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A COMBINATION OF TWO CONJUGATE GRADIENT METHODS UNDER A NEW LINE SEARCH WITH ITS APPLICATION IN IMAGE RESTORATION PROBLEMS

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A combined conjugate gradient algorithm is introduced for solving unconstrained optimization problems. In the suggested approach, the conjugate gradient parameter is defined as a combination of PRP (Polak-Ribiére-Polyak) and BRB (Rahali-Belloufi-Benzine) conjugate gradient parameters. To improve the convergence properties, we have adopted a new inexact line search technique that fits in with the suggested approach. The proposed line search technique can be useful for other gradient descent methods. We have established the existence of a step length that meets the new line search conditions. The generated descent direction and the convergence properties of the suggested approach are studied under the new line search conditions and the proposed method converges globally under mild assumptions. Our approach is evaluated on various test functions, and a comparison with similar recent algorithms is carried out. Furthermore, the proposed algorithm is applied for restoring images with different noise levels.

Keywords: unconstrained optimization, conjugate gradient methods, inexact line search, global convergence, image processing.

1. Introduction

In this study, we are interested in the following unconstrained optimization problem:

$$f^* = \min_{x\in\mathbb{R}^n} f(x), \qquad \text{(P)}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Numerous practical problems in real-life applications can be expressed as unconstrained optimization problems that involve differentiable cost functions (see, Ziadi and Bencherif-Madami, 2024; 2025; Yousif and Saleh, 2024). The solution of these problems becomes difficult when their dimensions are high. Scientists have explored various techniques, such as Newton's method, quasi-Newton methods, and conjugate gradient (CG) methods, to find the most efficient way to solve a particular problem. The conjugate gradient methods (CG) have become one of the favorite approaches thanks to their straightforward iterative process and low memory requirements (see, Chen *et al*., 2024; Sulaiman *et al*., 2024); they are also widely employed in numerous applications, in particular in image processing (Khudhur and Halil, 2024; Souli *et al*., 2025), neural networks (Saleh, 2023), grid computing (Collignon and Van Gijzen, 2010), molecular physics (Ziadi *et al*., 2017) and statistical modeling (Mehamdia *et al*., 2025). Starting from a point $x_0 \in \mathbb{R}^n$, a sequence of points $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ is generated by the following recursive scheme:

$$x_{k+1} = x_k + \alpha_k d_k, \quad k \in \mathbb{N}, \qquad (1)$$

where d_k is the descent direction and α_k is the step length that ensures that $f(x_{k+1}) \le f(x_k)$.

The determination of the step length α_k is crucial for ensuring global convergence. Usually, it is determined using inexact line searches, which are guaranteed to take

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steps that should be neither too long nor too short, such as the weak Wolfe line search

$$\begin{aligned} f(x_k + \alpha_k d_k) - f(x_k) &\le \alpha_k \delta d_k^t g_k, \\ g(x_k + \alpha_k d_k)^t d_k &\ge \sigma d_k^t g_k, \end{aligned}$$

or the strong Wolfe line search

$$\begin{aligned} f(x_k + \alpha_k d_k) - f(x_k) &\le \alpha_k \delta d_k^t g_k, \\ \left|g(x_k + \alpha_k d_k)^t d_k\right| &\le -\sigma d_k^t g_k, \end{aligned}$$

where $0 < \delta < \sigma < 1$. The descent search direction d_k is typically computed by the following iterative formula:

$$d_0 = -g_0, \;\; d_{k+1} = -g_{k+1} + \beta_k d_k, \quad k \in \mathbb{N}^*,$$

where $\beta_k \in \mathbb{R}$ is the conjugate parameter that characterizes the various conjugate gradient variants.

The most famous classical conjugate gradient methods include HS (Hestenes and Stiefel, 1952), FR (Fletcher and Reeves) (Fletcher, 1997), PRP (Polak-Ribière-Polyak) (Polyak, 1969; Polak and Ribière, 1969), CD (conjugate descent) (Fletcher, 1997), LS (Liu and Storey, 1991), and DY (Dai and Yuan) (Dai and Yuan, 2001), where their parameter β_k is given respectively as follows:

$$\begin{aligned} \beta_k^{\text{HS}} &= \frac{g_{k+1}^t y_k}{d_k^t y_k}, & \beta_k^{\text{PRP}} &= \frac{g_{k+1}^t y_k}{\|g_k\|^2}, \\ \beta_k^{\text{LS}} &= -\frac{g_{k+1}^t y_k}{g_k^t d_k}, & \beta_k^{\text{FR}} &= \frac{\|g_{k+1}\|^2}{\|g_k\|^2}, \\ \beta_k^{\text{CD}} &= -\frac{\|g_{k+1}\|^2}{g_k^t d_k}, & \beta_k^{\text{DY}} &= \frac{\|g_{k+1}\|^2}{d_k^t y_k}, \end{aligned}$$

where $y_k = g_{k+1} - g_k$ and $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. The DY, CD and FR versions have better theoretical convergence properties, but practically, they are less effective. Conversely, the LS, HS and PRP methods are more efficient in practice, but they may not always be convergent.

Apart from the six classical methods mentioned above, other methods have proved their efficiency and obtained good theoretical convergence properties. Wei *et al.* (2006) proposed a competitive CG method that converges globally under weak Wolfe conditions and for which β_k is

$$\beta_k^{\text{WYL}} = \frac{g_{k+1}^t (g_{k+1} - \frac{\|g_{k+1}\|}{\|g_k\|} g_k)}{\|g_k\|^2}.$$

Inspired by the WYL formula, Zhang (2009) suggested a modification of the β_k^{WYL} as follows

$$\beta_k^{\text{NHS}} = \frac{\|g_{k+1}\|^2 - \frac{\|g_{k+1}\|}{\|g_k\|} |g_{k+1}^t g_k|}{d_k^t y_k}.$$

Table 1. Hybrid conjugate gradient methods with convex combinations.

Formula	References
$\beta^{\text{hLSDY}} = \lambda\beta^{DY} + (1-\lambda)\beta^{LS}$	Liu and Li (2014)
$\beta^{\text{hHSDY}} = \lambda\beta^{DY} + (1-\lambda)\beta^{HS}$	Andrei (2009)
$\beta^{\text{hHSCD}} = \lambda\beta^{CD} + (1-\lambda)\beta^{HS}$	Zheng *et al.* (2020)
$\beta^{\text{hLSCD}} = \lambda\beta^{CD} + (1-\lambda)\beta^{LS}$	Djordjevic (2017)

This modification is efficient and the algorithm converges globally under the strong Wolfe conditions. Also, Hamoda *et al.* (2016) proposed a CG variant, which has superior convergence characteristics and whose parameter β_k is defined as

$$\beta_k^{\text{HRM}} = \frac{g_{k+1}^t \left(g_{k+1} - \frac{\|g_{k+1}\|}{\|g_k\|} g_k\right)}{\theta\|g_k\|^2 + (1-\theta)\|d_k\|^2}$$

with $0 \le \theta \le 1$.

Recently, there have been suggestions for new, effective conjugate gradient formulas. Rahali *et al.* (2021) have introduced a quite efficient CG method where the conjugate coefficient β_k is defined as

$$\beta_k^{\text{BRB}} = \frac{\|g_{k+1}\|^2}{\|d_k\|^2}, \tag{2}$$

In order to achieve effective performance and good convergence properties, numerous combinations of CG methods have been proposed. We gather some famous hybrid conjugate coefficients based on a convex combination in Table 1.

Building on the above discussion, in order to combine a good practical performance and powerful global convergence properties of both of BRB and PRP methods, we propose to combine them into a new hybrid method named hPB (hybrid PRP-BRB). To achieve good convergence properties, we adopt a new inexact line search technique where the step length α_k meets the following conditions:

$$f(x_k + \alpha_k d_k) - f(x_k) \le \alpha_k \delta d_k^t g_k \frac{\|g_k\|^2}{\|d_k\|^2}, \tag{3}$$

$$\left|g(x_k + \alpha_k d_k)^t d_k\right| \le -\sigma d_k^t g_k \frac{\|g_k\|^2}{\|d_k\|^2}, \tag{4}$$

where

$$\sigma \in \left(0, \frac{\mu - 1}{\mu^2(\mu^2 + 1.2)}\right],$$

with $\mu > 1$ and $0 < \delta < \sigma$. Since $\mu > 1$, it follows that $0 < \delta < \sigma < 1$.

The new step length matches the combined algorithm and the approach converges globally under mild assumptions. The proposed method inherits the good practical performance characteristics of BRB and PRP methods and the algorithm is successfully applied to a broad set of test functions (with varied analytical expressions and structures) that range from the simplest to the hardest, as well as to image processing.

The new hybrid conjugate parameter β_k^{hPB} and the pseudocode of the hPB algorithm are described in the next section. Then, in Section 3 we study the descent direction properties. The convergence analysis and the global convergence are established in Section 5. The performance of the suggested approach is presented in the last section with some conclusions.

2. Proposed hPB (hybrid PRP-BRB) method

The new hybrid conjugate gradient parameter, β^{hPB}, is computed as follows:

$$\beta_k^{\text{hPB}} = \begin{cases} \beta_k^{\text{PRP}} & \text{if } \beta_k^{\text{PRP}} > 0, \\ \lambda_k\, \beta_k^{\text{BRB}} + (1-\lambda_k)\beta_k^{\text{PRP}} & \text{if } \lambda_k \beta_k^{\text{BRB}} + (1-\lambda_k)\beta_k^{\text{PRP}} > 0 \text{ and } \beta_k^{\text{PRP}} \le 0, \\ \beta_k^{\text{BRB}} & \text{otherwise.} \end{cases} \tag{5}$$

It is chosen positive, where the hybridization parameter λ_k is defined in such a way that the descent direction d_{k+1} fulfills the conjugacy condition, that is, $d_{k+1}^t y_k = 0$. Indeed, in the case where

$$\beta_k^{\text{hPB}} = \lambda_k\, \beta_k^{\text{BRB}} + (1-\lambda_k)\beta_k^{\text{PRP}},$$

it follows that

$$\begin{aligned} d_{k+1} &= -g_{k+1} + \beta_k^{\text{hPB}} d_k, \qquad (6)\\ &= -g_{k+1} + (\lambda_k\, \beta_k^{\text{PRP}} + (1-\lambda_k)\beta_k^{\text{BRB}})d_k, \\ &= -g_{k+1} + \left(\lambda_k \frac{g_{k+1}^t y_k}{\|g_k\|^2} \right.\\ &\qquad \left. +(1-\lambda_k)\frac{\|g_{k+1}\|^2}{\|d_k\|^2}\right) d_k. \qquad (7) \end{aligned}$$

By pre-multiplying both the sides of (7) by y_k^t, it follows that

$$y_k^t d_{k+1} = -y_k^t g_{k+1} + \left(\lambda_k \frac{g_{k+1}^t y_k}{\|g_k\|^2} +(1-\lambda_k)\frac{\|g_{k+1}\|^2}{\|d_k\|^2}\right) y_k^t d_k.$$

Using the conjugacy condition, it follows that

$$\lambda_k = \frac{\vartheta_k - \psi_k}{\phi_k}, \tag{8}$$

where

$$\begin{aligned} \psi_k &= \|g_{k+1}\|^2 \|g_k\|^2 y_k^t d_k, \\ \vartheta_k &= y_k^t g_{k+1} \|g_k\|^2 \|d_k\|^2, \\ \phi_k &= y_k^t g_{k+1} y_k^t d_k \|d_k\|^2 - \psi_k. \end{aligned}$$

For each iteration k, we set

$$\lambda_k = \max\left\{0, \min\left\{\frac{\vartheta_k - \psi_k}{\phi_k}, 1\right\}\right\},$$

and as the search progresses, if for an iteration $k \in \mathbb{N}$ we have $\phi_k = 0$, we set $\beta_k^{\text{hPB}} = \beta_k^{\text{BRB}}$. The main steps of the hPB method are sketched in Algorithm 1.

3. Sufficient descent property of hPB algorithm

Before proving the intended property, we first prove the following auxiliary result.

Theorem 1. *The sequences $\{g_k\}_{k\in\mathbb{N}}$ and $\{d_k\}_{k\in\mathbb{N}}$, generated by the hPB algorithm under the new line search conditions* (3) *and* (4) *with $\sigma \in \left(0, \frac{\mu-1}{\mu^2(\mu^2+1.2)}\right]$ and $\mu > 1$, satisfy*

$$\frac{\|g_k\|}{\|d_k\|} \le \mu, \quad \forall k \in \mathbb{N}. \tag{9}$$

Proof. It is clear that in the case where Powell's restart criterion holds (i.e., $|g_k^t g_{k+1}| \ge 0.2\,\|g_{k+1}\|^2$), the descent direction d_k is defined as $d_k = -g_k$ and the relation (9) holds.

Now, if the Powell condition does not hold, we prove the above relation by induction. For $k = 0$, the condition (9) holds since $d_0 = -g_0$. Assume that the relation (9) holds for $k \ge 1$, and let us prove it for $k+1$. Since

$$d_{k+1} = -g_{k+1} + \beta_k^{\text{hPB}} d_k, \tag{10}$$

by multiplying both the sides of (10) by g_{k+1}^t, we obtain

$$d_{k+1}^t g_{k+1} = -\|g_{k+1}\|^2 + \beta_k^{\text{hPB}} d_k^t g_{k+1}. \tag{11}$$

On the other hand, we have

$$\left|\beta_k^{\text{hPB}}\right| \le \frac{\|g_{k+1}\|^2}{\|d_k\|^2} + 1.2\frac{\|g_{k+1}\|^2}{\|g_k\|^2}. \tag{12}$$

Indeed,

$$\begin{aligned} \left|\beta_k^{\text{hPB}}\right| &\le \left|\beta_k^{\text{BRB}}\right| + \left|\beta_k^{\text{PRP}}\right| \\ &\le \frac{\|g_{k+1}\|^2}{\|d_k\|^2} + \frac{\|g_{k+1}\|^2 + \left|g_{k+1}^t g_k\right|}{\|g_k\|^2} \\ &\le \frac{\|g_{k+1}\|^2}{\|d_k\|^2} + 1.2\frac{\|g_{k+1}\|^2}{\|g_k\|^2}. \end{aligned}$$

Algorithm 1. The hPB algorithm.

Require: Choose a scalar $\mu > 1$ and the parameters δ and σ such that $0 < \delta < \sigma < \frac{\mu-1}{\mu^2(\mu^2+1.2)}$. Choose a scalar $\epsilon > 0$ sufficiently small to stop the algorithm.
Initialization
Set $k = 0$ and select a starting point $x_0 \in \mathbb{R}^n$.
Set $g_0 = \nabla f(x_0)$ and $d_0 = -g_0$.

while $\|g_k\| \geq \epsilon$ **do**
  Compute the step legth α_k using the new inexact line search technique (3) and (4).
  Compute $x_{k+1} = x_k + \alpha_k d_k$, $g_{k+1} = \nabla f(x_{k+1})$.
  if $|g_k^t g_{k+1}| \geq 0.2\,\|g_{k+1}\|^2$ **then**
    Set $d_{k+1} = -g_{k+1}$ (The restart criterion of Powell holds).
  else
    Set $y_k = g_{k+1} - g_k$.
    if $\beta_k^{\text{PRP}} > 0$ **then**
      Set $\beta_k^{\text{hPB}} = \beta_k^{\text{PRP}}$.
    else
      Compute $\phi_k = y_k^t g_{k+1} y_k^t d_k \|d_k\|^2 - \|g_{k+1}\|^2 \|g_k\|^2 y_k^t d_k$.
      if $\phi_k = 0$ **then**
        Set $\beta_k^{\text{hPB}} = \beta_k^{\text{BRB}}$.
      else
        Compute the hybridization parameter λ_k following (8).
        Set $\lambda_k = \max\left\{0, \min\left\{\dfrac{\vartheta_k - \psi_k}{\phi_k}, 1\right\}\right\}$.
        if $\lambda_k \beta_k^{\text{BRB}} + (1-\lambda_k)\beta_k^{\text{PRP}} > 0$ **then**
          Set $\beta_k^{\text{hPB}} = \lambda \beta_k^{\text{BRB}} + (1-\lambda_k)\beta_k^{\text{PRP}}$.
        else
          Set $\beta_k^{\text{hPB}} = \beta_k^{\text{BRB}}$.
        end if
      end if
    end if
    Compute the descent direction $d_{k+1} = -g_{k+1} + \beta_k d_k$.
  end if
  Set $k = k + 1$.
end while

Hence, from (11) and condition (4) it follows that

$$\begin{aligned}
\|g_{k+1}\|^2 &\leq \left|d_{k+1}^t g_{k+1}\right| + \left|\beta_k^{\text{hPB}}\right| \left|d_k^t g_{k+1}\right| \\
&\leq \left|d_{k+1}^t g_{k+1}\right| - \sigma \left|\beta_k^{\text{hPB}}\right| d_k^t g_k \frac{\|g_k\|^2}{\|d_k\|^2} \\
&\leq \left|d_{k+1}^t g_{k+1}\right| + \sigma \left|\beta_k^{\text{hPB}}\right| \left|d_k^t g_k\right| \frac{\|g_k\|^2}{\|d_k\|^2} \\
&\leq \left|d_{k+1}^t g_{k+1}\right| + \sigma \left|\beta_k^{\text{hPB}}\right| \|d_k\| \|g_k\| \frac{\|g_k\|^2}{\|d_k\|^2} \\
&\leq \|d_{k+1}\|\, \|g_{k+1}\| + \sigma \left|\beta_k^v hPB\right| \frac{\|g_k\|^3}{\|d_k\|} \\
&\leq \|d_{k+1}\|\, \|g_{k+1}\| \\
&\quad + \sigma \left(1.2 \frac{\|g_k\|}{\|d_k\|} + \frac{\|g_k\|^3}{\|d_k\|^3}\right) \|g_{k+1}\|^2 . \qquad (13)
\end{aligned}$$

Furthermore, since

$$\sigma \leq \frac{\mu - 1}{\mu^2(\mu^2 + 1.2)},$$

we get

$$\begin{aligned}
1 - \sigma &\left(\frac{\|g_k\|^3}{\|d_k\|^3} + 1.2 \frac{\|g_k\|}{\|d_k\|}\right) \\
&\geq 1 - \sigma(\mu^3 + 1.2\mu) \geq 1 - \frac{\mu - 1}{\mu} = \frac{1}{\mu} > 0. \qquad (14)
\end{aligned}$$

Dividing both the sides of (13) by $\|g_{k+1}\| \cdot \|d_{k+1}\|$, we get

$$\frac{\|g_{k+1}\|}{\|d_{k+1}\|} \leq 1 + \sigma \left(\frac{\|g_k\|^3}{\|d_k\|^3} + 1.2 \frac{\|g_k\|}{\|d_k\|}\right) \frac{\|g_{k+1}\|}{\|d_{k+1}\|},$$

which yields

$$\frac{\|g_{k+1}\|}{\|d_{k+1}\|} \left(1 - \sigma \left(\frac{\|g_k\|^3}{\|d_k\|^3} + 1.2 \frac{\|g_k\|}{\|d_k\|}\right)\right) \leq 1.$$

Then from (14) it results that

$$\frac{\|g_{k+1}\|}{\|d_{k+1}\|} \leq \left(1 - \sigma \left(\frac{\|g_k\|^3}{\|d_k\|^3} + 1.2 \frac{\|g_k\|}{\|d_k\|}\right)\right)^{-1} \leq \mu.$$

The proof is complete. ■

Now, we are in a position to prove the sufficient descent property.

Theorem 2. *The sequences $\{g_k\}_{k \in \mathbb{N}}$ and $\{d_k\}_{k \in \mathbb{N}}$, generated by the hPB algorithm under the new line search conditions* (3) *and* (4) *with $\sigma \in \left(0, \frac{\mu-1}{\mu^2(\mu^2+1.2)}\right]$ and $\mu > 1$ satisfy*

$$g_k^t d_k \leq -\xi \|g_k\|^2, \quad \forall k \in \mathbb{N}. \qquad (15)$$

where $\xi > 0$.

Proof. In the case where the Powell restart criterion holds (i.e., $|g_k^t g_{k+1}| \geq 0.2\|g_{k+1}\|^2$), it is evident the descent direction d_k is given by $d_k = -g_k$ and the relation (15) holds.

In the case where the Powell condition does not hold, we prove the above relation by induction. Indeed, for $k = 0$, the search direction is $d_0 = -g_0$. This implies $d_0^t g_0 = -\|g_0\|^2$, and relation (15) holds.

Suppose that (15) is true for $k \geq 1$. For $k + 1$, by multiplying the two sides of (10) by g_{k+1}^t, we get

$$
\begin{aligned}
d_{k+1}^t g_{k+1} &= -\|g_{k+1}\|^2 + \beta_k^{\text{hPB}} d_k^t g_{k+1} \\
&\le -\|g_{k+1}\|^2 + \left|\beta^P owellhPB\right| \left|d_k^t g_{k+1}\right| \\
&\le -\|g_{k+1}\|^2 + \sigma\left(\frac{1.2\,\|g_{k+1}\|^2}{\|g_k\|^2} + \frac{\|g_{k+1}\|^2}{\|d_k\|^2}\right) \\
&\quad \cdot \|g_k\|\|d_k\| \frac{\|g_k\|^2}{\|d_k\|^2}, \\
&\qquad \text{(using relations (4) and (12))} \\
&\le -\|g_{k+1}\|^2 + \sigma\left(1.2\frac{\|g_k\|}{\|d_k\|} + \frac{\|g_k\|^3}{\|d_k\|^3}\right)\|g_{k+1}\|^2 \\
&\le -\|g_{k+1}\|^2 \left(1 - \sigma(\mu^3 + 1.2\mu)\right) \\
&\le -\frac{1}{\mu}\|g_{k+1}\|^2, \\
&\qquad \text{(using relations (9) and (14)).}
\end{aligned}
$$

Then the proof is complete for $\xi = 1/\mu$. ■

4. Convergence analysis

Before analyzing the convergence of the proposed approach, we first show that it is well defined. In the next theorem, we prove the existence of a step length α ($0 < \alpha < \infty$) that meets the conditions (3) and (4), where $0 < \delta < \sigma$, with $\sigma \in \left(0, \frac{\mu-1}{\mu^2(\mu^2+1.2)}\right]$ and $\mu > 1$.

Theorem 3. *Let f be a twice continuously differentiable function that is bounded below. If $g_k^t d_k < 0$, then there exists a strictly positive real value α that meets the conditions* (3) *and* (4).

Proof. Define the following function:

$$h(\alpha) = f(x_k + \alpha d_k) - f(x_k) - \delta\alpha g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2}.$$

Using a standard Taylor expansion, from (9) it results that

$$
\begin{aligned}
h(\alpha) &= f(x_k + \alpha d_k) - f(x_k) - \delta\alpha g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2} \\
&= \left(f(x_k) + \alpha g_k^t d_k + o(\alpha)\right) \\
&\quad - f(x_k) - \delta\alpha g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2}, \\
&\le \alpha\left(1 - \delta\mu^2\right) g_k^t d_k + o(\alpha) \\
&\le \alpha\left(1 - \frac{\mu - 1}{\mu^2 + 1.2}\right) g_k^t d_k + o(\alpha) < 0
\end{aligned}
$$

Furthermore, since f is bounded from below, we get $\lim_{\alpha\to+\infty} h(\alpha) = +\infty$ and $h(0) = 0$. Hence, the function $h(\cdot)$ changes its sign and then there exists a real value $\tau > 0$ with $h(\tau) = 0$. It is clear that $h(\alpha)$ has a negative sign over the interval $[0, \tau]$, and its global minimum cannot occur at the endpoints, since $h(0) = h(\tau) = 0$. Therefore, there exists $\alpha^* \in (0, \tau)$, such that $h(\alpha^*) < 0$ and $h'(\alpha^*) = 0$. Hence

$$
\begin{aligned}
h(\alpha^*) &= f(x_k + \alpha^* d_k) - f(x_k) \\
&\quad - \delta\alpha^* g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2} < 0,
\end{aligned}
$$

so that

$$f(x_k + \alpha^* d_k) < f(x) + \delta\alpha^* g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2},$$

and the first condition (3) holds.

On the other hand, we have

$$h'(\alpha) = g_{k+1}^t d_k - \delta g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2}.$$

Since $h'(\alpha^*) = 0$, we deduce that

$$\sigma g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2} \le \delta g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2} = g_{k+1}^t d_k < 0.$$

Therefore,

$$\left|g_{k+1}^t d_k\right| \le -\sigma g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2}.$$

Thus, the second condition (4) also holds. ■

The global convergence property is crucial for any conjugate gradient method. To establish it, we make the following assumptions:

Assumption 1. The level set $\Omega = \{x \in \mathbb{R}^n \mid f(x) \le f(x_0)\}$ is bounded for any starting point x_0.

Assumption 2. The function f is continuously differentiable and its gradient g satisfies the Lipschitz condition in some closed neighborhood $\mathcal{N}$ of Ω, i.e., $\exists L > 0$ such that

$$\|g(x) - g(y)\| \le L\,\|x - y\|, \quad \forall x, y \in \mathcal{N}. \tag{16}$$

The above assumptions imply the existence of a real number $\Gamma \ge 0$ such that

$$\|g(x)\| \le \Gamma, \quad \forall x \in \Omega. \tag{17}$$

To establish that the hPB algorithm converges globally (see Theorem 4 below), we need to prove the following results, which will be needed below.

Lemma 1. *Consider the sequences $\{g_k\}_{k\in\mathbb{N}}$, $\{\alpha_k\}_{k\in\mathbb{N}}$ and $\{d_k\}_{k\in\mathbb{N}}$, generated by the hPB algorithm. For some $\varpi > 0$, we have*

$$g_k^t d_k \ge -\varpi\,\|g_k\|^2, \quad \forall k \in \mathbb{N}. \tag{18}$$

Proof. Multiplying (6) by g_{k+1}, we get

$$\left|d_{k+1}^t g_{k+1}\right| \le \|g_{k+1}\|^2 + \left|\beta_k^{\mathrm{hPB}}\right| \left|d_k^t g_{k+1}\right|.$$

Then, from (9) it follows that

$$\begin{aligned}\left|d_{k+1}^t g_{k+1}\right| &\le \|g_{k+1}\|^2 + \sigma\left(1.2\frac{\|g_k\|}{\|d_k\|}\right.\\ &\left.\quad+\frac{\|g_k\|^3}{\|d_k\|^3}\right)\|g_{k+1}\|^2,\\ &\le \left(1+\sigma\left(1.2\mu+\mu^3\right)\right)\|g_{k+1}\|^2,\end{aligned}$$

which means that

$$-\varpi\|g_{k+1}\|^2 \le d_{k+1}^t g_{k+1} \le \varpi\|g_{k+1}\|^2,$$

where $\varpi = 1+\sigma\left(1.2\mu+\mu^3\right)$. This completes the proof. ■

Next, we state the following important result:

Lemma 2. *Under the above assumptions, the sequence of steplengths $\{\alpha_k\}_{k\in\mathbb{N}}$ generated by the hPB algorithm under the new line search conditions* (3) *and* (4) *with $\sigma \in \left(0, \frac{\mu-1}{\mu^2(\mu^2+1.2)}\right]$ and $\mu > 1$, satisfies*

$$\alpha_k \ge \frac{\sigma\frac{\|g_k\|^2}{\|d_k\|^2}-1}{L\|d_k\|^2}\, d_k^t g_k > 0, \quad \forall k\in\mathbb{N}.$$

Proof. From Theorem 1, it results that

$$\sigma\frac{\|g_k\|^2}{\|d_k\|^2} < \frac{\mu-1}{\mu^2+1.2} < 1.$$

Hence, from condition (4) it follows that

$$\begin{aligned}0 &< \left(\sigma\frac{\|g_k\|^2}{\|d_k\|^2}-1\right) d_k^t g_k\\ &< d_k^t(g_{k+1}-g_k) \le L\alpha_k\|d_k\|^2,\end{aligned}$$

which completes the proof. ■

Lemma 3. *Under Assumption 1, the sequences $\{g_k\}_{k\in\mathbb{N}}$ and $\{d_k\}_{k\in\mathbb{N}}$ produced by the hPB algorithm satisfy*

$$\sum_{k\ge 0}\frac{(g_k^t d_k)^2}{\|d_k\|^2} < +\infty. \tag{19}$$

Proof. Using the condition (3) and relation (18), we have

$$\begin{aligned}f(x_k) - f(x_k+\alpha_k d_k) &\ge -\delta\alpha_k g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2}\\ &\ge \frac{\delta\alpha_k}{\varpi}\frac{(g_k^t d_k)^2}{\|d_k\|^2}.\end{aligned}$$

Then we get

$$\frac{\delta\alpha_k}{\varpi}\frac{(g_k^t d_k)^2}{\|d_k\|^2} \le f(x_k) - f(x_{k+1}).$$

Let $m = \min\{\alpha_k : k\in\mathbb{N}\}$. Under Assumption 1, by summing these inequalities from $k=0$ to $+\infty$, we get

$$\frac{\delta m}{\varpi}\sum_{k\ge 0}\frac{(g_k^t d_k)^2}{\|d_k\|^2} \le \sum_{k\ge 0}\frac{\delta\alpha_k}{\varpi}\frac{(g_k^t d_k)^2}{\|d_k\|^2} < +\infty.$$

and thus (19) holds. ■

Now, we are in a position to address the global convergence of the hPB algorithm.

Theorem 4. *Under the above assumptions, the hPB algorithm converges globally in the sense that,*

$$\liminf_{k\to+\infty}\|g_k\| = 0. \tag{20}$$

Proof. Assume that the assertion (20) does not hold. Then there exists a positive value $r > 0$ such that

$$\|g_k\| > r, \quad \forall k\in\mathbb{N}. \tag{21}$$

Let $m = \min\{\alpha_k : k\in\mathbb{N}\}$ and $D = \max\{\|x-y\| : x,y\in\Omega\}$. From relation (1), it results that

$$\|d_k\|^2 = \frac{\|x_{k+1}-x_k\|^2}{\alpha_k^2} \le \frac{D^2}{m^2}.$$

On the other hand, from (12), (21), (17) and (9) we get

$$\begin{aligned}d_{k+1} &\le \|g_{k+1}\| + \left|\beta_k^{\mathrm{hPB}}\right|\|d_k\|\\ &\le \Gamma + \left(\frac{1.2\Gamma^2}{r^2}+\frac{\mu^2\Gamma^2}{r^2}\right)\frac{D}{m} = M.\end{aligned}$$

Hence

$$\sum_{k\ge 0}\frac{1}{\|d_k\|^2} = +\infty.$$

But, from (21), (15) and (19) it results that

$$\begin{aligned}C^2 r^4\sum_{k\ge 0}\frac{1}{\|d_k\|^2} &\le \sum_{k\ge 0}\frac{C^2\|g_k\|^4}{\|d_k\|^2}\\ &\le \sum_{k\ge 0}\frac{(g_k^t d_k)^2}{\|d_k\|^2} < +\infty.\end{aligned}$$

Therefore,

$$\sum_{k\ge 0}\frac{1}{\|d_k\|^2} < +\infty,$$

which contradicts the claim (21). Thus, the assertion (20) is true. ■

5. Computational experiments

Here, we present a series of computational performances concerning the hPB algorithm, applied to 40 test functions with 270 test problems, as listed in Table 2 and taken from (Andrei, 2008), with dimensions ranging from 2 to 100000, as well as on restoring four images with different noise levels. All the codes are written and implemented in Matlab version R2018a. In all experiments, the hPB algorithm is implemented with the setting parameters $\mu = 1.6$, $\delta = 10^{-4}$ and $\sigma = 10^{-3}$.

In order to illustrate the fact that the hPB method performs better by using the proposed inexact line search (3) and (4), we compare it with hPB-SW (hPB method using strong Wolfe conditions with $\delta = 10^{-4}$ and $\sigma = 10^{-3}$). Also, to demonstrate the effectiveness of the suggested approach, we compared it with HRM (Hamoda *et al.*, 2016) with $\theta = 0.4$, PRP^{+} (Gilbert and Nocedal, 1992), PRP (Polyak, 1969), BRB (Rahali *et al.*, 2021) and NHS (Zhang, 2009). For this comparison, the same starting point is assigned for each test problem, and each implementation is considered successful if a point x_k where $\|g(x_k)\|_\infty \leq 10^{-6}$ is reached within 2000 iterations, with CPU-time less than 500 seconds; otherwise, the implementation is assigned as failure.

To examine and compare the performances of the implemented algorithms, we will use a graphical comparison. Figures 1–4 illustrate the performances of the methods (according to their CPU times, numbers of iterations, function and gradient evaluations needed to reach the stopping criterion) using the performance profile of Dolan and Moré (2002). The performance of each method is plotted by a curve that refers to the percentage of solved problems within a factor of τ. The curve that is shaped on top corresponds to the code that solves a majority of the test problems within the given factor τ; for more details, see (Dolan and Moré, 2002).

Figures 1–4 illustrate the fact that the hPB outperforms the others. Specifically, it is faster for approximately 48% of the test problems and successfully solves around 96% of them, followed by hPB-sw and BRB with 95% and 93%, respectively. These outcomes prove the competitiveness and rapid convergence of the hPB algorithm in the majority of testing problems.

5.1. Image restoration problems. In optimization fields, image restoration problems are considered among the most difficult ones. They aim to restore the original image from one that has been corrupted by impulse noise. For this comparison, four test images (`Man.png`, `Boat.png`, `Hill.jpg` and `Bridge.bmp`) of size 512×512 are chosen to evaluate the effectiveness of the hPB algorithm against the same variants used in the previous comparisons. The image quality is assessed by two factors: the peak signal-to-noise ratio (PSNR) and its relative error (Err),

$$\mathrm{PSNR} = 10\log_{10}\frac{M\times N\times 255^2}{\sum_{i,j}(x^r_{i,j}-x^*_{i,j})^2},$$

$$\mathrm{Err} = \frac{\|x^r - x^*\|}{\|x^*\|},$$

where M and N are the sizes of the image, $x^r_{i,j}$ represents the pixel values of the restored image and $x^*_{i,j}$ denotes the original pixel values. The setting parameters of the proponent algorithms are set similarly to the previous test, and each computation will stop if any of the following criteria is fulfilled,

$$\mathrm{Iter} > 300 \quad \text{or} \quad \frac{|f(x_{k+1} - f(x_k)|}{|f(x_k)|} < 10^{-4}.$$

Figures 5–7 show the images restored for 30 %, 50% and 70 % of noise. The performance of each algorithm is measured by the restored image quality, the elapsed time and the number of iterations. The numerical outcomes are reported in Tables 3–5. The algorithm with a high PSNR, minimal error, with less CPU time, is considered best.

Upon examining the results in Tables 3–5 and Figures 5–7, it becomes evident that the hPB algorithm delivers good performance. In fact, as illustrated in Table 3 and Fig. 5, we can observe that the NHS method failed to restore images with 30% of noise, whereas the other methods were successful with a PSNR greater than 25, and the hPB method has the highest PSNR value for the Man and Boat images. On the other hand, the visual outcomes of Fig. 6 with 50% of noise show that the NHS method also failed to remove the noise with a PSNR less than 25, whereas the PRP^{+} method failed to restore the image of Man, while the HRM, hPB, BRB, PRP and hPB-sw methods succeeded for restoring all the images and the bold values in Table 4 indicate the superiority of the hPB method for restoring the images of Man, Boat and Hill. For 70% of noise, as shown in Fig. 7 and Table 5, the NHS method failed to restore the original images; the HRM, PRP^{+}, BRB and hPB-sw methods succeeded to remove the noise from two images, while the hPB and PRP methods succeeded in restoring the images of Man, Boat and Hill, where the highest PSNR value corresponds to the hPB method.

On the whole, the numerical and visual outcomes of removing $30\%, 50\%$, and 70% of noise show a satisfactory performance of the hPB algorithm. Notably, the bold values in Tables 3–5 highlight the efficiency of the proposed algorithm, as it produces higher PSNR values and takes a short time to restore a majority of the test images.

Table 2. List of test problems.

Function	Dimension n
Extended Maratos	500, 700, 1000, 1500, 2000,4000,9000, 9500, 60000, 10000, 15000
Arwhead	50, 60, 70, 80, 100, 150, 200, 800, 1000, 5000
ENGVAL1	100, 600, 700, 800, 1000, 1500, 1600,1800, 10000
Diagonal 1	2, 4, 10, 100, 1000, 10000, 100000
FLETCHCR	2, 4
Generalized Tridiagonal 1	2, 10, 20, 30, 40, 300, 500, 700, 10000
Diagonal 2	2, 4, 10, 800, 1000, 80000
Extended White and Holst	1000, 2000, 3000, 4000, 5000, 6000
Diagonal 7	500, 700, 1000, 1500, 2000, 7000, 8000
Extended Rosenbrock	10, 20, 30, 100 ,1200, 3000, 4000, 5000
Quadratic QF2	100, 200,1000,5000, 7000, 9000, 10000, 50000,
Diagonal 8	100, 200, 300, 400, 500, 1000, 1500, 2000
Extended Freudenstein and Roth	10, 100, 1000, 4000, 9000, 10000, 20000, 50000, 60000, 80000
Diagonal 3	2, 4, 6, 10, 50, 100, 200, 400, 700
Extended DENSCHNF	10, 100, 10000, 25000, 30000 50000, 70000, 80000, 90000
Perturbed Quadratic	2
POWER	2
QUARTC	2
Extended Himmelblau	4, 6, 8, 10, 9000, 10000
Raydan 1	2
Raydan 2	1000, 4000, 50000,80000
Perturbed quadratic diagonal	2
Extended DENSCHNB	10, 90, 100, 2000, 3000, 4000, 5000, 6000, 7000, 9000
HIMMELBG	2000, 2500, 2700, 3000, 6000, 30000, 50000, 80000
LIARWHD	10, 20, 40, 50, 4000,5000, 5500, 10000, 20000, 80000
Extended quadratic exponential EP1	40000, 50000, 60000, 70000
Diagonal 5	100, 200, 700, 1000, 1500, 2000, 2200, 2500
Generalized Rosenbrock	2, 50, 800, 1000
Extended BD1	4, 800, 900, 2000, 3000, 5000 20000, 40000, 60000, 70000, 80000
Hager	2, 4, 6, 10, 50, 80, 150, 300
NONSCOMP	2, 4, 1000, 5000, 70000
Extended quadratic penalty QP1	4, 6, 8, 10, 50, 100, 700, 1000, 1500
Quadratic QF1	5000, 6000, 8000, 9000, 20000, 50000, 70000, 80000
HIMMELLH	10, 50, 300, 500, 10000,50000, 60000, 80000, 10000
Extended quadratic penalty QP2	40, 60, 70, 200
DIXON3DQ	2, 4, 6, 600
Diagonal 4	20000, 30000, 40000, 50000, 60000, 70000
Extended PSC1	2, 4, 6, 8, 10, 100, 1000,10000
Almost Perturbed Quadratic	2, 4, 6
Extended Tridiagonal 1	6, 10, 20, 30, 80, 90, 100, 150, 300, 500, 700, 1000, 5000, 6000

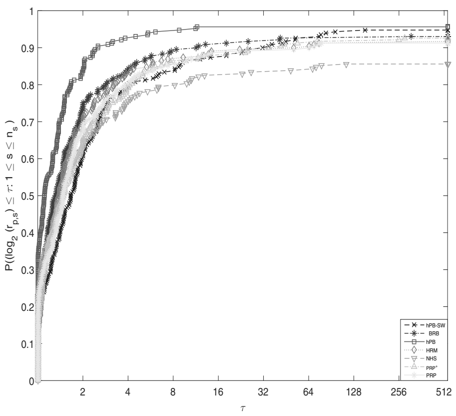

Fig. 1. Performance profiles based on CPU time.

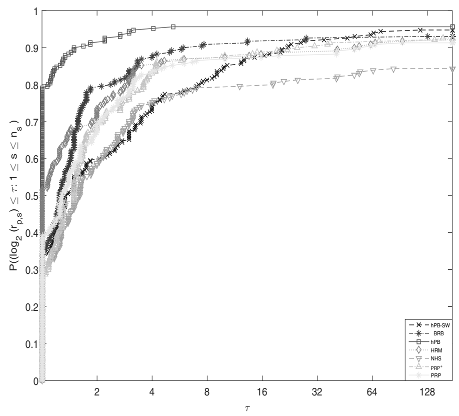

Fig. 2. Performance profiles based on the number of iterations.

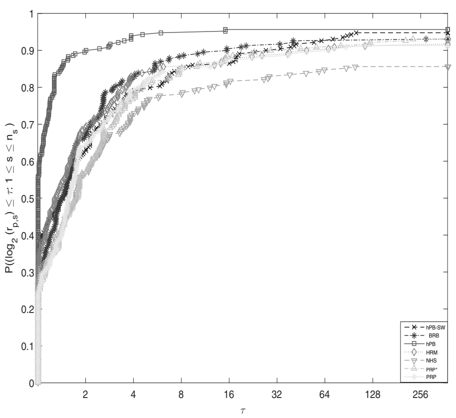

Fig. 3. Performance profiles based on the number of function evaluations.

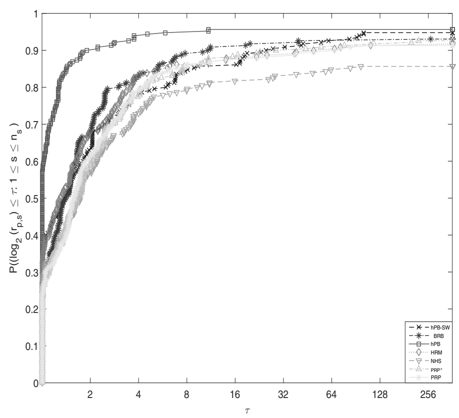

Fig. 4. Performance profiles based on the number of gradient evaluations.

6. Conclusions

In this work, a new hybrid conjugate gradient method, named hPB (hybrid PRP-BRB), is introduced for unconstrained nonconvex minimization. To achieve good convergence properties, we adopt a new inexact line search technique to determine the step length. Mathematical results concerning the convergence of the hPB method are established and the algorithm converges globally under mild assumptions. The performance of the algorithm is examined on various test functions and applied for restoring images with different noise levels. Moreover, a comparison with similar and recent algorithms is carried out, and the preliminary results indicate the robustness and competitiveness of the suggested approach.

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Fig. 5. Images restored by HRM, NHS, hPB, PRP$^+$, PRP, BRB and hPB-SW subject to 30% salt-and-pepper noise.

Fig. 6. Images restored by HRM, NHS, hPB, PRP$^+$, PRP, BRB and hPB-SW subject to 50% salt-and-pepper noise.

Fig. 7. Images restored by HRM, NHS, hPB, PRP$^+$, PRP, BRB and hPB-SW subject to 70% salt-and-pepper noise.

Table 3. Results of restoring images with 30% salt-and-pepper noise.

Methods \ Images		Man	Boat	Hill	Bridge
HRM	Iter	14	**14**	18	19
	CPU	14.4333	16.1088	19.6281	18.3902
	PSNR	31.4316	33.5530	34.8439	28.5705
	Err	0.055800	0.038858	0.0376	0.075280
NHS	Iter	26	26	24	23
	CPU	15.7869	14.9272	14.8145	14.4291
	PSNR	15.7231	17.5183	17.0234	16.2491
	Err	0.3405	0.2462	0.2931	0.3110
hPB	Iter	13	17	17	14
	CPU	16.5572	**11.2373**	16.3538	16.4848
	PSNR	**31.5644**	**33.7000**	**34.98430**	28.4354
	Err	**0.054953**	**0.038206**	**0.0370**	0.076460
PRP$^+$	Iter	**9**	**14**	17	**13**
	CPU	**11.7255**	17.2765	17.4224	**12.5381**
	PSNR	29.8526	32.7850	34.9240	28.3708
	Err	0.066923	0.042450	0.0373	0.077031
PRP	Iter	11	11	16	14
	CPU	16.0896	14.6281	16.4851	15.5467
	PSNR	31.3374	33.1674	34.7344	28.5131
	Err	0.0564	0.0406	0.0381	0.0757
BRB	Iter	13	15	**15**	14
	CPU	17.8876	19.6420	**16.0520**	17.0935
	PSNR	31.4483	32.7617	34.8471	28.5141
	Err	0.055693	0.042564	0.0376	0.075770
hPB-SW	Iter	13	15	14	15
	CPU	18.3206	17.9993	16.6398	19.4814
	PSNR	28.5141	31.4483	34.8471	**32.7617**
	Err	0.07577	0.055693	0.0376	**0.042564**

Table 4. Results of restoring images with 50% salt-and-pepper noise.

Methods \ Images		Man	Boat	Hill	Bridge
HRM	Iter	19	**14**	21	**14**
	CPU	31.6329	22.9351	29.2036	21.4267
	PSNR	29.0482	30.0439	32.5694	26.2812
	Err	0.073418	0.058202	0.0489	0.097982
NHS	Iter	31	34	30	30
	CPU	24.3502	24.8263	**21.1211**	23.0874
	PSNR	13.4363	15.2070	14.7059	14.1568
	Err	0.4430	0.3212	0.3828	0.3957
hPB	Iter	19	24	**15**	18
	CPU	26.5356	22.2738	22.325	23.2843
	PSNR	**29.1751**	**31.1843**	**32.6828**	26.7130
	Err	**0.072353**	**0.051041**	**0.0483**	0.093230
PRP$^+$	Iter	**5**	17	20	16
	CPU	**16.5329**	22.4427	29.8420	**21.0229**
	PSNR	15.3421	30.9165	32.4165	26.5201
	Err	0.355716	0.052639	0.0498	0.095324
PRP	Iter	20	17	21	24
	CPU	36.9955	24.7400	30.9204	38.2190
	PSNR	28.9228	30.8359	32.5124	26.4954
	Err	0.0744	0.0531	0.0492	0.0955
BRB	Iter	24	17	**15**	22
	CPU	29.8683	**21.7964**	22.5881	27.3311
	PSNR	29.1394	31.1089	30.7900	26.6435
	Err	0.072651	0.051485	0.0600	0.093978
hPB-sw	Iter	19	24	**15**	22
	CPU	24.5807	22.4704	22.6654	19.7673
	PSNR	26.6435	29.1394	30.7900	**31.1089**
	Err	0.093978	0.072651	0.0600	**0.051485**

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Table 5. Results of restoring images with 70% salt-and-pepper noise.

Methods \ Images		Man	Boat	Hill	Bridge
HRM	Iter	10	26	19	25
	CPU	28.8301	49.0126	44.8694	48.7749
	PSNR	22.3136	28.1401	27.8223	24.2470
	Err	0.159416	0.072465	0.0845	0.123838
NHS	Iter	34	37	34	31
	CPU	29.7937	30.6846	**29.0856**	**22.3260**
	PSNR	11.3935	12.9901	12.5343	**12.1899**
	Err	0.5604	0.4146	0.4915	0.4963
hPB	Iter	40	34	33	37
	CUP	30.8349	28.3926	31.1979	30.1883
	PNSR	**26.3222**	**28.2927**	**29.8827**	24.5192
	Err	**0.100484**	**0.071202**	**0.0667**	0.120017
PRP$^+$	Iter	29	24	16	21
	CPU	50.3015	35.5985	29.1123	35.9517
	PSNR	26.1898	28.2480	29.3224	24.1335
	Err	0.102028	0.071570	0.0711	0.125468
PRP	Iter	20	15	**14**	15
	CPU	34.7941	28.4139	32.8371	41.2126
	PSNR	26.1118	27.5034	25.6558	22.0517
	Err	0.1029	0.0779	0.1085	0.1594
BRB	Iter	**5**	21	30	**20**
	CPU	**20.0740**	27.3770	38.0558	31.7151
	PSNR	13.1600	28.2074	29.7744	24.1351
	Err	0.457310	0.071905	0.0675	0.125444
hPB-sw	Iter	20	**5**	30	21
	CPU	32.7535	**20.4136**	38.0359	29.5325
	PSNR	24.1351	13.16	29.7744	**28.2074**
	Err	0.125444	0.45731	0.0675	**0.071905**

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