REGIONAL BOUNDARY OBSERVABILITY: A NUMERICAL APPROACH

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In this paper we review the concept of regional boundary observability, developed in (Michelitti, 1976), by means of sensor structures. This leads to the so-called boundary strategic sensors. A characterization of such sensors which guarantees regional boundary observability is given. The results obtained are applied to a two-dimensional system, and various cases of sensors are considered. We also describe an approach which leads to the estimation of the initial boundary state, which is illustrated by simulations.

Keywords: observability, regional observability, regional boundary observability, boundary strategic sensors

1. Introduction

For a given distributed parameter system defined on a spatial domain Ω, we are interested in the knowledge of system states on the whole domain (Gilliam and Martin, 1988; Kobayashi, 1980). The regional observability concept, introduced by (El Jai *et al.*, 1995), is focused on state observation on a given part ω of Ω. This concept was extended in (Zerrik *et al.*, 1999) to the case where ω is located on the boundary of Ω. The approach, based on appropriate optimization techniques, shows how to construct the initial state on a part of the boundary, but the procedure can be adapted, in time, to observe the current boundary state on the same portion of the boundary.

The introduction of this concept is motivated by real situations. This is the case, e.g., in the energy exchange problem, where the aim is to determine the energy exchanged in a casting plasma on a plane target which is perpendicular to the direction of the flow from measurements carried out by thermocouples (Fig. 1). It can also be of great help for a system which is not observable on the whole boundary $\partial\Omega$ of Ω, but observable on a part $\Gamma \subset \partial\Omega$. For example, consider the system defined on $\Omega =]0,1[\times]0,1[$ by

$$\begin{cases} \dfrac{\partial h}{\partial t}(x_1,x_2,t) = \dfrac{\partial^2 h}{\partial x_1^2}(x_1,x_2,t) \\ \qquad\qquad + \dfrac{\partial^2 h}{\partial x_2^2}(x_1,x_2,t) & \text{in } \Omega\times]0,T[, \\ \dfrac{\partial h}{\partial \nu}(\xi,\eta,t) = 0 & \text{on } \partial\Omega\times]0,T[, \\ h(x_1,x_2,0) = h_0(x_1,x_2) & \text{in } \Omega. \end{cases} \tag{1}$$

The measurements are given by the output function

$$z(t) = \int_{\Gamma_0} h(\xi,\eta,t) f(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{2}$$

where the boundary sensor (Γ_0, f) is located in the subdomain $\Gamma_0 = \{0\}\times[0,1]$ and $f(\xi,\eta) = \cos\pi\eta$ is the spatial distribution of the sensing measurements.

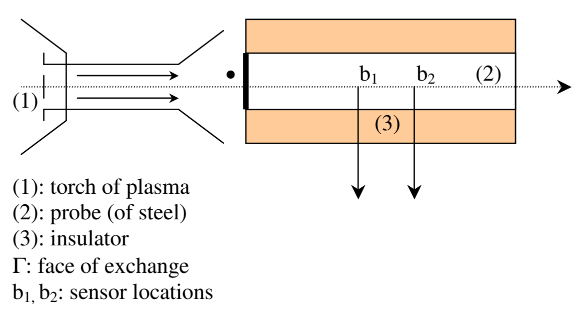


Fig. 1. The problem of estimating the energy exchanged on Γ.

The state $h_0(x_1,x_2) = \cos(\pi x_1)\cos(2\pi x_2)$ is not observable on $\partial\Omega$ but it is observable on $\Gamma = [0,1]\times\{0\}$. This shows that the regional boundary case is more general.

A regional boundary observability analysis has been made from a purely theoretical viewpoint (Zerrik *et al.*, 1999), but the study may also become concrete, in some sense, by using the structure of sensors, which form an important link between the system and its environment, have a passive role and allow the system evolution to be measured. Their structure depends on the geometry, the location of the support and the spatial distribution of the sensing measurements.

Our interest is mainly focused on regional boundary observability linked with sensor structures, their location and number. Thus we give some fundamental results related to sensors so that regional boundary observability can be achieved. This leads to the so-called boundary strategic sensors. The achieved results are also applied to two-dimensional diffusion systems. Secondly, we give a reconstruction method of the initial boundary conditions. This is the purpose of this paper, which is organized as follows.

The second section is devoted to the presentation of the system under consideration, a brief recall of the regional boundary observability concept and its characterization in terms of sensor structures. In Section 3 an application to a two-dimensional diffusion process is considered. Examples of various situations are also given and specific results are summarized in tabular form. In Section 4 we develop a technical approach which leads to a state reconstruction algorithm. In the last section the proposed approach is successfully tested through computer simulations.

2. Regional Boundary Observability and Sensors

2.1. Problem Statement

Let Ω be a regular bounded open set of $\mathbb{R}^n$ $(n \geq 2)$ with boundary $\partial\Omega$ and Γ be a nonempty subset of $\partial\Omega$, with positive Lebesgue measure. For a given $T > 0$, set $Q = \Omega\times]0,T[$ and $\Sigma = \partial\Omega\times]0,T[$. The system considered is described by the equation

$$(S)\begin{cases} \dfrac{\partial y}{\partial t}(x,t) = Ay(x,t) & \text{in } Q, \\ \dfrac{\partial y}{\partial \nu_A}(\xi,t) = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{3}$$

where A is a second-order linear differential operator with compact resolvent, which generates a strongly continuous semi-group $(S(t))_{t\geq 0}$ on the state space $X = H^1(\Omega)$. A^* indicates the adjoint operator of A and $\partial y/\partial\nu_A$ denotes the co-normal derivative associated with A.

Here y_0 is supposed to be in $H^1(\Omega)$ and unknown, and the measurements are obtained through the output function

$$z(t) = Cy(x,t), \tag{4}$$

C: $H^1(\Omega) \longrightarrow \mathbb{R}^p$ being a bounded linear operator depending on the structure of the sensors considered. The observation space is $\mathcal{O} = L^2(0,T;\mathbb{R}^p)$.

We define the operator:

$$K: \begin{array}{l} X \longrightarrow \mathcal{O}, \\ h \longrightarrow CS(\cdot)h, \end{array}$$

which is, in the zonal case, linear and bounded with an adjoint K^*: $\mathcal{O} \longrightarrow X$ given by

$$K^*z^* = \int_0^T S^*(s)C^*z^*(s)\,\mathrm{d}s.$$

We also consider the trace operator of order zero γ_0: $H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, which is linear, surjective and continuous. γ_0^* denotes its adjoint and χ: $H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^{\frac{1}{2}}(\Gamma)$ is the operator restriction to Γ when its adjoint is denoted by χ^*.

Let $y_0^1 = \chi\gamma_0 y_0$ be the restriction of the trace of y_0 to Γ. Boundary regional observability explores the state reconstruction in the case where the subregion ω is a subset of the boundary $\partial\Omega$. More precisely, we only have to reconstruct the component y_0^1 of the unknown initial state. In this case we have to pay particular attention to space functions and the operators considered.

Definition 1. The system (3) together with the output (4) is said to be *exactly* (resp. *approximately*) *regionally boundary observable* on Γ if

$$\mathrm{Im}(\chi\gamma_0 K^*) = H^{\frac{1}{2}}(\Gamma) \text{ (resp. } \mathrm{Ker}(K\gamma_0^*\chi^*) = \{0\}). \tag{5}$$

Conventionally, we shall say that the system is *exactly* (resp. *approximately*) $\mathcal{B}$*-observable* ($\mathcal{B}$ for the boundary) on Γ.

This is a natural definition of regional boundary observability extending those given in (Curtain and Zwart, 1995; El Jai and Pritchard, 1988) to the case where we restrict the state reconstruction to the boundary subregion Γ.

Obviously, we have the following assertions:

1. If the system (3) is exactly $\mathcal{B}$-observable, then it is approximately $\mathcal{B}$-observable.

2. If the system (4) is exactly (resp. approximately) $\mathcal{B}$-observable on Γ, then it is exactly (resp. approximately) $\mathcal{B}$-observable on every subset Γ_1 of Γ.

2.2. Boundary Strategic Sensor

The aim of this section is to give a characterization of sensors (number and location) in order for a system to be regionally approximately boundary observable.

Consider the system (3) and assume that the measurements are made by p sensors $(D_i, f_i)_{1\leq i\leq p}$. The output function is then given by $z(t) = (z_1(t), \ldots, z_p(t))$

with $z_i(t) = y(b_i, t),\ b_i \in \overline{\Omega}$ for $1 \le i \le p$ in the pointwise case, and $z_i(t) = \int_{D_i} y(x,t) f_i(x)\,\mathrm{d}x,\ D_i \subset \overline{\Omega}$ for $1 \le i \le p$ in the zonal case.

Definition 2. A sequence of sensors $(D_i, f_i)_{1 \le i \le p}$ is said to be *boundary strategic* on Γ if the corresponding observed system is approximately $\mathcal{B}$-observable on Γ. (In what follows we shall say that such sensors are Γ-strategic.)

Assume that there exists a complete set of eigenfunctions $(\varphi_{m_j})_{m \in I;\, j=1,\dots,r_m}$ of A in $H^1(\Omega)$ associated with the eigenvalues λ_m of multiplicities r_m and $r = \sup_{m \in I} r_m$ is finite. For $x = (x_1, \dots, x_n) \in \Omega$ and $m = (m_1, \dots, m_n) \in I$, let $\overline{x} = (x_1, \dots, x_{n-1})$ and $\overline{m} = (m_1, \dots, m_{n-1})$. Suppose that the functions $\psi_{\overline{m}_j}(\overline{x}) = \chi\gamma_0\varphi_{m_j}(x),\ m \in I$, form a complete set in $H^{\frac{1}{2}}(\Gamma)$.

Proposition 1. *The sequence of sensors* $(D_i, f_i)_{1 \le i \le p}$ *is* Γ-*strategic if and only if*

1. $p \ge r$ *and*
2. $\operatorname{rank} G_m = r_m$,

where

$$(G_m)_{ij} = \begin{cases} \langle \varphi_{m_j}, f_i \rangle_{L^2(D_i)} & \text{in the zonal case,} \\ \varphi_{m_j}(b_i) & \text{in the pointwise case,} \end{cases}$$

for $1 \le i \le p,\ 1 \le j \le r_m$.

Proof. For brevity, the proof is limited to the case of zonal sensors. The techniques used constitute extensions of those given in (Fattorini, 1968) and are based on algebraic manipulations.

For $z^* \in H^{\frac{1}{2}}(\Gamma)$ we have

$$\begin{aligned} &K\gamma_0^*\chi^* z^*(t) \\ &= \Big(\sum_{m \in I} e^{\lambda_m t} \sum_{j=1}^{r_m} \langle \varphi_{m_j}, f_i \rangle_{L^2(D_i)} \langle \varphi_{m_j}, \gamma_0^*\chi^* z^* \rangle_{H^1(\Omega)}\Big)_{i=1,p} \\ &= \Big(\sum_{m \in I} e^{\lambda_m t} \sum_{j=1}^{r_m} \langle \varphi_{m_j}, f_i \rangle_{L^2(D_i)} \langle \chi\gamma_0\varphi_{m_j}, z^* \rangle_{H^{\frac{1}{2}}(\Gamma)}\Big)_{i=1,p} \\ &= \Big(\sum_{m \in I} e^{\lambda_m t} \sum_{j=1}^{r_m} \langle \varphi_{m_j}, f_i \rangle_{L^2(D_i)} \langle \psi_{\overline{m}_j}, z^* \rangle_{H^{\frac{1}{2}}(\Gamma)}\Big)_{i=1,p}. \end{aligned}$$

If the system (3) together with the output (4) is not approximately $\mathcal{B}$-observable on Γ, there exists $z^* \ne 0$ such that $K\gamma_0^*\chi^* z^* = 0$, which gives

$$\sum_{j=1}^{r_m} \langle \psi_{\overline{m}_j}, z^* \rangle_{H^{\frac{1}{2}}(\Gamma)} \langle \varphi_{m_j}, f_i \rangle_{L^2(D_i)} = 0, \quad \forall\, m \in I, \quad 1 \le i \le p.$$

Consider

$$z_m = \begin{bmatrix} \langle \psi_{\overline{m}_1}, z^* \rangle_{H^{\frac{1}{2}}(\Gamma)} \\ \vdots \\ \langle \psi_{\overline{m}_{r_m}}, z^* \rangle_{H^{\frac{1}{2}}(\Gamma)} \end{bmatrix}.$$

Since $G_m z_m = 0$, it follows that $\operatorname{rank} G_m \ne r_m$ for any m.

Conversely, if $\operatorname{rank} G_m \ne r_m$, then there exists $m \in I$ such that

$$z_m = \begin{bmatrix} z_{m_1} \\ \vdots \\ z_{m_{r_m}} \end{bmatrix} \ne 0$$

and $G_m z_m = 0$.

Let $z^* \in H^{\frac{1}{2}}(\Gamma)$ be such that

$$\langle \psi_{\overline{j}_k}, z^* \rangle_{H^{\frac{1}{2}}(\Gamma)} = 0 \quad \text{for} \quad j \ne m$$

and

$$\langle \psi_{\overline{m}_k}, z^* \rangle_{H^{\frac{1}{2}}(\Gamma)} = z_{m_k} \quad \text{for} \quad 1 \le k \le r_m.$$

Therefore we have

$$\sum_{k=1}^{r_j} \langle f_i, \varphi_{j_k} \rangle_{D_i} \langle \psi_{\overline{j}_k}, z^* \rangle_{H^{\frac{1}{2}}(\Gamma)} = 0, \quad \forall\, j \ne m, \quad 1 \le i \le p$$

and

$$\sum_{k=1}^{r_m} \langle f_i, \varphi_{m_k} \rangle_{D_i} \langle \psi_{\overline{m}_k}, z^* \rangle_{H^{\frac{1}{2}}(\Gamma)} = 0, \quad 1 \le i \le p.$$

Thus there exists $z^* \ne 0$ (belonging to $H^{\frac{1}{2}}(\Gamma)$) such that $K\gamma_0^*\chi^* z^* = 0$, i.e. the system (3) together with the output (4) is not approximately $\mathcal{B}$- observable on Γ.

Note that the proposition implies that the required number of sensors is at least equal to the highest multiplicity of the eigenvalues. ■

Remark 1.

1. The above result remains also true for boundary sensors (pointwise or zonal cases).
2. By infinitesimally deforming the domain, the multiplicity of the eigenvalues can be reduced to one (Michelitti, 1976). Consequently, the $\mathcal{B}$-observability can be achieved by using only one sensor.

3. Applications to Sensor Location

In this section, we present an application of the above results to a two-dimensional system defined on $\Omega =]0, a[\times]0, d[$ by

$$\begin{cases} \dfrac{\partial y}{\partial t}(x_1, x_2, t) = \dfrac{\partial^2 y}{\partial x_1^2}(x_1, x_2, t) + \dfrac{\partial^2 y}{\partial x_2^2}(x_1, x_2, t) \text{ in } Q, \\ \dfrac{\partial y}{\partial \nu}(\xi_1, \xi_2, t) = 0 \text{ on } \Sigma, \\ y(x_1, x_2, 0) = y_0(x_1, x_2) \text{ in } \Omega. \end{cases} \tag{6}$$

The output function is given by (4) and the sensors are considered to be pointwise or zonal, and located in the interior of the system domain Ω or on its boundary.

The eigenfunctions associated with the system (6) are of the form

$$\varphi_{ij}(x_1, x_2) = \frac{2a_{ij}}{\sqrt{ad}} \cos\left(i\pi \frac{x_1}{a}\right) \cos\left(j\pi \frac{x_2}{d}\right)$$

with $a_{ij} = (1 - \lambda_{ij})^{-\frac{1}{2}}$. They correspond to the eigenvalues

$$\lambda_{ij} = -\left(\frac{i^2}{a^2} + \frac{j^2}{d^2}\right)\pi^2$$

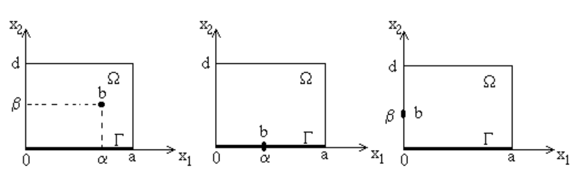
of multiplicity one if $a^2/d^2 \notin \mathbb{Q}$. In this case the system (6) can be $\mathcal{B}$-observable only by one sensor.

Let $\Gamma =]0, a[\times \{0\}$, be the subregion target. In this case $I = \mathbb{N}^2$, $(i, j) = i$, $(x_1, x_2) = x_1$, and the functions $\psi_i(x_1) = \sqrt{2/a}\cos(i\pi(x_1/a))$, $i \in \mathbb{N}$, form a complete set in $H^{\frac{1}{2}}(\Gamma)$.

The following results give information on the location of internal pointwise or zonal Γ-strategic sensors. The case of boundary sensors is given in Tables 1 and 2.

3.1. Internal Pointwise Sensor

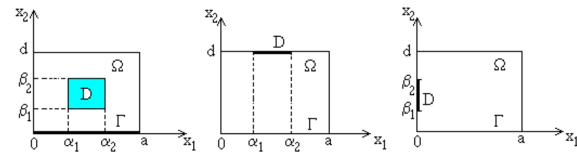
Consider the system (6) with the output function $z(t) = y(b, t)$ where the pointwise sensor (b, δ_b) is located inside the domain at a point $b = (\alpha, \beta) \in \Omega$ (Fig. 2(a)).



(a) (b) (c)

Fig. 2. Location of a pointwise sensor.

Corollary 1. *The sensor is not Γ-strategic if there exists $k, l \in \mathbb{N}^*$ such that $2k\alpha/a$ or $2l\beta/d$ is odd.*



(a) (b) (c)

Fig. 3. Location of a zone sensor.

This shows that the regional boundary observability depends on the location of the sensor. We note that in real applications a sensor is considered as pointwise if the support area of the measure distribution is very small compared with the system domain.

3.2. Internal Zone Sensor

Here we consider the system (6) with the output function $z(t) = \int_D y(x_1, x_2, t) f(x_1, x_2)\, dx_1\, dx_2$. Suppose that the sensor is located inside the domain Ω over $D =]\alpha_1, \alpha_2[\times]\beta_1, \beta_2[$ (cf. Fig. 3(a)), and $f \in L^2(D)$ defines the spatial distribution of the sensing measurements on D.

For $0 < \alpha_1 < \alpha_2 < a$ and $0 < \beta_1 < \beta_2 < d$, set

$$\eta_1 = \frac{\alpha_1 + \alpha_2}{2}, \quad \eta_2 = \frac{\beta_1 + \beta_2}{2},$$

$$\mu_1 = \frac{\alpha_2 - \alpha_1}{2}, \quad \mu_2 = \frac{\beta_2 - \beta_1}{2}.$$

Corollary 2.

1. *If f is uniformly distributed on D, then the sensor is not Γ-strategic if one of the following properties is satisfied: $\mu_1/a \in \mathbb{Q}$ or $\mu_2/d \in \mathbb{Q}$, or there exists $k, l \in \mathbb{N}^*$ such that $2k\eta_1/a$ or $2l\eta_2/d$ is odd.*

2. *If f is symmetric with respect to the point (η_1, η_2), then the sensor is not Γ-strategic if $\eta_1/a \in \mathbb{Q}$ or $\eta_2/d \in \mathbb{Q}$.*

3. *If f is symmetric with respect to the axis $x = \eta_1$ (or with respect to the axis $y = \eta_2$), then the sensor is not Γ-strategic if there exists $k \in \mathbb{N}^*$ such that $2k\eta_1/a$ is odd (resp. there exists $l \in \mathbb{N}^*$ such that $2l\eta_2/d$ is odd).*

Note that the regional boundary observability depends on the geometry of the sensor support and the measurement function. For the case where the sensor is located on the boundary, we obtain analogous results (see Tables 1 and 2).

Remark 2.

1. Note that the sensor support given above corresponds to a real geometry of a sensor in the diffusion system. The hypothesis of symmetry and uniform distribution are physically realistic. For example, this would be the case if the sensor was evenly distributed over its support ($f = \delta\chi_D$, where χ is the characteristic function and $D \subset \Omega$ is the zone in which the measurements are carried out).

2. From a practical point of view, the distributed system is most often approximated by a finite-dimensional system. Then the conditions of the $\mathcal{B}$-observability on Γ can be also verified for the finite-dimensional system. For instance, in the pointwise sensor case, if the system is approximated by a tree-dimensional system, then the condition of the non $\mathcal{B}$-observability on Γ is $\alpha/a \in I_3$ where $I_3 = \{1/6, 1/4, 1/2, 3/4, 5/6\}$ for all $\beta \in]0, d[$.

3.3. Recapitulating Tables

In this subsection we present the established results of the previous subsections in tabular form. The cases of boundary sensors (pointwise and zonal) are also considered.

Table 1. Pointwise sensor.

Sensor location	Non -strategic cases
$\alpha \in]0, a[$ and $\beta = 0$ or $\beta = d$	$\exists\, k \in \mathbb{N}^* \mid 2k\alpha/a$ is odd
$\beta \in]0, d[$ and $\alpha = 0$ or $\alpha = a$	$\exists\, l \in \mathbb{N}^* \mid 2l\beta/d$ is odd
$\alpha \in]0, a[$ and $\beta \in]0, d[$	$\exists\, k \in \mathbb{N}^* \mid 2k\alpha/a$ is odd or $\exists\, l \in \mathbb{N}^* \mid 2l\beta/d$ is odd

For the pointwise case, the regional boundary observability depends on the location of the sensor. For the zonal case, it depends on the form of the sensor support and its location (location of the support centre), as well as on the measurement function.

Table 2. Zonal sensor.

Sensor location	Non -strategic cases
$D =]\alpha_1, \alpha_2[\times\{0\}$ or $=]\alpha_1, \alpha_2[\times\{d\}$	$\star f$ uniformly distributed on D $\mu_1/a \in \mathbb{Q}$ or $\exists\, k \in \mathbb{N}^* \mid 2k\eta_1/a$ is odd. $\star f$ symmetric with respect to $(\eta_1, 0)$ or (η_1, d) $\eta_1/a \in \mathbb{Q}$ $\star f$ symmetric with respect to the axis $x = \eta_1$ $\exists\, k \in \mathbb{N}^\star \mid 2k\eta_1/a$ is odd
$D = \{0\}\times]\beta_1, \beta_2[$ or $= \{a\}\times]\beta_1, \beta_2[$	$\star f$ uniformly distributed on D $\mu_2/d \in \mathbb{Q}$ or $\exists\, l \in \mathbb{N}^* \mid 2l\eta_2/d$ is odd. $\star f$ symmetric with respect to $(0, \eta_2)$ or (a, η_2), $\eta_2/d \in \mathbb{Q}$ $\star f$ symmetric with respect to the axis $y = \eta_2$ $\exists\, l \in \mathbb{N}^\star \mid 2l\eta_2/d$ is odd
$D =]\alpha_1, \alpha_2[\times]\beta_1, \beta_2[$	$\star f$ uniformly distributed on D $\mu_1/a \in \mathbb{Q}$ or $\mu_2/d \in \mathbb{Q}$ or $\exists\, k \in \mathbb{N}^\star \mid 2k\eta_1/a$ is odd or $\exists\, l \in \mathbb{N}^\star \mid 2l\eta_2/d$ is odd $\star f$ symmetric with respect to (η_1, η_2) $\eta_1/a \in \mathbb{Q}$ or $\eta_2/d \in \mathbb{Q}$ $\star f$ symmetric with respect to the axis $x = \eta_1$ (resp. $y = \eta_2$) $\exists\, k \in \mathbb{N}^\star \mid 2k\eta_1/a$ is odd (resp. $\exists\, l \in \mathbb{N}^\star \mid 2l\eta_2/d$ is odd).

4. Reconstruction Method

In this section, we present an approach which allows the determination of the regional boundary initial condition y_0^1 on Γ, based on the internal regional observability. The method is an extension of those given in (El Jai and Pritchard, 1988; El Jai *et al.*, 1993; Kobayashi 1980).

Let us consider the system (3) with the output (4) on the same assumptions as in Section 2, and let ω be an open subset of Ω, regular and of positive Lebesgue measure such that $\Gamma \subset \partial\Omega \cap \partial\omega$. If the system (3) together with (4) is approximately observable on ω, then it is approximately $\mathcal{B}$-observable on Γ (Zerrik *et al.*, 1999).

This result links the internal regional observability on ω developed by El Jai *et al.* in (1993) to the regional $\mathcal{B}$-observability on Γ (which is part of $\partial\omega$).

Let the initial state be decomposed in the following form:

$$y_0 = \begin{cases} y_0^2 & \text{on } \omega, \\ y_0^3 & \text{on } \Omega \setminus \omega. \end{cases}$$

The problem consists in reconstructing the initial state y_0^2 on ω and determining its trace y_0^1 on Γ. Let us go further in the state reconstruction by considering various types of sensors.

4.1. Pointwise Measurements

In this case the output function is given by

$$z(t) = y(b,t), \tag{7}$$

where $b \in \Omega$ denotes the given location of the sensor. The problem consists in constructing the component y_0^2 of the initial state on ω with the knowledge of (3)–(7). For that purpose, we consider the set

$$G = \{g \in H^1(\Omega) \text{ such that } g = 0 \text{ on } \Omega\backslash\omega\}. \tag{8}$$

For a given $\varphi_0 \in G$, the system

$$\begin{cases} \dfrac{\partial\varphi}{\partial t}(x,t) = A\varphi(x,t) & \text{in } Q, \\ \dfrac{\partial\varphi}{\partial\nu_A}(\xi,t) = 0 & \text{on } \Sigma, \\ \varphi(x,0) = \varphi_0(x) & \text{in } \Omega \end{cases} \tag{9}$$

has a unique solution $\varphi \in L^2(0,T;H^1(\Omega)) \cap C^0(0,T;L^2(\Omega))$ and the mapping

$$\varphi_0 \in G \to \|\varphi_0\|_G^2 = \int_0^T \varphi^2(b,t)\,\mathrm{d}t \tag{10}$$

defines a semi-norm on G.

If the system (9) is approximately observable on ω, the mapping (10) defines a norm on G (Zerrik *et al.*, 1999). We also denote by G the completion of G.

For $\varphi_0 \in G$, (9) gives φ, which allows us to consider the system

$$\begin{cases} \dfrac{\partial\Psi_1}{\partial t}(x,t) = -A^*\Psi_1(x,t) \\ \qquad\qquad -\varphi(b,t)\delta(x-b) & \text{in } Q, \\ \dfrac{\partial\Psi_1}{\partial\nu_{A^*}}(\xi,t) = 0 & \text{on } \Sigma, \\ \Psi_1(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{11}$$

Let Ψ_1 be the solution of (11) and consider the operator

$$\wedge : \begin{array}{l} G \longrightarrow G^*, \\ \varphi_0 \longrightarrow P(\Psi_1(0)), \end{array} \tag{12}$$

where P denotes the projection on G^*.

Now consider the system

$$\begin{cases} \dfrac{\partial\Psi_2}{\partial t}(x,t) = -A^*\Psi_2(x,t) \\ \qquad\qquad -y(b,t)\delta(x-b) & \text{in } Q, \\ \dfrac{\partial\Psi_2}{\partial\nu_{A^*}}(\xi,t) = 0 & \text{on } \Sigma, \\ \Psi_2(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{13}$$

If φ_0 is such that φ leads to $\Psi_1(0) = \Psi_2(0)$ on ω, then the system (13) looks like the adjoint of the system to be observed (3)–(7) and, consequently, the observation problem on ω is equivalent to solving the equation

$$\wedge\,\varphi_0 = P(\Psi_2(0)). \tag{14}$$

If we assume that the operator A has a complete set of eigenfunctions (φ_i) in $H^1(\Omega)$, then we have the following result:

Proposition 2. *If the system (3) together with the output function (7) is approximately observable on ω, then (14) has a unique solution $\varphi_0 \in G$ and the regional boundary initial state to be observed on Γ is given by $y_0^1 = \chi\gamma_0\varphi_0$.*

Sketch of the proof. The proof proceeds in the following two steps:

Step 1. We show that the map $\varphi_0 \longrightarrow \|\varphi_0\|_G^2$ defines a norm on G using the fact that the sensor (b,δ_b) is ω-strategic (Amouroux *et al.*, 1994).

Step 2. We prove that the operator $\wedge$ is an isomorphism. It is sufficient to multiply the result by φ and integrate it on Q. Using the Green formula, we obtain $\langle\varphi_0, \wedge\varphi_0\rangle = \|\varphi_0\|_G^2$, which proves that $\wedge$ is an isomorphism and then (14) has a unique solution. For more details, see (Zerrik *et al.*, 1999).

4.2. Case of Zone Measurements

Let us come back to the system (3) and suppose that the measurements are given by an internal zone sensor defined by (D,f) with $D \subset \Omega$ and $f \in L^2(D)$. The system is augmented with the output function

$$z(t) = \int_D y(x,t)f(x)\,\mathrm{d}x. \tag{15}$$

In this case we consider the system (9), G being given by (8), and the mapping

$$\varphi_0 \in G \to \|\varphi_0\|_G^2 = \int_0^T \langle f, \varphi(t)\rangle_{L^2(D)}^2\,\mathrm{d}t, \tag{16}$$

which is a semi-norm on G. Thus with the system

$$\begin{cases} \dfrac{\partial\Psi_1}{\partial t}(x,t) = -A^*\Psi_1(x,t) - \langle f, \varphi(x,t)\rangle_{L^2(D)}^2 \\ \qquad\qquad \times f(x)\chi_D & \text{in } Q, \\ \dfrac{\partial\Psi_1}{\partial\nu_{A^*}}(\xi,t) = 0 & \text{on } \Sigma, \\ \Psi_1(x,T) = 0 & \text{in } \Omega \end{cases} \tag{17}$$

we introduce the operator $\wedge$: $\varphi_0 \in G \longrightarrow P(\Psi_1(0))$ and consider the system

$$\begin{cases} \dfrac{\partial \Psi_2}{\partial t}(x,t) = -A^*\Psi_2(x,t) \\ \qquad\qquad -z(t)f(x)\chi_D & \text{in } Q, \\ \dfrac{\partial \Psi_2}{\partial \nu_{A^*}}(\xi,t) = 0 & \text{on } \Sigma, \\ \Psi_2(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{18}$$

The observation problem on ω reduces to solving the equation

$$\wedge\, \varphi_0 = P(\Psi_2(0)). \tag{19}$$

Proposition 3. *If the system (3) together with the output function (15) is approximately observable on ω, then (19) has a unique solution $\varphi_0 \in G$ and the regional boundary initial state to be observed on Γ is given by $y_0^1 = \chi\gamma_0\varphi_0$.*

The proof is (with some minor technical modification) similar to the pointwise case. For more details, we refer the reader to (Zerrik *et al.*, 1999).

5. Simulations

We have seen that the regional boundary observability is equivalent, in all cases, to solving the equation

$$\wedge\, \varphi_0 = P\big(\Psi_2(0)\big). \tag{20}$$

The numerical approximation of (20) is realized easily when one can have a basis $(\tilde{\varphi}_i)$ of $H^1(\Omega)$ and the idea is to calculate the components $\wedge_{ij}$ of the operator $\wedge$. Then we approximate the solution of (20) by the linear system

$$\sum_{j=0}^{N} \wedge_{ij}\varphi_{0,j} = \Psi_{2,i} \quad \text{for } i = 0,\dots,N, \tag{21}$$

where N is the order of approximation and $\Psi_{2,i}$ are the components of $P(\Psi_2(0))$ in the basis $(\tilde{\varphi}_i)$ considered. Assume that $(\tilde{\varphi}_i)$ is the set of eigenfunctions of the operator A^* associated with the eigenvalues λ_i of multiplicity one.

In the case of pointwise measurements we have

$$\langle \wedge\varphi_0,\varphi_0\rangle = \sum_{i,j=0}^{\infty} \langle\tilde{\varphi}_i,\varphi_0\rangle_{H^1(\omega)}\langle\tilde{\varphi}_j,\varphi_0\rangle_{H^1(\omega)} \times \frac{-1+e^{(\lambda_j+\lambda_i)T}}{\lambda_j+\lambda_i}\,\tilde{\varphi}_i(b)\tilde{\varphi}_j(b).$$

Then the components of $\wedge$ are given by

$$\wedge_{ij} = \frac{-1+e^{(\lambda_j+\lambda_i)T}}{\lambda_j+\lambda_i}\tilde{\varphi}_i(b)\tilde{\varphi}_j(b). \tag{22}$$

In the zonal case the same developments lead to the components of $\wedge$:

$$\wedge_{ij} = \frac{e^{(\lambda_i+\lambda_j)T}-1}{\lambda_j+\lambda_i}\langle f,\tilde{\varphi}_i\rangle_{L^2(D)}\langle f,\tilde{\varphi}_j\rangle_{L^2(D)}. \tag{23}$$

Summing up, in the pointwise case, the regional reconstruction is obtained via the following simplified algorithm:

Step 1. Choose a sensor location b, an error test ε, an initial state y_0.

Step 2. Choose an approximation order N.

Step 3. Solve (13) to obtain $\Psi_2(x,0)$.

Step 4. Solve (21) to obtain φ_0 where $\wedge_{ij}$'s are given by (22).

Step 5. If $\|y_0-\varphi_0\|^2_{L^2(\omega)} > \varepsilon$, go to Step 2, otherwise φ_0 corresponds to the initial state to be observed on ω.

This algorithm converges since the developments are based on a Dirichlet series. To avoid instabilities in numerical calculations, which are suspected to arise, we must take some care with the numerical method for solving the linear system (21) and also with the choice of the sensor location.

5.1. Example

For an illustrative application, consider the parabolic system on $\Omega =]0,1[\times]0,1[$ given by

$$\begin{cases} \dfrac{\partial y}{\partial t}(x_1,x_2,t) = 0.01\left[\dfrac{\partial^2 y}{\partial x_1^2}(x_1,x_2,t)\right. \\ \qquad\qquad \left. + \dfrac{\partial^2 y}{\partial x_2^2}(x_1,x_2,t)\right] & \text{in } Q, \\ \dfrac{\partial y}{\partial \nu}(\xi,\eta,t) = 0 & \text{on } \Sigma, \\ y(x_1,x_2,0) = y_0(x_1,x_2) & \text{in } \Omega. \end{cases} \tag{24}$$

The measurements are given by a pointwise sensor $z(t) = y(b,t)$, with $b = (0.30, 0.65)$ and $T = 2$. Here the boundedness of multiplicities r_m does not hold, but it holds for $\Omega^\star =]0, 1+\varepsilon[\times]0,1[$ ($\varepsilon \notin \mathbb{Q}$ small enough), which constitutes a good approximation of Ω (Michelitti, 1976)), and for which the outlined results are applicable. We note that numerically an irrational number does not exist but it can be considered as irrational if the truncation number exceeds the desired precision.

Let $\Gamma = \{0\}\times[0,1]$ and

$$y_0^1(\eta) = 2\left(\frac{\eta^3}{3} - \frac{\eta^2}{2} + 0.1\right)$$

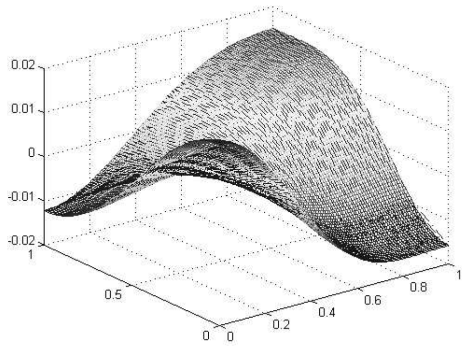


Fig. 4. True initial state y_0^2 on ω.

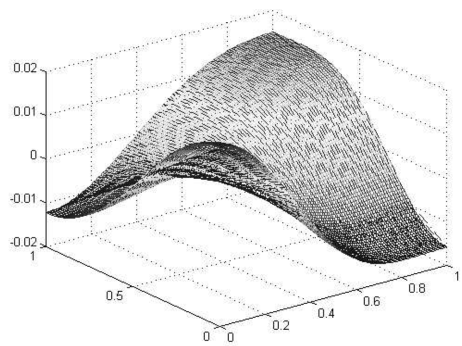


Fig. 5. Estimated state $y_{0,e}^2$ on ω.

be the initial state to be observed on Γ. Let $\omega =]0, 0.24[\times]0, 1[$ and

$$y_0^2(x_1, x_2) = 2\left(\frac{x_1^3}{3} - \frac{x_1^2}{2} + 0.1\right)\left(\frac{x_2^3}{3} - \frac{x_2^2}{2} + 0.1\right)$$

be an extension of y_0^1 to ω. Applying the delineated approach, we obtain the results given in Figs. 4 and 5.

Figure 6 shows that the estimated boundary state is very close to the true initial boundary state on Γ. The initial state y_0^1 is estimated with the reconstruction error $\|y_0^1 - y_{0,e}^1\|^2_{L^2(\Gamma)} = 4.951 \times 10^{-7}$.

5.2. Subregion-Pointwise Actuator

The following simulation results show the evolution of the estimated state error with respect to the sensor location $b = (b_1, b_2)$ when b_1 is fixed at 0.81 and $b_2 \in]0, 1[$.

Figure 7 reveals the following facts:

- For a given subregion Γ, there is an optimal sensor location (optimal in the sense that it leads to a solution which is very close to the initial boundary state).
- When a sensor is located sufficiently far from the subregion Γ, the estimated state error is constant for any locations.

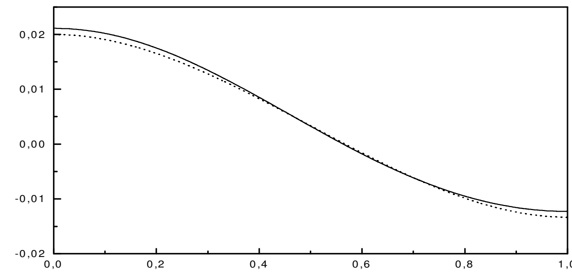


Fig. 6. True (y_0^1) (dashed line) and estimated $y_{0,e}^1$ (continuous line) initial state on Γ.

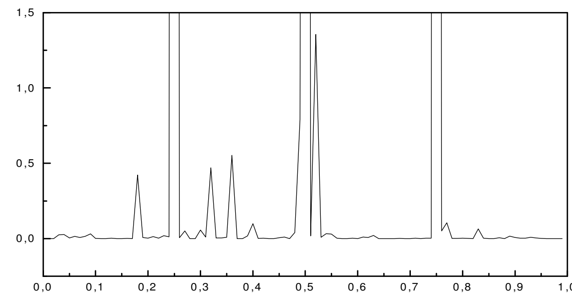


Fig. 7. Evolution of the estimated state error with respect to the sensor locations b_2.

- The worst locations correspond to a non Γ-strategic sensor as developed in the previous sections, where $b_2 \in \{1/4, 1/2, 3/4\}$ (the order of system approximation is two).

5.3. Relation Subregion Area—Estimated State Error

The state reconstruction error depends on the subregion area. Table 3 shows that both the error and the subregion area increase or decrease. The $\mathcal{B}$-observability is realized by means of one pointwise sensor located at $b = (0.44, 0.65)$. These results are similar for other types of sensors.

Table 3. The evolution of the boundary observability error with respect to the subregion Γ area.

Subregion Γ	$\|y_o^1 - y_{0,e}^1\|^2_{L^2(\Gamma)}$
$\{0\} \times [0, 1]$	1.285×10^{-3}
$\{0\}\times]0.10, 0.95[$	4.418×10^{-4}
$\{0\}\times]0.20, 0, 85[$	2.786×10^{-4}
$\{0\}\times]0.30, 0, 65[$	2.255×10^{-4}
$\{0\}\times]0.30, 0.60[$	2.211×10^{-4}
$\{0\}\times]0.33, 0.60[$	9.997×10^{-5}
$\{0\}\times]0.35, 0.60[$	3.002×10^{-6}
$\{0\}\times]0.40, 0.50[$	3.744×10^{-8}

6. Conclusion

The concept developed in this paper is related to the regional boundary observability in connection with the sensor structure. It permits us to avoid some 'bad' sensor locations. Various interesting results concerning the choice of the sensor structure are given and illustrated in specific situations. Furthermore, we have developed a technical approach which leads to an implementable state reconstruction algorithm. The results can be used for the estimation of the current boundary state by updating the initial time. This can be employed for the knowledge of boundary conditions on a portion of the boundary. An approach for real applications is now under consideration.

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