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METHODS FOR COMPUTATION OF SOLUTIONS TO REGULAR DISCRETE-TIME LINEAR SYSTEMS

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Two new methods for the computation of solutions to regular discrete-time linear systems are presented. The first of them is an extension of the Dias-Mesquista method for regular discrete-time linear systems. The other is based on an expansion in a series of the inverse matrix $[Ez - A]^{-1}$. The methods are compared with the Weierstrass-Kronecker decomposition method and the Drazin inverse method. Relationships between the coefficient matrices of the four methods are established. A new mixed method is presented.

1. Introduction

Generalised (descriptor, singular) continuous-time and discrete-time linear systems have been considered in many papers and books (see References). An interesting survey of regular (singular) linear systems has been given by Lewis (1986).

Consider a discrete-time linear system described by the equation

$$Ex_{i+1} = Ax_i + Bu_i, \qquad i = 0, 1, \ldots \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the local semistate vector, $u_i \in \mathbb{R}^m$ is the input vector and E, A, B are real matrices of appropriate dimensions. It is assumed that $\det E = 0$ and

$$\det[Ez - A] \neq 0 \tag{2}$$

for some $z \in \mathbb{C}$ (the field of complex numbers).

System (1) is called regular if (2) holds and it is called standard if E is equal to the identity matrix. It is well-known (Aplevich, 1991; Campbell, 1976; Dai, 1989; Gantmacher, 1959; Kaczorek, 1993; Lewis, 1986; Wonham, 1979) that if (2) holds, then eqn. (1) has a unique solution for any input sequence $\{u_i\}$ and admissible initial conditions x_0.

Four different methods of finding the solution x_i to (1) will be presented and a comparative study of them will be given. First, the method based on the Weierstrass-Kronecker decomposition of the regular pencil $[Ez - A]$ will be presented. Next, the method based on the Drazin inverse (Campbell *et al.*, 1976; Campbell, 1980; Gantmacher, 1959; Kaczorek, 1993) will be considered. The third method will be an

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extension for discrete-time linear systems of the method given by Dias and Mesquista (1990). The fourth method will be based on expansion in a series of the inverse matrix $[Ez - A]^{-1}$.

2. Weierstrass-Kronecker Decomposition Method

It is well-known (Aplevich, 1991; Gantmacher, 1959; Kaczorek, 1993) that if (2) holds, then there exist non-singular matrices $P, Q \in \mathbb{R}^{n\times n}$ such that

$$P[Ez - A]Q = \begin{bmatrix} I_{n_1}z - A_1 & 0 \\ 0 & Nz - I_{n_2} \end{bmatrix} \tag{3}$$

where I_k is the $k\times k$ identity matrix, n_1 is the degree of $\det[Ez-A]$, $A_1 \in \mathbb{R}^{n_1\times n_1}$ and $N \in \mathbb{R}^{n_2\times n_2}$ $(n_2 = n - n_1)$ is a nilpotent matrix with index q, i.e. $N^{q-1} \neq 0$ and $N^q = 0$.

There exist several methods for computing the matrices P and Q (Aplevich, 1991; Dai, 1989; Gantmacher, 1959; Kaczorek, 1993; Lewis, 1986). Among them one which is worth recommending is as follows. Let z_i be the i-th root of $\det[Ez-A] = 0$ and

$$m_i := \dim \operatorname{Ker}[Ez_i - A] \tag{4}$$

where Ker denotes the kernel (null space).

Compute the eigenvectors v_{ij}^1 defined by

$$[Ez_i - A]v_{ij}^1 = 0 \qquad \text{for} \quad j = 1, \dots, m_i \tag{5}$$

and next v_{ij}^{k+1} from

$$[Ez_i - A]v_{ij}^{k+1} = -Ev_{ij}^k \qquad \text{for} \quad k \geq 1 \tag{6}$$

Let $m_\infty := \dim \ker E = n - \operatorname{rank} E$. Compute the infinite eigenvectors $v_{\infty j}^1$ defined by

$$Ev_{\infty j}^1 = 0 \qquad \text{for} \quad j = 1, \dots, m_\infty \tag{7}$$

and next $v_{\infty j}^{k+1}$ from

$$Ev_{\infty j}^{k+1} = Av_{\infty j}^k \qquad \text{for} \quad k \geq 1 \tag{8}$$

Arrange the eigenvectors as the columns of the matrices

$$Q = \begin{bmatrix} v_{ij}^k & v_{\infty j}^k \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} Ev_{ij}^k & Av_{\infty j}^k \end{bmatrix} \tag{9}$$

Using (5)–(8) it is easy to check that

$$[Ez - A]\begin{bmatrix} v_{ij}^k & v_{\infty j}^k \end{bmatrix} = \begin{bmatrix} Ev_{ij}^k & Av_{\infty j}^k \end{bmatrix} \begin{bmatrix} I_{n_1}z - A_1 & 0 \\ 0 & Nz - I_{n_2} \end{bmatrix} \tag{10}$$

Premultiplying (10) by $P = \left[Ev_{ij}^k \quad Av_{\infty j}^k\right]^{-1}$ we obtain (3) with P, Q defined by (9). Premultiplying (1) by P, defining

$$\begin{bmatrix} x_i^1 \\ x_i^2 \end{bmatrix} := Q^{-1} x_i, \qquad \dim x_i^1 = n_1, \qquad \dim x_i^2 = n_2$$

and using (3) we obtain

$$x_{i+1}^1 = A_1 x_i^1 + B_1 u_i, \qquad i = 0, 1, \ldots \tag{11}$$

and

$$N x_{i+1}^2 = x_i^2 + B_2 u_i, \qquad i = 0, 1, \ldots \tag{12}$$

where

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} := PB, \qquad B_1 \in \mathbb{R}^{n_1 \times m}, \qquad B_2 \in \mathbb{R}^{n_2 \times m}$$

The solutions x_i^1, x_i^2 to (11) and (12) are respectively given by

$$x_i^1 = A_1^i x_0^1 + \sum_{k=0}^{i-1} A_1^{i-k-1} B_1 u_k \tag{13}$$

and

$$x_i^2 = -\sum_{k=0}^{q-1} N^k B_2 u_{i+k} \tag{14}$$

Therefore the solution x_i to (1) is given by

$$x_i = Q \begin{bmatrix} A_1^i Q_1 x_0 + \sum\limits_{k=0}^{i-1} A_1^{i-k-1} B_1 u_k \\ -\sum\limits_{k=0}^{q-1} N^k B_2 u_{i+k} \end{bmatrix} \tag{15}$$

where $Q^{-1} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, Q_1 \in \mathbb{R}^{n_1 \times n}$.

From (13) and (14) it follows that the set of admissible initial conditions x_0^1 and x_0^2 is given by

$$S_{x_0^1, x_0^2} := \left\{ \mathbb{R}^{n_1} \oplus \operatorname{Im}\left[B_2, NB_2, \ldots, N^{q-1} B_2\right] \right\} \tag{16}$$

where Im denotes the image (range). The method will be illustrated by two examples with a non-singular and singular matrix A.

Example 1. Consider eqn. (1) with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{17}$$

In this case

$$\det[Ez - A] = \begin{vmatrix} z-1 & 0 & -1 \\ 0 & z-1 & 0 \\ -1 & 0 & 0 \end{vmatrix} = 1 - z = 0$$

and $n_1 = 1$, $n_2 = 2$, $z_1 = 1$, $m_1 = \dim \ker[Ez_1 - A] = 1$.

Using (5), (7) and (8) we obtain

$$[Ez_1 - A]v_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} v_1 = 0, \quad v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad Ev_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$Ev_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_2 = 0, \qquad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad Av_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$Ev_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_3 = Av_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad Av_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Then from (9) and (3) we get

$$Q = [v_1\, v_2\, v_3] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad P^{-1} = [Ev_1, Av_2, Av_3] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$P[Ez - A]Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z-1 & 0 & -1 \\ 0 & z-1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} z-1 & 0 & 0 \\ 0 & -1 & z \\ 0 & 0 & -1 \end{bmatrix}$$

Therefore

$$A_1 = 1, \quad N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad q = 2, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and from (15) we have

$$x_i = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_0^1 \\ -u_{i+1} \\ -u_i \end{bmatrix} = \begin{bmatrix} -u_i \\ x_0^1 \\ -u_{i+1} \end{bmatrix} \qquad i = 0, 1, \dots \tag{18}$$

Example 2. Consider eqn. (1) with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{19}$$

In this case

$$\det[Ez - A] = \begin{vmatrix} z-1 & 0 & -1 \\ 0 & z-1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = z(z-1) = 0$$

and $n_1 = 2$, $n_2 = 1$, $z_1 = 1$, $z_2 = 0$, $m_1 = \dim\ker[Ez_1 - A] = 1$.

In a similar way as in Example 1 we compute

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad Ev_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [Ez_2 - A]v_2 = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} v_2 = 0$$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad Ev_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$Ev_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_3 = 0, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad Av_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and

$$Q = [v_1\, v_2\, v_3] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad P^{-1} = [Ev_1, Ev_2, Av_3] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

From (9) and (3) we have

$$P[Ez - A]Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z-1 & 0 & -1 \\ 0 & z-1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} z-1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Therefore

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad N = 0, \quad q = 1, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB = \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$$

and from (15) we obtain

$$x_i = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ 2u_{i-1} \\ u_i \end{bmatrix} = \begin{bmatrix} 2u_{i-1} \\ a \\ u_i - 2u_{i-1} \end{bmatrix} \tag{20}$$

where a is any real number (a is equal to the first component of $x_0^1 = [x_{20}\, x_{10}]^T$, T denotes the transposition).

Using (16) it is easy to check that the set of admissible initial conditions x_0 in this case is given by

$$S_{x_0} = Q\Big\{\mathbb{R}^{n_1} \oplus \mathrm{Im}[B_2, NB_2, \ldots, N^{q-1}B_2]\Big\} = \mathbb{R}^3 \tag{21}$$

3. Drazin Inverse Method

The smallest non-negative integer q is called the index of A if $\operatorname{rank} A^q = \operatorname{rank} A^{q+1}$. A matrix A^D is called the Drazin inverse of a square matrix if: i) $AA^D = A^D A$, ii) $A^D A A^D = A^D$, iii) $A^D A^{q+1} = A^q$, where q is the index of A (Campbell *et al.*, 1976; Kaczorek, 1993). The Drazin inverse A^D of a square matrix A always exists and is unique. If $\det A \neq 0$, then $A^D = A^{-1}$, where A^{-1} is the classical inverse of A. There exist several methods for computing A^D of A (Campbell *et al.*, 1976; Kaczorek, 1993). Two of them will be presented here.

The first method is based on the factorisation VM of A^q (Kaczorek, 1993). Let $A \in \mathbb{R}^{n \times m}$ and

$$A^q = VM^T, \quad V \in \mathbb{R}^{n \times r}, \quad M^T \in \mathbb{R}^{r \times n} \tag{22}$$

where $\ker V = \{0\}$ and $M^T M = I_r$, $r = \operatorname{rank} A^q$. The Drazin inverse of A is given by (Kaczorek, 1993)

$$A^D = V[M^T A V]^{-1} M^T \tag{23}$$

Example 3. Find the Drazin inverse of the matrix

$$E = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} \tag{24}$$

The index of (24) is equal to $q = 1$ since $r = \operatorname{rank} E^2 = 2$ and by (22) we have

$$E = VM^T \quad \text{for} \quad V = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ -\frac{1}{2} & 0 \end{bmatrix}, \quad M^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Using (23) we obtain

$$E^D = V[M^T E V]^{-1} M^T$$

$$= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix} \tag{25}$$

The second method is based on the following alhorithm (Campbell and Meyer, 1979):

Step 1. Set $S_0 = I_n$ and compute recursively $S_j = AS_{j-1} + \beta_{n-j} I_n$, $\beta_{n-j} = -\frac{1}{j}\operatorname{tr}[AS_{j-1}]$ (tr denotes the trace) until some $S_i = 0$ but $S_{i-1} \neq 0$.

Step 2. Let k be a number such that $\beta_{n-k} \neq 0$ and $\beta_{n-k-1} = \beta_{n-k-2} = \ldots = \beta_{n-i-1} = 0$.

Step 3. Let $l := n - k$ and compute S_{k-1}^{l+1}.

Step 4. Compute

$$A^D = \frac{(-1)^{l+1}}{\beta_l^{l+1}} A^l S_{k-1}^{l+1} \tag{26}$$

Example 4. Using the above algorithm compute E^D for (24).

Step 1. $\beta_2 = -\operatorname{tr} E = -\frac{3}{2}$, $\quad S_1 = ES_0 + I_n\beta_2 = E + I_3(-\frac{3}{2}) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{3}{2} \end{bmatrix}$,

$\beta_1 = -\frac{1}{2}\operatorname{tr}[ES_1] = \frac{1}{2}$, $\quad S_2 = ES_1 + I_n\beta_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$,

$\beta_0 = -\frac{1}{3}\operatorname{tr}[ES_2] = 0.$

Step 2. $k = 2$.

Step 3. $l = n - k = 1$ and $S_{k-1}^{l+1} = S_1^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ \frac{5}{4} & 0 & \frac{9}{4} \end{bmatrix}$ (27)

Step 4. Using (26) and (27) we obtain

$$E^D = \frac{(-1)^{l+1}}{\beta_1^{l+1}} E^l S_{k-1}^{l+1} = \frac{(-1)^2}{\beta_1^2} ES_1^2$$

$$= \frac{1}{(\frac{1}{2})^2} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ \frac{5}{4} & 0 & \frac{9}{4} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

If (2) holds, then there exists a scalar $c \in \mathbb{C}$ such that

$$\det[Ec - A] \neq 0$$

and we may find

$$\overline{E} := [Ec - A]^{-1}E \quad \text{and} \quad \overline{A} := [Ec - A]^{-1}A \tag{28}$$

It is easy to show that $\overline{E}\,\overline{A} = \overline{AE}$ and $\ker\overline{A} \cap \ker\overline{E} = \{0\}$ (Campbell *et al.*, 1976; Kaczorek, 1993).

Premultiplying (1) by $[Ec - A]^{-1}$ and using (28) we obtain

$$\overline{E}x_{i+1} = \overline{A}x_i + \overline{B}u_i, \qquad i = 0, 1, \ldots \tag{29}$$

where

$$\overline{B} := [Ec - A]^{-1}B \tag{30}$$

The solution x_i to (29) (and also to (1)) is given by (Campbell *et al.*, 1976; Kaczorek, 1993)

$$x_i = (\overline{E}^D\overline{A})^i\overline{E}^D\overline{E}x_0 + \sum_{k=0}^{i-1}\overline{E}^D(\overline{E}^D\overline{A})^{i-k-1}\overline{B}u_k$$

$$+(\overline{EE}^D - I_n)\sum_{k=0}^{q-1}(\overline{EA}^D)^k\overline{A}^D\overline{B}u_{i+k} \tag{31}$$

where q is the index of $\overline{E}(E)$.

The set of admissible initial conditions x_0 is given by

$$S_{X_0} := \mathrm{Im}[H_0, H_1, \ldots, H_q] \tag{32}$$

where

$$H_k := \begin{cases} (I_n - \overline{EE}^D)(\overline{EA}^D)^k\overline{A}^D\overline{B} & \text{for } k = 0, 1, \ldots q-1 \\ \overline{EE}^D & \text{for } k = q \end{cases} \tag{33}$$

Example 5. Find the solution x_i to (1) with (19).
Choosing $c = 2$ and using (28) and (30) we obtain

$$[Ec - A] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad [Ec - A]^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

and

$$\overline{E} := [Ec - A]^{-1}E = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix}$$

$$\overline{A} = [Ec - A]^{-1}A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \quad \overline{B} = [Ec - A]^{-1}B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Taking into account the result of Example 3 and that

$$\overline{A}^D = \overline{A}\,\overline{A}^D\overline{B} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \overline{E}^D\overline{A} = \overline{E}\overline{A}^D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overline{E}\overline{E}^D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

from (31) we obtain

$$x_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^i \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} x_0$$

$$+ \sum_{k=0}^{i-1} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{i-k-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_k$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix} \left(\sum_{k=0}^{1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^k \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} u_{i+k} \right)$$

$$= \begin{bmatrix} 2u_{i-1} \\ a \\ u_i - 2u_{i-1} \end{bmatrix} \tag{34}$$

where a is any real number (a is the second component of x_0).

Through (22) and (33) the set of admissible initial conditions is given by

$$S_{x_0} = \operatorname{Im}[H_0, H_1]$$

$$= \operatorname{Im}\left[(I_n - \overline{E}\overline{E}^D)\overline{A}^D\overline{B}, \overline{E}\overline{E}^D\right] = \operatorname{Im} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \mathbb{R}^3 \tag{35}$$

Note that (34) and (35) are the same as (20) and (21), respectively.

4. Extension of the Dias and Mesquista Method for Discrete-Time Systems

Without loss of generality it is assumed that

$$E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{36}$$

where $r = \operatorname{rank} E$.

Consider the family of (A, E)-invariant subspaces defined by (Armentano, 1984; Dias and Mesquista, 1990; Kaczorek, 1993; Wonham, 1979)

$$\boldsymbol{Z} := \left\{ \mathcal{Z} : A\mathcal{Z} \subset E\mathcal{Z}; \mathcal{Z} \subset \mathbb{R}^n \right\} \tag{37}$$

The supremal element $\mathcal{Z}^* := \sup \boldsymbol{Z}$ can be computed in a finite number of steps $\mu \leq n$ by the algorithm

$$\mathcal{Z}_k = A^{-1}(E\mathcal{Z}_{k-1}), \quad k = 1, \ldots, n \quad \mathcal{Z}_0 = \mathbb{R}^n$$

$$\mathcal{Z}^* = Z_\mu = Z_{\mu+1}$$

where $A^{-1}\mathcal{Z}_k := \{x \in \mathbb{R}^n : Ax \in \mathcal{Z}_k\}$. (38)

Let the columns of V form a basis for $\mathcal{Z}^*, \mathcal{Z}^* = \operatorname{Im} V$ and

$$x_i = V z_i + \sum_{k=-i}^{p} L_k u_{i+k} \tag{39}$$

where z_i, L_k and p will be defined later. Substituting (39) and (36) into (1) we obtain

$$\begin{aligned} V_1 z_{i+1} = {} & [A_1\, A_2] V z_i + \Big([A_1\, A_2] L_0 + B_1 - [I_r\, 0] L_{-1} \Big) u_i \\ & + \sum_{k=1}^{p} \Big([A_1\, A_2] L_k - [I_r\, 0] L_{k-1} \Big) u_{i+k} - [I_r\, 0] L_p u_{i+p+1} \\ & + \sum_{k=1}^{i-1} \Big([A_1\, A_2] L_{-k} - [I_r\, 0] L_{-k-1} \Big) u_{i-k} + [A_1\, A_2] L_{-t} u_0 \end{aligned} \tag{40}$$

$$\begin{aligned} 0 = {} & [A_3\, A_4] V z_i + \Big([A_3\, A_4] L_0 + B_2 \Big) u_i \\ & + \sum_{k=1}^{p} \Big([A_3\, A_4] \Big) L_k u_{i+k} + \sum_{k=1}^{i} [A_3\, A_4] L_{-k} u_{i-k} \end{aligned} \tag{41}$$

where

$$V := \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \qquad V_1 \in \mathbb{R}^{r \times l}, \qquad l = \dim \mathcal{Z}^* \tag{42}$$

It can be easily shown that $l = \deg \det[Ez - A] \le r$.

Define $\overline{Q} := \begin{bmatrix} \overline{Q}_1 \\ \overline{Q}_1 \end{bmatrix}$, $\overline{Q}_1 \in \mathbb{R}^{l \times r}$, $\overline{Q}_2 \in \mathbb{R}^{(r-l) \times r}$ such that $\det \overline{Q} \neq 0$

and

$$\overline{Q} V_1 = \begin{bmatrix} I_l \\ 0 \end{bmatrix} \tag{43}$$

Premultiplying (40) by $\overline{Q}$ and using (43) we obtain

$$z_{i+1} = \overline{A}_1 z_i, \qquad \overline{A}_1 := \overline{Q}_1 [A_1 \, A_2] V \tag{44}$$

if

$$\overline{Q}_2 [A_1 \, A_2] V z_i = 0 \qquad \text{for an arbitrary } z_i \tag{45}$$

$$[A_1 \, A_2] L_0 + B_1 - [I_r \, 0] L_{-1} = 0 \tag{46}$$

$$[A_1 \, A_2] L_k - [I_r \, 0] L_{k-1} = 0 \qquad \text{for} \quad k = 1, ..., p \tag{47}$$

$$[A_1 \, A_2] L_{-k} - [I_r \, 0] L_{-k-1} = 0 \qquad \text{for} \quad k = 1, ..., i-1 \tag{48}$$

$$[I_r \, 0] L_p = 0, \quad [A_1 \, A_2] L_{-i} = 0 \tag{49}$$

since $\det \overline{Q} \neq 0$.

Note that (41) is satisfied if

$$[A_3 \, A_4] V z_i = 0 \qquad \text{for an arbitrary } z_i \tag{50}$$

$$[A_3 \, A_4] L_0 + B_2 = 0 \tag{51}$$

$$[A_3 \, A_4] L_k = 0 \qquad \text{for} \quad k = 1, ..., p \tag{52}$$

$$[A_3 \, A_4] L_{-k} = 0 \qquad \text{for} \quad k = 1, ..., i \tag{53}$$

In a similar way as in (Dias and Mesquista, 1990) it can be shown that

$$\begin{bmatrix} Q_2 [A_1 \, A_2] \\ A_3 A_4 \end{bmatrix} V = 0 \tag{54}$$

since $V z_i \in \mathcal{Z}^*$. Therefore (45) and (50) are satisfied for an arbitrary z_i.

Two cases will be considered for $\det A \neq 0$ and $\det A = 0$.

Case $\det A \neq 0$.
It will be shown that if $\det A \neq 0$, then it can be assumed in (39) that $k = -i = 0$. Assuming $k = -i = 0$ and $L_{-1} = 0$ from (46) and (51) we obtain

$$L_0 = -A^{-1}B \tag{55}$$

Next from (47), (49), and (52) we have

$$L_k = A^{-1}EL_{k-1}, \qquad k = 1, ..., p \tag{56}$$

and

$$EL_p = 0 \tag{57}$$

Using (55) and (56) we may compute $L_0, L_1, ..., L_p$ and the procedure stops when (57) is satisfied.

The solution z_i of (44) has the form

$$z_i = \overline{A}_1^{\,i} z_0 \tag{58}$$

Substitution of (58) into (39) for $k = -i = 0$ yields

$$x_i = V\overline{A}_1^{\,i} z_0 + \sum_{k=0}^{p} L_k u_{i+k} \tag{59}$$

Therefore if $\det A \neq 0$, then the solution x_i to (1) with (36) is given by (59).

Remark. Note that if $\operatorname{rank} A = \operatorname{rank}[A, B]$ and $\operatorname{rank}[A_1, A_2] = r$, it can be also assumed that $k = -i = 0$ in (39), since the equations $AL_0 = -B$ and $AL_k = EL_{k-1}$ have solutions L_0 and L_k for $k = 1, ..., p$.

Example 6. Find the solution x_i to (1) with (17).
Using (38) we obtain

$$Z_1 = \overline{A}^1(E\mathbb{R}^3) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \left(\operatorname{Im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = \operatorname{Im} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Z_2 = \overline{A}^1(EZ_1) = \operatorname{Im} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad Z_3 = \overline{A}^1(EZ_2) = \operatorname{Im} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\mathcal{Z}^* = \mathrm{Im}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; V = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; V_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; \overline{Q} = \begin{bmatrix} \overline{Q}_1 \\ \overline{Q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

From (55), (56), and (57) we have

$$L_0 = -\overline{A}^{1}B = -\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$L_1 = \overline{A}^{1}EL_0 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1}\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

and $EL_1 = 0$, $p = 1$.

Using (59) we obtain $\overline{A}_1 = Q_1[A_1\,A_2]V = [0\,1]\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1$ and

$$x_i = V\overline{A}_1^i z_0 + \sum_{k=0}^{1} L_k u_{i+k} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} z_0 + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} u_{i+1} = \begin{bmatrix} -u_i \\ z_0 \\ -u_{i+1} \end{bmatrix} \tag{60}$$

Note that (60) agrees with (18).

Case $\det A = 0$.
Note that if (2) holds, then for (36) the matrix $[A_3\,A_4]$ has a full row rank and from (51) we have

$$L_0 = -\begin{bmatrix} A_3^T \\ A_4^T \end{bmatrix}[A_3A_3^T + A_4A_4^T]^{-1}B_2 + \left(I_n - \begin{bmatrix} A_3^T \\ A_4^T \end{bmatrix}[A_3A_3^T + A_4A_4^T]^{-1}[A_3A_4]\right)K_1 \tag{61}$$

where K_1 is an arbitrary matrix.

The matrix K_1 is chosen so that $[I_r\;0]L_0 = 0$. If there exists K_1 such that $[I_r\;0]L_0 = 0$, then $p = 0$. Next, from (46) we may compute

$$L_{-1} = \begin{bmatrix} I_r \\ 0 \end{bmatrix}\left([A_1\,A_2]L_0 + B_1\right) + \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}K_2 \tag{62}$$

The matrix K_2 is chosen so that $[A_1\,A_2]L_{-1} = 0$. If there exists K_2 such that $[A_1\,A_2]L_{-1} = 0$, then $i = 1$.

Combining the equations

$$\overline{Q}_2([A_1\,A_2]L_k - [I_r\;0]L_{k-1}) = 0, \qquad k = 1, ..., p$$

$$\overline{Q}_2([A_1\,A_2]L_{-k} - [I_r\;0]L_{-k-1}) = 0, \qquad k = 1, ..., i-1$$

with (52) and (53), respectively we obtain

$$HL_k = \overline{E}L_{k-1}, \qquad k = 1, ...p \tag{63}$$

and

$$HL_{-k} = \overline{E}L_{-k-1}, \qquad k = 1, ..., i-1 \tag{64}$$

where

$$H := \begin{bmatrix} \overline{Q}_2[A_1\,A_2] \\ A_3A_4 \end{bmatrix} = \begin{bmatrix} \overline{Q}_2 & 0 \\ 0 & I_{n-r} \end{bmatrix} A$$

$$\overline{E} := \begin{bmatrix} \overline{Q}_2 & 0 \\ 0 & I_{n-r} \end{bmatrix} E = \begin{bmatrix} \overline{Q}_2[I_r\;0] \\ 0 \quad 0 \end{bmatrix} \tag{65}$$

In a similar way as in (Dias and Mesquista, 1990) it can be shown that matrix (65) has a full row rank. Therefore from (63),(64) we have

$$L_k = H_R\overline{E}L_{k-1}, \qquad k = 1, ..., p \tag{66}$$

and

$$L_{-k} = H_R\overline{E}L_{-k-1}, \qquad k = 1, ..., i-1 \tag{67}$$

where

$$H_R := H^T[HH^T]^{-1}$$

The Moore-Penrose generalized inverse H^g can also be used (Dias and Mesquista, 1990). Using (66), (67) we may compute $L_k(L_{-k})$ for $k = 1, ..., p\,(k = 1, ..., i-1)$. The algorithm stops when (49) is satisfied.

Example 7. Find the solution x_i to (1) with(19).

Using (38) we obtain

$$Z_1 = \overline{A}^1(E\,\mathrm{IR}^3) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}^{-1}\left(\mathrm{Im}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \mathrm{Im}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Z_2 = \overline{A}^1(EZ_1) = \operatorname{Im}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$\mathcal{Z}^* = \operatorname{Im}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \overline{Q} = \overline{Q}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

From (61) we have

$$L_0 = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.5 & 0 & -0.5 \\ 0 & 1 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}\begin{bmatrix} k_{11} \\ k_{12} \\ k_{13} \end{bmatrix} = \begin{bmatrix} 0.5(1+k_{11}-k_{13}) \\ k_{12} \\ 0.5(1-k_{11}+k_{13}) \end{bmatrix}$$

and

$$[I_r\ 0]L_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0.5(1+k_1-k_3) \\ k_2 \\ 0.5(1-k_1+k_3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

gives $k_1 = -1,\ k_2 = k_3 = 0$. Hence

$$L_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad p = 0$$

Using (62) we obtain

$$L_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} k_{21} \\ k_{22} \\ k_{23} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ k_{23} \end{bmatrix}$$

and

$$[A_1\ A_2]L_{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ k_{23} \end{bmatrix} = 0$$

gives $k_{23} = -2$.

Hence

$$L_{-1} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad t = 1$$

Thus from (39) we obtain

$$x_i = V\overline{A}_1^i z_0 + \sum_{k=-1}^{0} L_k u_{i+k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^i \begin{bmatrix} z_{10} \\ z_{20} \end{bmatrix}$$

$$+ \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} u_{i-1} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_i = \begin{bmatrix} 2u_{i-1} \\ z_{20} \\ u_i - 2u_{i-1} \end{bmatrix} \tag{68}$$

Note that (68) agrees with (20).

5. Method of Expansion in a Series

Let $X(z)$ be the z-transform of x_i defined by

$$X(z) := \sum_{i=0}^{\infty} x_i z^{-i} \tag{69}$$

Using the z-transformation eqn. (1) can be written in the form

$$[Ez - A]X(z) = zEx_0 + BU(z) \tag{70}$$

where $U(z)$ is the z-transform of u_i.

From (70) we have

$$X(z) = [Ez - A]^{-1} zEx_0 + [Ez - A]^{-1} BU(z)z \tag{71}$$

Note that if the degree of $\det[Ez-A]$ is less than $\operatorname{rank} E$, then the matrix $[Ez-A]^{-1}$ may be improper and it can be decomposed into a polynomial part $P(z)$ and a strictly proper part $T_{sp}(z)$

$$[Ez - A]^{-1} = P(z) + T_{sp}(z) \tag{72}$$

where

$$P(z) = \sum_{i=0}^{p} P_i z^i \tag{73}$$

$$T_{sp}(z) = \sum_{i=0}^{\infty} T_i z^{-(i+1)} \tag{74}$$

$$p \le \operatorname{rank} E - \deg \det[Ez - A]$$

Substituting (72)–(74) into (71) we obtain

$$X(z) = \sum_{i=0}^{p} P_i E x_0 z^{i+1} + \sum_{i=0}^{\infty} T_i E x_0 z^{-1} + \sum_{i=0}^{p} P_i B z^i U(z) + \sum_{i=0}^{\infty} T_i B z^{-(i+1)} U(z) \tag{75}$$

The inverse z-transformation of (75) yields

$$x_i = T_i E x_{0i} + \sum_{k=0}^{p} P_k B u_{i+k} + \sum_{k=0}^{i-1} T_k B u_{i-k-1} \qquad i \ge 0 \tag{76}$$

Note that all terms with positive powers z of (75) have been neglected since we are interested in the solution for $i \ge 0$. Therefore the solution x_i to (1) is given by (76).

Example 8. Find the solution x_i to eqn. (1) with (19).

In this case

$$[Ez - A]^{-1} = \begin{bmatrix} z-1 & 0 & -1 \\ 0 & z-1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} z^{-1} & 0 & z^{-1} \\ 0 & (z-1)^{-1} & 0 \\ -z^{-1} & 0 & 1 - z^{-1} \end{bmatrix} = P_0 + T_{sp}(z)$$

where

$$P_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_{sp}(z) = \begin{bmatrix} z^{-1} & 0 & z^{-1} \\ 0 & (z-1)^{-1} & 0 \\ -z^{-1} & 0 & -z^{-1} \end{bmatrix} = \sum_{k=0}^{\infty} T_k z^{-(k+1)}$$

$$T_0 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad T_k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{for} \quad k = 1, 2, \ldots$$

Using (76) we obtain

$$x_i = T_iEx_0 + \sum_{k=0}^{0} P_kBu_{k+1} + \sum_{k=0}^{i-1} T_kBu_{i-k-1}$$

$$= \begin{bmatrix} 0 \\ x_{20} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_i + \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} u_{i-1} = \begin{bmatrix} 2u_{i-1} \\ x_{20} \\ u_i - 2u_{i-1} \end{bmatrix}$$

This result agrees with the previous ones.

6. Comparison of Methods

Solution (15) may be written in the form

$$x_i = Q\begin{bmatrix} A_1^iQ_1 \\ 0 \end{bmatrix} x_0 + \sum_{k=0}^{i-1} Q\begin{bmatrix} A_1^{i-k-1}B_1 \\ 0 \end{bmatrix} u_k - \sum_{k=0}^{q-1} Q\begin{bmatrix} 0 \\ N^kB_2 \end{bmatrix} u_{i+k} \tag{77}$$

A comparison of (77) with (31) yields

$$Q\begin{bmatrix} A_1^kQ_1 \\ 0 \end{bmatrix} = (\overline{E}^D\overline{A})^k\overline{E}^D\overline{E}, \qquad k = 0, 1, \ldots \tag{78}$$

$$Q\begin{bmatrix} A_1^kB_1 \\ 0 \end{bmatrix} = \overline{E}^D(\overline{E}^D\overline{A})^k\overline{B}, \qquad k = 0, 1, \ldots \tag{79}$$

$$Q\begin{bmatrix} 0 \\ N^kB_2 \end{bmatrix} = (I - \overline{E}\overline{E}^D)(\overline{E}\overline{A}^D)^k\overline{A}^D\overline{B}, \qquad k = 0, 1, \ldots \tag{80}$$

From (78)–(80) for $k = 0$ we have

$$\begin{aligned} &Q\begin{bmatrix} Q_1 \\ 0 \end{bmatrix} = \overline{E}^D\overline{E}, \qquad Q\begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \overline{E}^D\overline{B} \\ &Q\begin{bmatrix} 0 \\ B_2 \end{bmatrix} = (I - \overline{E}\overline{E}^D)\overline{A}^D\overline{B} \end{aligned} \tag{81}$$

Knowing $\overline{E}, \overline{E}^D, \overline{B}, \overline{A}^D$ and Q we may find from (81) Q_1, B_1 and B_2.

From a comparison of (77) and (39) with (58) we obtain

$$Q\begin{bmatrix} A_1^kQ_1 \\ 0 \end{bmatrix} x_0 = V\overline{A}_1^k z_0, \qquad k = 0, 1, \ldots \tag{82}$$

$$Q\begin{bmatrix} A_1^{k-1}Q_1 \\ 0 \end{bmatrix} = L_k, \qquad k = 0, 1, \dots, -i \tag{83}$$

$$Q\begin{bmatrix} 0 \\ N^k B_2 \end{bmatrix} = L_k, \qquad k = 0, 1 \dots, p = q - 1 \tag{84}$$

From (82) for $k = 0$ we have

$$\operatorname{Im} Q\begin{bmatrix} Q_1 \\ 0 \end{bmatrix} = \operatorname{Im} V \tag{85}$$

A comparison of (31) and (39) with (58) yields

$$(\overline{E}^D\overline{A})^k\overline{E}^D\overline{E}x_0 = V\overline{A}_1^k z_0, \qquad k = 0, 1, \dots \tag{86}$$

$$\overline{E}^D(\overline{E}^D\overline{A})^{k-1}\overline{B} = L_k, \qquad k = -1, \dots, -i \tag{87}$$

$$(I_n - \overline{E}\overline{E}^D)(\overline{E}\overline{E}^D)^k\overline{A}^D\overline{B} = L_k, \qquad k = 0, 1, \dots, p = q - 1 \tag{88}$$

From a comparison of (77) and (76) it follows that

$$Q\begin{bmatrix} A_1^k Q_1 \\ 0 \end{bmatrix} = T_k E, \qquad k = 0, 1, \dots \tag{89}$$

$$Q\begin{bmatrix} A_1^k B_1 \\ 0 \end{bmatrix} = T_k B, \qquad k = -1, \dots, -i \tag{90}$$

$$Q\begin{bmatrix} 0 \\ N^k B_2 \end{bmatrix} = -P_k B, \qquad k = 0, 1, \dots, p = q - 1 \tag{91}$$

Similarly, from a comparison of (76) and (39) with (58) we obtain

$$T_k E x_0 = V\overline{A}_1^k z_0 \tag{92}$$

$$L_k = T_k B, \qquad k = -1, \dots, -i \tag{93}$$

$$L_k = -P_k B, \qquad k = 0, 1, \dots, p = q - 1 \tag{94}$$

Note that from the above comparisons we may find new formulae for the solution x_i to (1). For example, using (92) and (76) we may obtain the solution in the form

$$x_i = V\overline{A}_1^i z_0 + \sum_{k=0}^{q-1} P_k B u_{k+1} + \sum_{k=0}^{i-1} T_k B u_{i-k-1} \tag{95}$$

where $V, \overline{A}_1, P_k$ and T_k are defined by (38), (44), and (73)–(74), respectively.

7. Concluding Remarks

An extension for regular discrete-time linear systems of the Dias and Mesquista method (Dias and Mesquista, 1990) has been given. In (Kaczorek, 1995) it has been shown that the Dias and Mesquista method should be modified by adding an additional term containing an integral for regular continuous-time linear systems. A method based on the expansion in a series of the inverse matrix $[Ez - A]^{-1}$ has been proposed. These two new methods have been compared with two well-known methods based on the Weierstrass-Kronecker decomposition and on the Drazin inverse. The methods have been illustrated by numerical examples with a non-singular and singular matrix A. Relationships between the coefficient matrices of the four methods have been established. Moreover, a new mixed method of finding the solution (95) to eqn. (1) has been presented.

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