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UNCERTAIN SYSTEMS AND MINIMUM ENERGY CONTROL[†]

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The paper considers two minimum energy control problems for an uncertain linear infinite-dimensional system with a bounded input operator. Uncertainty in the system description is modelled by unknown bounded perturbations of the system operator and the input operator. We present a new approach to computing estimates for the deviation of the final state of the perturbed system from the final state of the unperturbed system. This approach involves differential Lyapunov equations and a novel concept of the so-called *composite semigroup*.

1. Introduction

The purpose of this paper is to sketch a mathematical framework for an analysis of minimum energy control problems for uncertain linear infinite-dimensional systems with bounded input operators. The uncertainty in the model is deterministic and is described by unknown additive bounded perturbations to the system operator (a semigroup generator) and the input operator. Although we assume that the input operator and perturbations are bounded, our approach can be extended to a wide range of classes of systems with an unbounded input operator as well as unbounded perturbations, e.g. as in (Emirsajlow *et al.*, 1995) and (Weiss, 1994). Since this would also involve substantial work, the results will be published elsewhere.

In order to state our problems precisely we need to introduce the following notation and basic assumptions.

- H is a real Hilbert space identified with its dual. H plays the role of the state space. A is a linear operator on H generating a strongly continuous semigroup $\mathbf{T}(t) \in \mathcal{L}(H)$, $t \geq 0$, which describes the free dynamics of the system. The domain of A, denoted by $D(A)$, is a Hilbert space when equipped with the scalar product $\langle \cdot, \cdot \rangle_{D(A)} = \langle A\cdot, A\cdot \rangle_H + \langle \cdot, \cdot \rangle_H$. $D(A^*)^*$ is a Hilbert space defined as the dual to the domain $D(A^*)$ of A^*. ($D(A^*)$ is a Hilbert space defined analogously as $D(A)$).

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- U, the control space, is a real Hilbert space identified with its dual. $B \in \mathcal{L}(U,H)$ is the input operator.
- $\Delta_A \in \mathcal{L}(H)$ and $\Delta_B \in \mathcal{L}(U,H)$ are unknown additive perturbations of the system operator A and the input operator B, respectively.

With the pair $\{A, B\}$ we associate a *nominal control system* Σ described by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \in [0,\infty) \tag{1}$$

where the state function $x(\cdot) \in C_{loc}[0,\infty;H)$, the control $u(\cdot) \in L^2_{loc}(0,\infty;U)$. We interpret a solution of the differential equation (1) in the mild sense which means that for all $x_0 \in H$ it is given by the integral formula

$$x(t) = \mathbf{T}(t)x_0 + \int_0^t \mathbf{T}(t-r)Bu(r)\,\mathrm{d}r, \quad t \in [0,\infty) \tag{2}$$

In order to make the situation more realistic we assume that in fact the system dynamics are *uncertain* and can be modelled as the following *perturbed control system* Σ_Δ

$$\dot{x}_\Delta(t) = A_\Delta x_\Delta(t) + B_\Delta u(t), \quad x_\Delta(0) = x_0 \tag{3}$$

where $A_\Delta = A + \Delta_A$ and $B_\Delta = B + \Delta_B$.

It is a well-known result, e.g. see (Kato, 1966; Pazy,1983), that for every $\Delta_A \in \mathcal{L}(H)$ the operator $A_\Delta = A + \Delta_A$ generates a strongly continuous semigroup $\mathbf{T}_\Delta(t) \in \mathcal{L}(H), t \geq 0$, and $D(A_\Delta) = D(A)$. For every $u(\cdot) \in L^2_{loc}(0,\infty;U)$ and $x_0 \in H$ there exists a mild solution $x_\Delta(\cdot) \in C_{loc}[0,\infty;H)$ of (3) given by the integral formula

$$x_\Delta(t) = \mathbf{T}_\Delta(t)x_0 + \int_0^t \mathbf{T}_\Delta(t-r)B_\Delta u(r)\,\mathrm{d}r, \quad t \in [0,\infty) \tag{4}$$

In order to state the two *minimum energy control problems* under consideration we assume the we are given a fixed time interval $[0,\tau]$, where $\tau \in (0,\infty)$, a final state $x_1 \in H$ and a number $\alpha \in (0,\infty)$. This allows us to define the following two *sets of feasible controls*

$$\mathcal{U}_0 = \{u(\cdot) \in L^2(0,\tau;U) : x(\tau) = x_1\} \tag{5}$$

and

$$\mathcal{U}_\alpha = \{u(\cdot) \in L^2(0,\tau;U) : \|x(\tau) - x_1\|_H^2 \leq \alpha\} \tag{6}$$

Then our minimum energy control problems take the following forms.

(E) : Find a control $u_E(\cdot) \in \mathcal{U}_0$ which minimizes the energy

$$E(u) = \int_0^\tau \|u(t)\|_U^2\,\mathrm{d}t \tag{7}$$

(A) : Find a control $u_A(\cdot) \in \mathcal{U}_\alpha$ which minimizes the energy (7).

It is convenient to recall the following notions.

Definition 1. For $t \in [0,\tau)$, the set $\mathcal{R}(t) \subset H$, defined by

$$\mathcal{R}(t) = \{x \in H : x = \int_t^\tau \mathbf{T}(\tau - t)Bu(t)\,\mathrm{d}t,\; u \in L^2(0,\tau;U)\} \tag{8}$$

is called *the reachability set* of Σ on $[t,\tau]$.

Definition 2. For $t \in [0,\tau)$, a system Σ is said to be *exactly controllable* on $[t,\tau]$ if $\mathcal{R}(t) = H$ and *approximately controllable* on $[t,\tau]$ if $\overline{\mathcal{R}(t)} = H$.

The following result is obvious.

Corollary 1. *If a system Σ is exactly controllable on $[0,\tau]$, then for all $x_0, x_1 \in H$ Problem (E) possesses a unique solution $u_E(\cdot) \in \mathcal{U}_0$ and if Σ is approximately controllable on $[0,\tau]$, then for all $x_0, x_1 \in H$ and $\alpha \in (0,\infty)$ Problem (A) possesses a unique solution $u_A(\cdot) \in \mathcal{U}_\alpha$.* ■

Since perturbations $\Delta_A \in \mathcal{L}(H)$ and $\Delta_B \in \mathcal{L}(U,H)$ are unknown, it is possible to compute the controls $u_E(\cdot)$ and $u_A(\cdot)$ only for the nominal system Σ. If we now apply these controls to the perturbed system Σ_Δ, then in general $x_\Delta(\tau) \neq x(\tau)$. The main purpose of this paper is to develop techniques for estimating the distance

$$\|x_\Delta(\tau) - x(\tau)\|_H = ? \tag{9}$$

in terms of the nominal system parameters A, B (in fact, norms of some related operators) and norms $\|\Delta_A\|_{\mathcal{L}(H)}$, $\|\Delta_B\|_{\mathcal{L}(U,H)}$.

A basic estimate for (9) will be derived in Section 7 by combining auxilary estimates derived in Sections 4 and 6. Before this, in Section 2, we recall from (Emirsajlow 1989a; 1989b) explicit formulae for the controls $u_E(\cdot)$ and $u_A(\cdot)$. Then, in Section 3, we make use of these formulae to derive expressions for the difference $x_\Delta(\tau) - x(\tau)$ in both problems, i.e. Problem (E) and (A). Section 5 is devoted to differential Lyapunov equations where the notion of a composite semigroup is introduced. We complete the paper with Section 8 presenting a simple example and Section 9 containing some concluding remarks.

2. Explicit Formulae for Controls $u_E(\cdot)$ and $u_A(\cdot)$

Detailed proofs of all the results presented in this section can be found in (Emirsajlow 1989a; 1989b) and are therefore omitted.

Definition 3. An operator $M(t) \in \mathcal{L}(H)$, $t \in [0,\tau]$, defined by

$$M(t) = \int_t^\tau \mathbf{T}(\tau - r)BB^*\mathbf{T}^*(\tau - r)\,\mathrm{d}r = \int_t^\tau \mathbf{T}(r - t)BB^*\mathbf{T}^*(r - t)\,\mathrm{d}r \tag{10}$$

is called the *controllability gramian* of Σ on $[t,\tau]$.

It follows from this definition that for every $t \in [0,\tau]$ the operator $M(t) \in \mathcal{L}(H)$ is self-adjoint and non-negative. It is also easy to show that if the system Σ is exactly controllable on $[t,\tau]$, then $M(t)$, is coercive, and if Σ is approximately controllable on $[t,\tau]$ then $M(t)$ is positive. Consequently, for every $\varepsilon \in (0,\infty)$ there always exists an inverse

$$K_\varepsilon(t) = M_\varepsilon^{-1}(t) = (M(t)+\varepsilon I)^{-1} \in \mathcal{L}(H), \quad t \in [0,\tau] \tag{11}$$

which is self-adjoint and coercive, and if Σ is exactly controllable on $[t,\tau]$, then there exists an inverse

$$K(t) = M^{-1}(t) \in \mathcal{L}(H), \quad t \in [0,\tau) \tag{12}$$

which is self-adjoint and coercive.

The following results hold for Problems (E) and (A) and will be useful in the following sections.

Theorem 1. *If Σ is exactly controllable on $[0,\tau]$, then there exists a unique solution of Problem (E) given by*

$$u_E(t) = B^*\mathbf{T}^*(\tau - t)q_0 \tag{13}$$

where $q_0 \in H$ is a unique solution to the equation

$$M(0)q_0 = x_1 - \mathbf{T}(\tau)x_0 \tag{14}$$

and

$$u_E(t) = B^*\mathbf{T}^*(\tau - t)K(0)(x_1 - \mathbf{T}(\tau)x_0) \tag{15}$$

Moreover, the minimum energy is given by

$$E(u_E) = \langle M(0)q_0, q_0\rangle_H = \langle K(0)(x_1 - \mathbf{T}(\tau)x_0), x_1 - \mathbf{T}(\tau)x_0\rangle_H \tag{16}$$

■

Theorem 2. *If Σ is approximately controllable on $[0,\tau]$, then there exists a unique solution of Problem (A) given by:*

(a) *For $\alpha \in (0, \|x_1 - \mathbf{T}(\tau)x_0\|_H^2)$*

$$u_A(t) = B^*\mathbf{T}^*(\tau - t)q_\varepsilon \tag{17}$$

where $q_\varepsilon \in H$ is a unique solution to the equation

$$M_\varepsilon(0)q_\varepsilon = x_1 - \mathbf{T}(\tau)x_0 \tag{18}$$

and

$$u_A(t) = B^*\mathbf{T}^*(\tau - t)K_\varepsilon(0)(x_1 - \mathbf{T}(\tau)x_0) \tag{19}$$

where $\varepsilon \in (0,\infty)$ uniquely satisfies the condition

$$\varepsilon^2\|K_\varepsilon(0)(x_1 - \mathbf{T}(\tau)x_0)\|_H^2 = \alpha \tag{20}$$

Moreover, the minimum energy is given by

$$E(u_A) = \langle M(0)q_\varepsilon, q_\varepsilon\rangle_H = \langle K_\varepsilon(0)(x_1 - \mathbf{T}(\tau)x_0), x_1 - \mathbf{T}(\tau)x_0\rangle_H - \alpha/\varepsilon \tag{21}$$

(b) *For* $\alpha \in [\|x_1 - \mathbf{T}(\tau)x_0\|_H^2, \infty)$

$$u_A(t) \equiv 0 \tag{22}$$

■

3. Expressions for the Difference $x_\Delta(\tau) - x(\tau)$

Applying $u_E(\cdot)$ to Σ_Δ leads to the following expression for the final state

$$\begin{aligned} x_\Delta(\tau) &= \int_0^\tau \mathbf{T}_\Delta(\tau - r)B_\Delta B^*\mathbf{T}^*(\tau - r)q_0\,\mathrm{d}r + \mathbf{T}_\Delta(\tau)x_0 \\ &= \int_0^\tau \mathbf{T}_\Delta(\tau - r)B_\Delta B^*\mathbf{T}^*(\tau - r)K(0)(x_1 - \mathbf{T}(\tau)x_0)\,\mathrm{d}r + \mathbf{T}_\Delta(\tau)x_0 \end{aligned} \tag{23}$$

Hence it follows that if we define an operator $M_\Delta(t) \in \mathcal{L}(H)$, $t \in [0, \tau]$, by

$$\begin{aligned} M_\Delta(t) &= \int_t^\tau \mathbf{T}_\Delta(\tau - r)B_\Delta B^*\mathbf{T}^*(\tau - r)\,\mathrm{d}r \\ &= \int_t^\tau \mathbf{T}_\Delta(r - t)B_\Delta B^*\mathbf{T}^*(r - t)\,\mathrm{d}r \end{aligned} \tag{24}$$

then we easily obtain the following expression for the difference $x_\Delta(\tau) - x(\tau)$ in Problem (E)

$$\begin{aligned} x_\Delta(\tau) - x(\tau) &= (M_\Delta(0) - M(0))q_0 + (\mathbf{T}_\Delta(\tau) - \mathbf{T}(\tau))x_0 \\ &= (M_\Delta(0) - M(0))K(0)(x_1 - \mathbf{T}(\tau)x_0) + (\mathbf{T}_\Delta(\tau) - \mathbf{T}(\tau))x_0 \end{aligned} \tag{25}$$

In turn, in Problem (A) we obtain

$$\begin{aligned} x_\Delta(\tau) - x(\tau) &= (M_\Delta(0) - M(0))q_\varepsilon + (\mathbf{T}_\Delta(\tau) - \mathbf{T}(\tau))x_0 \\ &= (M_\Delta(0) - M(0))K_\varepsilon(0)(x_1 - \mathbf{T}(\tau)x_0) + (\mathbf{T}_\Delta(\tau) - \mathbf{T}(\tau))x_0 \end{aligned} \tag{26}$$

In the above expresssions (25) and (26) both states $x_0, x_1 \in H$ are allowed to be arbitrary. This implies that also states $q_0, q_\varepsilon \in H$ can be arbitrary. Thus, in order to estimate the norm

$$\|x_\Delta(\tau) - x(\tau)\|_H \tag{27}$$

in both cases we have to find a method of estimating the operator norms

$$\|M_\Delta(0) - M(0)\|_{\mathcal{L}(H)} \tag{28}$$

and

$$\|\mathbf{T}_\Delta(\tau) - \mathbf{T}(\tau)\|_{\mathcal{L}(H)} \tag{29}$$

In the next section we derive an estimate on (29). In Section 5 we give a useful characterization of the operators $M(\cdot)$ and $M_\Delta(\cdot)$ by means of differential Lyapunov equations and then, in Section 6, we derive an estimate for (28).

4. An Estimate for $\|\mathbf{T}_\Delta(\tau) - \mathbf{T}(\tau)\|_{\mathcal{L}(H)}$

Fix $\tau > 0$. In order to state the main result of this section we need to introduce three operators $L_\tau \in \mathcal{L}(C([0,\tau];H))$, $N_\tau \in \mathcal{L}(C([0,\tau];H),H)$, $T_\tau \in \mathcal{L}(H,C([0,\tau];H))$, defined as follows

$$(L_\tau h)(t) = \int_0^t \mathbf{T}(t-r)h(r)\,\mathrm{d}r, \quad h \in C([0,\tau];H),\ t \in [0,\tau] \tag{30}$$

$$N_\tau h = (L_\tau h)(\tau) = \int_0^\tau \mathbf{T}(\tau-r)h(r)\,\mathrm{d}r, \quad h \in C([0,\tau];H) \tag{31}$$

$$(T_\tau h)(t) = \mathbf{T}(t)h, \quad h \in H,\ t \in [0,\tau] \tag{32}$$

Unless stated otherwise, the norms of these operators will be understood in the sense of the above definitions.

Theorem 3. *If* $\Delta_A \in \mathcal{L}(H)$ *is such that*

$$\|\Delta_A\|_{\mathcal{L}(H)} < \|L_\tau\|^{-1} \tag{33}$$

then

$$\|\mathbf{T}_\Delta(\tau) - \mathbf{T}(\tau)\|_{\mathcal{L}(H)} \le \frac{\|N_\tau\|\|T_\tau\|\|\Delta_A\|_{\mathcal{L}(H)}}{1 - \|L_\tau\|\|\Delta_A\|_{\mathcal{L}(H)}} \tag{34}$$

Proof. Let us notice that

$$z(t) = \mathbf{T}(t)x_0, \quad \dot z(t) = Az(t),\ t \in [0,\infty],\ z(0) = x_0$$

and

$$z_\Delta(t) = \mathbf{T}_\Delta(t)x_0, \quad \dot z_\Delta(t) = A_\Delta z_\Delta(t),\ t \in [0,\infty],\ z_\Delta(0) = x_0$$

Hence

$$z_\Delta(t) = \mathbf{T}(t)x_0 + \int_0^t \mathbf{T}(t-r)\Delta_A z_\Delta(r)\,\mathrm{d}r, \quad t \in [0,\infty) \tag{35}$$

and making use of (30), (31), we obtain

$$z_\Delta(t) = (T_\tau x_0)(t) + (L_\tau \Delta_A z_\Delta)(t), \quad t \in [0,\tau]$$

It can be proven easily that the operator $(I - L_\tau\Delta_A) \in \mathcal{L}(C([0,\tau];H))$ is boundedly invertible for all $\Delta_A \in \mathcal{L}(H)$. This implies that

$$z_\Delta(t) = ((I - L_\tau\Delta_A)^{-1}T_\tau x_0)(t), \quad t \in [0,\tau] \tag{36}$$

It follows from (35) that

$$z_\Delta(\tau) - z(\tau) = N_\tau \Delta_A z_\Delta$$

and putting (36) into this expression gives

$$\mathbf{T}_\Delta(\tau)x_0 - \mathbf{T}(\tau)x_0 = N_\tau\Delta_A(I - L_\tau\Delta_A)^{-1}T_\tau x_0$$

Since $x_0 \in H$ can be arbitrary, we obtain

$$\mathbf{T}_\Delta(\tau) - \mathbf{T}(\tau) = N_\tau \Delta_A (I - L_\tau \Delta_A)^{-1} T_\tau \tag{37}$$

This is an exact formula for the difference $\mathbf{T}_\Delta(\tau) - \mathbf{T}(\tau)$, valid for all $\Delta_A \in \mathcal{L}(H)$. Making use of standard functional analysis results, see e.g. (Trenogin, 1980), we obtain (34). ■

5. Differential Lyapunov Equations and a Composite Semigroup

Differentiating (10) and (24) with respect to t leads to the following differential Lyapunov equations

$$\dot{M}(t)h = -AM(t)h - M(t)A^*h - BB^*h, \quad M(\tau) = 0,\ t \in [0, \tau] \tag{38}$$

and

$$\begin{aligned}\dot{M}_\Delta(t)h = & - AM_\Delta(t)h - M_\Delta(t)A^*h \\ & - B_\Delta B^*h - \Delta_A M_\Delta(t)h, \ M_\Delta(\tau) = 0,\ t \in [0, \tau]\end{aligned} \tag{39}$$

for $M(\cdot)$ and $M_\Delta(\cdot)$. Here $h \in D(A^*)$ and (38) and (39) hold in $D(A^*)^*$.

In order to prove that (38) and (39) have unique solutions $M(\cdot)$, $M_\Delta(\cdot) \in C([0,\tau];\mathcal{L}(H))$, where $\mathcal{L}(H)$ is equipped with a strong operator topology, we introduce a special type of a continuous semigroup which we call a *composite semigroup.*

A continuous *composite semigroup* $\mathcal{T}_t : \mathcal{L}(H) \to \mathcal{L}(H)$, $t \in [0, \infty)$, is defined by

$$\mathcal{T}_t X = T(t) X T^*(t), \quad X \in \mathcal{L}(H),\ t \in [0, \infty) \tag{40}$$

where $T(t) \in \mathcal{L}(H)$, $t \in [0, \infty)$, is generated by A. The *infinitesimal generator* $\mathcal{A} : \mathcal{L}(H) \supset D(\mathcal{A}) \to \mathcal{L}(H)$ of $\mathcal{T}_t$ is defined as usual, i.e. its *domain*

$$D(\mathcal{A}) = \left\{ X \in \mathcal{L}(H) : \lim_{t \to 0+} \frac{(\mathcal{T}_t X)h - Xh}{t} \right\} \tag{41}$$

where the limit exists in H for every $h \in H$, and then

$$(\mathcal{A}X)h = \lim_{t \to 0+} \frac{(\mathcal{T}_t X)h - Xh}{t} \tag{42}$$

where $X \in D(\mathcal{A})$ and $h \in H$.

It can be shown that the operator $\mathcal{A}$ satisfies

$$(\mathcal{A}X)h = AXh + XA^*h \tag{43}$$

for $X \in D(\mathcal{A})$, $h \in D(A^*)$. Using the composite semigroup we can rewrite (38), respectively (39), in the form

$$\dot{M}(t) = -\mathcal{A}M(t) - BB^*, \quad M(\tau) = 0,\ t \in [0, \tau] \tag{44}$$

respectively

$$\dot{M}_\Delta(t) = -\mathcal{A}M_\Delta(t) - \Delta_A M_\Delta(t) - B_\Delta B^*, \quad M_\Delta(\tau) = 0,\ t \in [0,\tau] \quad (45)$$

Now, standard results on linear differential equations on Banach spaces, e.g. (Pazy, 1983), imply the existence and uniqueness of a solution $M(\cdot) \in C([0,\tau]; D(\mathcal{A})) \cap C^1([0,\tau]; \mathcal{L}(H))$ of (44), where $D(\mathcal{A})$ is a Banach space equipped with the graph norm $\|X\|_{D(\mathcal{A})} = \|\mathcal{A}X\|_{\mathcal{L}(H)} + \|X\|_{\mathcal{L}(H)}$. In turn, the standard perturbation results for semigroups on Banach spaces, e.g. (Kato, 1966; Pazy, 1983), guarantee the existence and uniqueness of a solution $M_\Delta(\cdot) \in C([0,\tau]; D(\mathcal{A})) \cap C^1([0,\tau]; \mathcal{L}(H))$. In other words, the operator $\mathcal{A}_\Delta = \mathcal{A} + \Delta_A$, where $\Delta_A : \mathcal{L}(H) \to \mathcal{L}(H)$, generates a continuous semigroup $\mathcal{T}_{\Delta t} : \mathcal{L}(H) \to \mathcal{L}(H)$ and $D(\mathcal{A}_\Delta) = D(\mathcal{A})$.

6. An Estimate for $\|M_\Delta(0) - M(0)\|_{\mathcal{L}(H)}$

Using the results of Section 5 we obtain the following expressions for $M(\cdot)$ and $M_\Delta(\cdot)$, respectively,

$$M(t) = \int_t^\tau (\mathcal{T}_{\tau - r} BB^*)\,\mathrm{d}r \quad (46)$$

and

$$M_\Delta(t) = \int_t^\tau (\mathcal{T}_{\tau - r}\Delta_A M_\Delta(r))\,\mathrm{d}r + \int_t^\tau (\mathcal{T}_{\tau - r} B_\Delta B^*)\,\mathrm{d}r \quad (47)$$

where $t \in [0,\tau]$. Now, we need to introduce two operators $\mathbf{L}_\tau \in \mathcal{L}(C([0,\tau]; \mathcal{L}(H)))$, $\mathbf{N}_\tau \in \mathcal{L}(C([0,\tau]; \mathcal{L}(H)), \mathcal{L}(H))$ defined as follows

$$(\mathbf{L}_\tau X)(t) = \int_t^\tau (\mathcal{T}_{\tau - r} X(r))\,\mathrm{d}r \quad (48)$$

where $t \in [0,\tau]$ and $X(\cdot) \in C([0,\tau]; \mathcal{L}(H))$, and

$$\mathbf{N}_\tau X = (\mathbf{L}_\tau X)(0) = \int_0^\tau (\mathcal{T}_{\tau - r} X(r))\,\mathrm{d}r, \quad X(\cdot) \in C([0,\tau]; \mathcal{L}(H)) \quad (49)$$

Unless stated otherwise, the norms of $\mathbf{N}_\tau$ and $\mathbf{L}_\tau$ will be understood in the sense of the above definitions.

We are now ready to prove the following important theorem.

Theorem 4. *For every $\Delta_A \in \mathcal{L}(H)$ such that*

$$\|\Delta_A\|_{\mathcal{L}(H)} < \|\mathbf{L}_\tau\|^{-1} \quad (50)$$

and every $\Delta_B \in \mathcal{L}(U,H)$, the following estimate holds

$$\|M_\Delta(0) - M(0)\|_{\mathcal{L}(H)} \le \frac{\|\mathbf{N}_\tau\|\|\mathbf{L}_\tau BB^*\|\|\Delta_A\|_{\mathcal{L}(H)}}{1 - \|\mathbf{L}_\tau\|\|\Delta_A\|_{\mathcal{L}(H)}} + \|\mathbf{N}_\tau\|\|B^*\|\|\Delta_B\|_{\mathcal{L}(U,H)} + \frac{\|\mathbf{N}_\tau\|\|\mathbf{L}_\tau\|\|B^*\|\|\Delta_A\|_{\mathcal{L}(H)}\|\Delta_B\|_{\mathcal{L}(U,H)}}{1 - \|\mathbf{L}_\tau\|\|\Delta_A\|_{\mathcal{L}(H)}} \quad (51)$$

Proof. Using (48) we can rewrite (46) and (47) as

$$M(t) = (\mathbf{L}_\tau BB^*)(t), \quad t \in [0,\tau] \tag{52}$$

and

$$M_\Delta(t) = ((I - \mathbf{L}_\tau \Delta_A)^{-1}\mathbf{L}_\tau B_\Delta B^*)(t), \quad t \in [0,\tau] \tag{53}$$

where the bounded invertibility of $(I - \mathbf{L}_\tau \Delta_A) \in \mathcal{L}(C([0,\tau];\mathcal{L}(H)))$ can be easily justified. Hence,

$$\begin{aligned} M_\Delta(t) - M(t) &= (\mathbf{L}_\tau \Delta_A (I - \mathbf{L}_\tau \Delta_A)^{-1}\mathbf{L}_\tau B_\Delta B^*)(t) \\ &\quad + (\mathbf{L}_\tau \Delta_B B^*)(t), \quad t \in [0,\tau] \end{aligned} \tag{54}$$

and using (49), we finally get

$$\begin{aligned} M_\Delta(0) - M(0) &= \mathbf{N}_\tau \Delta_A (I - \mathbf{L}_\tau \Delta_A)^{-1}\mathbf{L}_\tau BB^* \\ &\quad + \mathbf{N}_\tau \Delta_B B^* + \mathbf{N}_\tau \Delta_A (I - \mathbf{L}_\tau \Delta_A)^{-1}\mathbf{L}_\tau \Delta_B B^* \end{aligned} \tag{55}$$

This is an exact formula for the difference $M_\Delta(0) - M(0)$, valid for all $\Delta_A \in \mathcal{L}(H)$ and $\Delta_B \in \mathcal{L}(U,H)$. Now, using the standard estimate on the operator $(I - \mathbf{L}_\tau \Delta_A)^{-1}$, see e.g. (Trenogin, 1980), we easily obtain (51). ■

7. Estimates for $||x_\Delta(\tau) - x(\tau)||_H$

Combining (25), (26) and using Theorems 3 and 4 we immediately obtain the required estimates on the distance $||x_\Delta(\tau) - x(\tau)||_H$ in both problems under consideration.

Theorem 5. *Suppose* $\Delta_A \in \mathcal{L}(H)$ *is such that*

$$||\Delta_A||_{\mathcal{L}(H)} < \min\left\{||L_\tau||^{-1}, ||\mathbf{L}_\tau||^{-1}\right\} \tag{56}$$

$\Delta_B \in \mathcal{L}(U,H)$ *and the assumptions of Theorems 1 and 2 hold for Problems (E) and (A), respectively. Then, the distance* $||x_\Delta(\tau) - x(\tau)||_H$ *in Problem (E) can be estimated as follows*

$$\begin{aligned} ||x_\Delta(\tau) - x(\tau)||_H \;\le\; & \left\{ \frac{||\mathbf{N}_\tau|| \, ||\mathbf{L}_\tau BB^*|| \, ||\Delta_A||_{\mathcal{L}(H)}}{1 - ||\mathbf{L}_\tau|| \, ||\Delta_A||_{\mathcal{L}(H)}} + ||\mathbf{N}_\tau|| \, ||B^*|| \, ||\Delta_B||_{\mathcal{L}(U,H)} \right. \\ & \left. + \frac{||\mathbf{N}_\tau|| \, ||\mathbf{L}_\tau|| \, ||B^*|| \, ||\Delta_A||_{\mathcal{L}(H)} ||\Delta_B||_{\mathcal{L}(U,H)}}{1 - ||\mathbf{L}_\tau|| \, ||\Delta_A||_{\mathcal{L}(H)}} \right\} ||q_0||_H \\ & + \frac{||N_\tau|| \, ||T_\tau|| \, ||\Delta_A||_{\mathcal{L}(H)}}{1 - ||L_\tau|| \, ||\Delta_A||_{\mathcal{L}(H)}} ||x_0||_H \end{aligned} \tag{57}$$

and in Problem (A), as follows

$$\|x_\Delta(\tau)-x(\tau)\|_H \leq \left\{\frac{\|\mathbf{N}_\tau\|\|\mathbf{L}_\tau BB^*\|\|\Delta_A\|_{\mathcal{L}(H)}}{1-\|\mathbf{L}_\tau\|\|\Delta_A\|_{\mathcal{L}(H)}}+\|\mathbf{N}_\tau\|\|B^*\|\|\Delta_B\|_{\mathcal{L}(U,H)} + \frac{\|\mathbf{N}_\tau\|\|\mathbf{L}_\tau\|\|B^*\|\|\Delta_A\|_{\mathcal{L}(H)}\|\Delta_B\|_{\mathcal{L}(U,H)}}{1-\|\mathbf{L}_\tau\|\|\Delta_A\|_{\mathcal{L}(H)}}\right\}\|q_\varepsilon\|_H + \frac{\|N_\tau\|\|T_\tau\|\|\Delta_A\|_{\mathcal{L}(H)}}{1-\|L_\tau\|\|\Delta_A\|_{\mathcal{L}(H)}}\|x_0\|_H \tag{58}$$

where we assume $q_\varepsilon = 0$ *in the case of part (b) of Theorem 2.* ■

Actual use of the estimates (57) and (58) requires computation of the operator norms $\|\mathbf{N}_\tau\|$, $\|\mathbf{L}_\tau\|$ and $\|N_\tau\|$, $\|L_\tau\|$, $\|T_\tau\|$ which is not necessarily easy. It is well-known, see e.g. (Pazy, 1983), that every strongly continuous semigroup has an exponential growth bound. Hence, there exist constants $C \geq 1$ and $\omega \in (-\infty,\infty)$ such that

$$\|\mathbf{T}(t)\|_{\mathcal{L}(H)} \leq Ce^{\omega t}, \quad t \in [0,\infty) \tag{59}$$

Using (59) we can obtain crude estimates for the above operator norms. In particular,

$$\|N_\tau\| \leq C\frac{e^{\omega\tau}-1}{\omega} \tag{60}$$

$$\|L_\tau\| \leq C\frac{e^{\omega\tau}-1}{\omega} \tag{61}$$

$$\|T_\tau\| \leq C_\tau \tag{62}$$

$$\|\mathbf{N}_\tau\| \leq C^2\frac{e^{2\omega\tau}-1}{2\omega} \tag{63}$$

$$\|\mathbf{L}_\tau\| \leq C^2\frac{e^{2\omega\tau}-1}{2\omega} \tag{64}$$

where C_τ is defined as follows

$$C_\tau = \begin{cases} C & \text{if} \quad \omega \leq 0 \\ Ce^{\omega\tau} & \text{if} \quad \omega > 0 \end{cases} \tag{65}$$

8. A Simple Example

The purpose of this section is to test on a simple scalar example how conservative estimates (57) and (58) are. In this example we make use of estimates on the operator norms derived in the previous section.

Let the nominal system Σ be described by the following scalar differential equation

$$\dot{x}(t) = ax(t) + bu(t), \quad x(0) = x_0,\ t \in [0,\infty) \tag{66}$$

where $a, b, x_0 \in \mathbb{R}^1$, and the perturbed system Σ_δ by

$$\dot{x_\delta}(t) = (a+\delta_a)x_\delta(t) + (b+\delta_b)u(t), \quad x_\delta(0) = x_0,\ t \in [0,\infty) \tag{67}$$

where $\delta_a, \delta_b \in \mathbb{R}^1$. Moreover, we assume we are given $\tau \in (0,\infty)$, $x_1 \in \mathbb{R}^1$ and $\alpha \in (0, |x_1 - e^{a\tau}x_0|)$. If $a \neq 0$ and $b \neq 0$, then Σ is exactly controllable on every interval $[0,\tau]$. In this simple case exact controllability and approximate controllability coincide.

Using Theorems 1 and 2 we obtain the following formulae for the controls $u_E(\cdot)$ and $u_A(\cdot)$

$$u_E(t) = be^{a(\tau-t)}q_0, \quad t \in [0,\tau] \tag{68}$$

where $q_0 \in \mathbb{R}^1$ is given by

$$q_0 = \frac{2a}{b^2}\,\frac{x_1 - e^{a\tau}x_0}{e^{2a\tau}-1} \tag{69}$$

and

$$u_A(t) = be^{a(\tau-t)}q_\varepsilon, \quad t \in [0,\tau] \tag{70}$$

where $q_\varepsilon \in \mathbb{R}^1$ is given by

$$\begin{aligned} q_\varepsilon &= \frac{2a}{b^2}\,\frac{x_1 - e^{a\tau}x_0 - \sqrt{\alpha}\,\mathrm{sign}(x_1 - e^{a\tau}x_0)}{e^{2a\tau}-1} \\ &= q_0 \frac{2a}{b^2}\,\frac{\sqrt{\alpha}\,\mathrm{sign}(x_1 - e^{a\tau}x_0)}{e^{2a\tau}-1} \end{aligned} \tag{71}$$

Applying u_E and u_A to Σ and Σ_δ allows us to calculate expressions for the final states $x_\delta(\tau)$ and $x(\tau)$. For Problem (E) we obtain

$$x(\tau) = x_1 \tag{72}$$

$$x_\delta(\tau) = e^{(a+\delta_a)\tau}x_0 + \frac{b(b+\delta_b)}{2a+\delta_a}(e^{(2a+\delta_a)\tau}-1)q_0 \tag{73}$$

and for Problem (A) we obtain

$$\begin{aligned} x(\tau) &= e^{a\tau}x_0 + \frac{b^2}{2a}(e^{2a\tau}-1)q_\varepsilon \\ &= x_1 - \sqrt{\alpha}\,\mathrm{sign}(x_1 - e^{a\tau}x_0) \end{aligned} \tag{74}$$

$$x_\delta(\tau) = e^{(a+\delta_a)\tau}x_0 + \frac{b(b+\delta_b)}{2a+\delta_a}(e^{(2a+\delta_a)\tau}-1)q_\varepsilon \tag{75}$$

Thus we can compute an exact value for the distance $|x_\delta(\tau) - x(\tau)|$ in both problems. In this example, C and ω, in estimate (59) are given by

$$C = 1, \quad \omega = a \tag{76}$$

and hence (see (60)–(64))

$$\|N_\tau\| \le \frac{e^{a\tau}-1}{a} \tag{77}$$

$$\|L_\tau\| \le \frac{e^{a\tau}-1}{a} \tag{78}$$

$$\|T_\tau\| \le C_\tau \tag{79}$$

$$\|\mathbf{N}_\tau\| \le \frac{e^{2a\tau}-1}{2a} \tag{80}$$

$$\|\mathbf{L}_\tau\| \le \frac{e^{2a\tau}-1}{2a} \tag{81}$$

where C_τ is defined as follows

$$C_\tau = \begin{cases} 1 & \text{if} \quad a<0 \\ e^{a\tau} & \text{if} \quad a>0 \end{cases} \tag{82}$$

Since in this case

$$\|\mathbf{L}_\tau BB^*\| = \frac{b^2}{2a}(e^{2a\tau}-1) \tag{83}$$

we can use formulae (57) and (58) to estimate the distance $|x_\delta(\tau)-x(\tau)|$ in Problems (E) and (A) whenever $\delta_a \in \mathbb{R}^1$ satisfies

$$|\delta_a| < \min\left\{\frac{a}{e^{a\tau}-1}, \frac{2a}{e^{2a\tau}-1}\right\} \tag{84}$$

Below we present results of computations performed for $a=-1$, $b=1$ $\tau=1$, $x_0=1$, $x_1=0$, $\alpha=0.01$ and three combinations of δ_a and δ_b satisfying the same bounds $|\delta_a|\le 0.05$ and $|\delta_b|\le 0.05$.

- $\delta_a = 0.05$, $\delta_b = 0.05$,

  (E): $|x_\delta(\tau)-x(\tau)| = 0.0063$, $\mathcal{E} = 0.0513$
  (A): $|x_\delta(\tau)-x(\tau)| = 0.0006$, $\mathcal{E} = 0.0430$

- $\delta_a = 0.05$, $\delta_b = -0.05$,

  (E): $|x_\delta(\tau)-x(\tau)| = 0.0312$, $\mathcal{E} = 0.0513$
  (A): $|x_\delta(\tau)-x(\tau)| = 0.0278$, $\mathcal{E} = 0.0430$

- $\delta_a = -0.05$, $\delta_b = 0.05$,

  (E): $|x_\delta(\tau)-x(\tau)| = 0.0298$, $\mathcal{E} = 0.0513$
  (A): $|x_\delta(\tau)-x(\tau)| = 0.0266$, $\mathcal{E} = 0.0430$

In the above expressions, $\mathcal{E}$-s denote estimates on the distance $|x_\delta(\tau)-x(\tau)|$ provided by Theorem 5, i.e. for the worst case situation. It follows from these computations that the estimates (57) and (58) are reasonably tight and may be of practical use.

9. Concluding Remarks

It was already mentioned in Section 7 that the actual use of estimates (57) and (58) requires computation of the operator norms $\|\mathbf{N}_\tau\|$, $\|\mathbf{L}_\tau\|$ and $\|N_\tau\|$, $\|L_\tau\|$, $\|T_\tau\|$. The crude estimates on these norms presented in Section 7 involve constants C and ω obtained from the exponential bound (59). This can be regarded as a disadvantage. Alternatively, we can define these operators in different spaces by replacing the space $C([0,\tau];\mathcal{L}(H))$ by $L^2(0,\tau;\mathcal{L}(H))$ and then trying to compute the resulting operator norms using parametrized optimization problems. Furthermore, practical usefulness of estimates (57) and (58) should be tested on examples which are more representative of infinite-dimensional systems than our simple scalar example.

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