K(\tau)\, \mathrm{e}^{-i\omega\tau}\, \mathrm{d}\tau, \tag{49}$$

and the spectral density matrix of $z(t)$ (Zhou *et al.*, 1994) has the form

$$S_z(\omega) = G(i\omega)S(\omega)G^*(i\omega).$$

Thereafter the gain (48) can be written as

$$\Theta^2 = \frac{\int_{-\infty}^{+\infty} \operatorname{tr}\left[\Lambda(\omega)S(\omega)\right] \mathrm{d}\omega}{\int_{-\infty}^{+\infty} \operatorname{tr} S(\omega)\, \mathrm{d}\omega}. \tag{50}$$

It can be shown that the matrix (49) is equal to

$$S(\omega) = 2\pi\, w(\omega)\, w^*(\omega)$$

where

$$w(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} w(t)\mathrm{e}^{-i\omega t}\, \mathrm{d}t.$$

The rank of matrix $S(\omega)$ is equal to 1 and the definition of the $\sigma$-entropy of a deterministic signal in the form (32) is incorrect because formally $\ln\det S(\omega) = -\infty$ for any input signal with $m > 1$. But the matrix $S(\omega)$ is Hermitian and hence it can be diagonalized by a unitary matrix $U(\omega)$:

$$U(\omega)\, S(\omega)\, U^*(\omega) = \begin{bmatrix} s(\omega) & 0 \\ 0 & 0_{m-1} \end{bmatrix},$$

where $0_{m-1}$ is the null matrix of size $(m-1)\times(m-1)$. Next add a small value which is proportional to $\varepsilon(\omega) > 0$:

$$\varepsilon(\omega)\, I + U(\omega)\, S(\omega)\, U^*(\omega) = \begin{bmatrix} \varepsilon(\omega)+s(\omega) & 0 \\ 0 & \varepsilon(\omega) I_{m-1} \end{bmatrix}.$$

This matrix is nonsingular and since

$$\ln\det[\varepsilon(\omega)\, I + S(\omega)] = \ln[\varepsilon(\omega) + s(\omega)] + (m-1)\ln\varepsilon(\omega),$$

we could define the $\sigma$-entropy as

$$\mathfrak{S}(S) = -\frac{1}{2}\int_{-\infty}^{+\infty} \varphi(\omega_0,\omega)\ln\det \frac{\varepsilon(\omega) I + S(\omega)}{\int_{-\infty}^{+\infty} \operatorname{tr} S(\omega)\,\mathrm{d}\omega}\,\mathrm{d}\omega, \tag{51}$$

where $\varepsilon(\omega)$ satisfies the condition

$$\int_{-\infty}^{+\infty} \varphi(\omega_0,\omega)\,\bigl|\ln\varepsilon(\omega)\bigr|\,\mathrm{d}\omega < \infty. \tag{52}$$

For the system (28) and the input signal (47), define the $\sigma$-entropy norm $|\!|\!|F|\!|\!|_s$ as the maximum of the gain (50) where the $\sigma$-entropy (51) does not exceed a given value $s$:

$$|\!|\!|F|\!|\!|_s^2 = \sup_{\substack{\mathfrak{S}(S)\le s\\ \varepsilon(\omega)\to 0}} \frac{\int_{-\infty}^{+\infty} \operatorname{tr}\bigl[\Lambda(\omega) S(\omega)\bigr]\,\mathrm{d}\omega}{\int_{-\infty}^{+\infty} \operatorname{tr} S(\omega)\,\mathrm{d}\omega}. \tag{53}$$

**Theorem 6.** *For any $s \ge 0$ the $\sigma$-entropy norm* (53) *is*

$$|\!|\!|F|\!|\!|_s^2 = \int_{-\infty}^{+\infty} \operatorname{tr} \frac{\varphi(\omega_0,\omega)\,\Lambda(\omega)\bigl[I - q\Lambda(\omega)\bigr]^{-1}}{\int_{-\infty}^{+\infty} \varphi(\omega_0,\omega)\operatorname{tr}\bigl[I - q\Lambda(\omega)\bigr]^{-1}\mathrm{d}\omega}\,\mathrm{d}\omega,$$

*where $q$ is the unique solution of the equation*

$$-\frac{1}{2}\int_{-\infty}^{+\infty} \varphi(\omega_0,\omega)\ln\det \frac{\varphi(\omega_0,\omega)\bigl[I - q\Lambda(\omega)\bigr]^{-1}}{\int_{-\infty}^{+\infty}\varphi(\omega_0,\omega)\operatorname{tr}\bigl[I - q\Lambda(\omega)\bigr]^{-1}\mathrm{d}\omega}\,\mathrm{d}\omega = s. \tag{54}$$

*Proof.* Equation (53) may be rewritten in the form

$$|\!|\!|F|\!|\!|_s^2 = \sup_{S(\omega)} \left\{ \int_{-\infty}^{+\infty} \operatorname{tr}\bigl[\Lambda(\omega) S(\omega)\bigr]\,\mathrm{d}\omega \;: -\frac{1}{2}\int_{-\infty}^{+\infty} \varphi(\omega_0,\omega)\ln\det\bigl[\varepsilon(\omega) I + S(\omega)\bigr]\,\mathrm{d}\omega \le s, \int_{-\infty}^{+\infty} \operatorname{tr} S(\omega)\,\mathrm{d}\omega = 1 \right\}.$$

The Lagrange function in this case is

$$L[S] = \int_{-\infty}^{+\infty} \operatorname{tr}\bigl[\Lambda(\omega) S(\omega)\bigr]\,\mathrm{d}\omega + \lambda_1\left\{ s + \frac{1}{2}\int_{-\infty}^{+\infty} \varphi(\omega_0,\omega)\ln\det\bigl[\varepsilon(\omega) I + S(\omega)\bigr]\,\mathrm{d}\omega \right\} + \lambda_2\left\{ 1 - \int_{-\infty}^{+\infty} \operatorname{tr} S(\omega)\,\mathrm{d}\omega \right\},$$

On the analogy of the proof of Theorem 4, the spectral density $S(\omega)$ at which the Lagrangian attains the maximum is

$$S(\omega) = p\,\varphi(\omega_0,\omega)\bigl(I - q\Lambda(\omega)\bigr)^{-1} - \varepsilon(\omega) I.$$

Now $\varepsilon(\omega) \to 0$ and according to the proof of Theorem 4

$$|\!|\!|F|\!|\!|_s^2 = \int_{-\infty}^{+\infty} \operatorname{tr} \frac{\varphi(\omega_0,\omega)\,\Lambda(\omega)\bigl[I - q\Lambda(\omega)\bigr]^{-1}}{\int_{-\infty}^{+\infty} \varphi(\omega_0,\omega)\operatorname{tr}\bigl[I - q\Lambda(\omega)\bigr]^{-1}\mathrm{d}\omega}\,\mathrm{d}\omega,$$

where $q$ is the unique solution of the equation:

$$-\frac{1}{2}\int_{-\infty}^{+\infty} \varphi(\omega_0,\omega)\ln\det \frac{\varphi(\omega_0,\omega)\bigl[I - q\Lambda(\omega)\bigr]^{-1}}{\int_{-\infty}^{+\infty}\varphi(\omega_0,\omega)\operatorname{tr}\bigl[I - q\Lambda(\omega)\bigr]^{-1}\mathrm{d}\omega}\,\mathrm{d}\omega = s.$$

■

**4.4. Deterministic signal with bounded power norm.** Consider the system (28) with the input deterministic signal $w(t)$ which satisfies the condition:

$$\bigl\|w(t)\bigr\|_{\mathcal{P}} = \sqrt{\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \bigl|w(t)\bigr|^2\,\mathrm{d}t} < \infty. \tag{55}$$

where $\|\cdot\|_{\mathcal{P}}$ is the power norm of a deterministic signal.

For the system (28) with the input signal(55), define the gain $\Theta$ as the ratio of the power norm of the system output $z(t)$ to the power norm of the input signal $w(t)$:

$$\Theta = \frac{\|z(t)\|_{\mathcal{P}}}{\|w(t)\|_{\mathcal{P}}} = \frac{\sqrt{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |z(t)|^2 \,\mathrm{d}t}}{\sqrt{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |w(t)|^2 \,\mathrm{d}t}}. \tag{56}$$

Define the autocorrelation matrix

$$K(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} w(t+\tau) w^{\mathrm{T}}(t) \,\mathrm{d}t.$$

Then the spectral density matrix is

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} K(\tau)\, \mathrm{e}^{-i\omega\tau} \,\mathrm{d}\tau$$

and the spectral density matrix of $z(t)$ (Zhou *et al.*, 1994) is

$$S_z(\omega) = G(i\omega) S(\omega) G^*(i\omega).$$

Thus the gain (56) can be written as

$$\Theta^2 = \frac{\int_{-\infty}^{+\infty} \operatorname{tr}\big[\Lambda(\omega) S(\omega)\big] \,\mathrm{d}\omega}{\int_{-\infty}^{+\infty} \operatorname{tr} S(\omega) \,\mathrm{d}\omega}. \tag{57}$$

On the analogy of Section 4.3, define the $\sigma$-entropy of the input signal $w(t)$:

$$\mathfrak{S}(S) = -\frac{1}{2} \int_{-\infty}^{+\infty} \varphi(\omega_0, \omega) \ln\det \frac{\varepsilon(\omega) I + S(\omega)}{\int_{-\infty}^{+\infty} \operatorname{tr} S(\omega) \,\mathrm{d}\omega} \,\mathrm{d}\omega \tag{58}$$

and the $\sigma$-entropy norm $|||F|||_s$ of the system (28) as the maximum of the gain (57) where the $\sigma$-entropy (58) of input signals does not exceed a given value $s$:

$$|||F|||_s^2 = \sup_{\substack{\mathfrak{S}(S)\le s\\ \varepsilon(\omega)\to 0}} \frac{\int_{-\infty}^{+\infty} \operatorname{tr}\big[\Lambda(\omega) S(\omega)\big] \,\mathrm{d}\omega}{\int_{-\infty}^{+\infty} \operatorname{tr} S(\omega) \,\mathrm{d}\omega}. \tag{59}$$

**Theorem 7.** *For any $s \ge 0$ the $\sigma$-entropy norm $|||F|||_s$ of the system* (28) *with the input signal* (55) *is equal to*

$$|||F|||_s^2 = \int_{-\infty}^{+\infty} \operatorname{tr} \frac{\varphi(\omega_0,\omega)\,\Lambda(\omega)\big[I - q\Lambda(\omega)\big]^{-1}}{\int_{-\infty}^{+\infty} \varphi(\omega_0,\omega) \operatorname{tr}\big[I - q\Lambda(\omega)\big]^{-1} \,\mathrm{d}\omega} \,\mathrm{d}\omega,$$

*where $q \in \big[0, \max_\omega \lambda_{\max}^{-1}(\omega)\big)$ is the unique solution of the equation*

$$-\frac{1}{2}\int_{-\infty}^{+\infty} \varphi(\omega_0,\omega) \ln\det \frac{\varphi(\omega_0,\omega)\big[I - q\Lambda(\omega)\big]^{-1}}{\int_{-\infty}^{+\infty} \varphi(\omega_0,\omega) \operatorname{tr}\big[I - q\Lambda(\omega)\big]^{-1} \,\mathrm{d}\omega} \,\mathrm{d}\omega = s.$$

The proof of Theorem 7 is similar that of Theorem 6.

**4.5. $\mathfrak{N}$ norm.** The obtained results are valid for deterministic or stochastic signals with the bounded $\mathcal{L}_2$ or power norm. Let us generalize this results and consider the system (28) and the deterministic or stochastic input signal with the bounded $\mathcal{L}_2$ or power norm:

$$\|w(t)\|_{\mathfrak{N}}^2 = \mathfrak{N}\big(w^{\mathrm{T}}(t) w(t)\big) < \infty,$$

where $\|\cdot\|_{\mathfrak{N}}$ is the $\mathfrak{N}$ norm of a signal and $\mathfrak{N}$ stands for the linear operator, which transforms the Euclidean norm $|w(t)|^2 = w(t)^{\mathrm{T}} w(t)$ into the $\mathcal{L}_2$ or power norm of the deterministic or stochastic signal in accordance with following rule:

$$\mathfrak{N}(\cdot) = \begin{cases} \int_{-\infty}^{+\infty} \mathrm{d}t & \text{deterministic signal, } \mathcal{L}_2 \text{ norm},\\ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{d}t & \text{deterministic signal, } \mathcal{P} \text{ norm},\\ \int_{-\infty}^{+\infty} \mathbf{E}[\cdot] \,\mathrm{d}t & \text{stochastic signal, } \mathcal{L}_2 \text{ norm},\\ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathbf{E}[\cdot] \,\mathrm{d}t & \text{stochastic signal, } \mathcal{P} \text{ norm}. \end{cases}$$

Define the gain $\Theta_{\mathfrak{N}}$ of the system

$$\Theta_{\mathfrak{N}}^2 = \frac{\|z(t)\|_{\mathfrak{N}}^2}{\|w(t)\|_{\mathfrak{N}}^2},$$

and further the autocorrelation matrix

$$K_{\mathfrak{N}}(\tau) = \mathfrak{N}\big(w(t+\tau) w^{\mathrm{T}}(t)\big),$$

the spectral density matrix

$$S_{\mathfrak{N}}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} K_{\mathfrak{N}}(\tau)\, \mathrm{e}^{-i\omega\tau}\, \mathrm{d}\tau,$$

the $\mathfrak{N}$ norm of the input signal

$$\|w\|_{\mathfrak{N}}^2 = \int_{-\infty}^{+\infty} \operatorname{tr} S_{\mathfrak{N}}(\omega)\, \mathrm{d}\omega$$

and, finally, the $\sigma$-entropy

$$\mathfrak{S}(S_{\mathfrak{N}}) = -\frac{1}{2} \int_{-\infty}^{+\infty} \varphi(\omega_0, \omega) \ln\det \frac{\varepsilon(\omega) I + S_{\mathfrak{N}}(\omega)}{\int_{-\infty}^{+\infty} \operatorname{tr} S_{\mathfrak{N}}(\omega)\, \mathrm{d}\omega}\, \mathrm{d}\omega.$$

Thus the $\sigma$-entropy norm $|||F|||_s$ can be written as

$$|||F|||_s^2 = \sup_{\substack{\mathfrak{S}(S) \le s \\ \varepsilon(\omega) \to 0}} \frac{\int_{-\infty}^{+\infty} \operatorname{tr}\big[\Lambda(\omega) S_{\mathfrak{N}}(\omega)\big]\, \mathrm{d}\omega}{\int_{-\infty}^{+\infty} \operatorname{tr} S_{\mathfrak{N}}(\omega)\, \mathrm{d}\omega}. \qquad (60)$$

**Theorem 8.** *For any $s \ge 0$ the $\sigma$-entropy norm (60) is*

$$|||F|||_s^2 = \int_{-\infty}^{+\infty} \operatorname{tr} \frac{\varphi(\omega_0, \omega)\, \Lambda(\omega) \big[I - q\Lambda(\omega)\big]^{-1}}{\int_{-\infty}^{+\infty} \varphi(\omega_0, \omega) \operatorname{tr}\big[I - q\Lambda(\omega)\big]^{-1}\, \mathrm{d}\omega}\, \mathrm{d}\omega,$$

*where $q \in \big[0, \max_\omega \lambda_{\max}^{-1}(\omega)\big)$ is the unique solution of the equation:*

$$-\frac{1}{2} \int_{-\infty}^{+\infty} \varphi(\omega_0, \omega) \ln\det \frac{\varphi(\omega_0, \omega) \big[I - q\Lambda(\omega)\big]^{-1}}{\int_{-\infty}^{+\infty} \varphi(\omega_0, \omega) \operatorname{tr}\big[I - q\Lambda(\omega)\big]^{-1}\, \mathrm{d}\omega}\, \mathrm{d}\omega = s.$$

The proof of this theorem is exactly the same as that of Theorem 6.

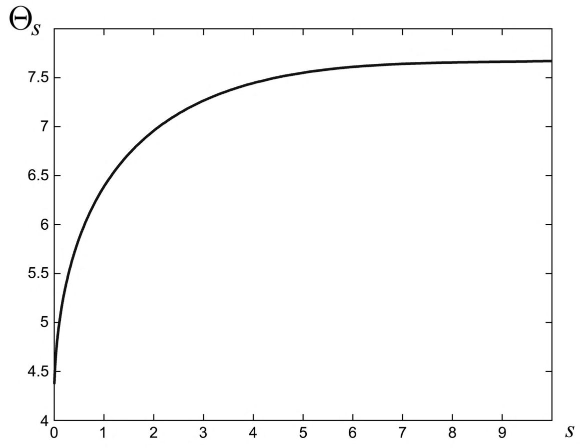

Fig. 1. Generalized $\sigma$-entropy gain.

## 5. Numerical example

Consider the numerical realization of the linear discrete system (20) with the nonzero initial condition:

$$A = \begin{bmatrix} 0.23596 & -0.85556 & -0.68156 \\ -0.77842 & 0.00756 & -0.26014 \\ 1.09960 & -0.93759 & -0.22880 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.52481 & 1.8551 \\ 1.12830 & -0.2773 \\ 0.55014 & 1.06661 \end{bmatrix},$$

$$C = \begin{bmatrix} -2.09920 & 0.37147 & 0.69535 \\ 0.63848 & -0.37418 & 0.87763 \end{bmatrix},$$

$$D = \begin{bmatrix} 1.03360 & 0.60107 \\ 0.41979 & -0.67402 \end{bmatrix}.$$

This discrete system is stable (the spectral radius $\rho(A) = 0.6602 < 1$) and Fig. 1 shows the results of calculating the generalized $\sigma$-entropy gain $\Theta_s$ as a function of $\sigma$-entropy $s$.

## 6. Conclusion

In this paper the $\sigma$-entropy and the $\mathfrak{N}$ norm of signals are introduced. The advantages of these important concepts are the following:

- The definition of the $\mathfrak{N}$ norm of the input signal is consistent with the definition of the spectral density matrix:
$$\|w\|_{\mathfrak{N}}^2 = \int_{-\infty}^{+\infty} \operatorname{tr} S_{\mathfrak{N}}(\omega)\, \mathrm{d}\omega.$$
- All the differences that are a consequence of the choice of the $\mathcal{L}_2$ or power norm of the deterministic or stochastic input signal are encapsulated in the matrix $S(\omega)$.

- The encapsulation of differences in the choice of the input signal makes it possible to determine in an invariant way (e.g., independently of the norm) the $\sigma$-entropy of the input signal and the $\sigma$-entropy norm of the system.

- The invariance of the $\sigma$-entropy norm yields invariant results of $\sigma$-entropy analysis.

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