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SEMI–ANALYTICAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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We describe the method that allows, in many cases, to solve effectively a wide class of ordinary differential equations describing a mechanical system. The method is based upon the idea of normalization procedure. We present its implementation and some examples of its applications.

1. Introduction

There are two general directions in the study of dynamic systems—analytical and numerical. The latter is used when the system is too complicated for analytical tools or when there are no appropriate methods available. It should also be mentioned that even in very simple systems we can encounter problems which cannot be solved analytically. On the other hand, some analytical methods are difficult in realization and in practice they can be applied only as algorithms of sophisticated computer algebra systems.

In this paper we discuss systems whose dynamics can be described by means of a set of ordinary differential equations and, more precisely, by the Hamilton's equations of motion. This limitation is not too restrictive. Firstly, every system of ordinary differential equations can be transformed into the Hamiltonian one (via doubling the number of variables). Secondly, the method which we apply for the Hamiltonian system has its version for general systems of ordinary differential equations.

The main idea is very simple. Instead of direct integration, we change variables in order to simplify equations of motion. This method will be constructive if we know in advance how to change variables. In many cases it is possible to select a class of 'acceptable' changes of variables in such a way that the obtained simplified system is explicitely solvable.

The idea described here is applied in many versions of perturbations method and is frequently used in many branches of applied mathematics. However, its application in pure analytical form is difficult. In practice, changes of variables are presented as infinite series of specific functions. A few such changes lead to very complicated expressions and even performing calculations with the help of general

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purposes computer algebra system (REDUCE, MACSYMA, MATHEMATICA) does not help much because of memory limitations. It is especially true in the case of IBM PC compatible computers.

Thus, it seems to be reasonable to make some restrictions. First, we should select a class of problems we want to solve, and next find an effective way of implementation of the method described above. It is important to notice that in many cases explicit analytic formulae for variables' changes are not necessary. Thus, we should know how to perform these transformations with the help of an appropriate procedure. In other words, coefficients of series can be given numerically.

We present here a computer realization of the concept. We chose for the study a neighborhood of an equilibrium in a Hamiltonian system. The reason for this is twofold. The adequate theory is most complete and relatively simple for this kind of problem (c.f. the paper of Bruno, 1988).

The approach presented here can be generalized for any invariant of the system, e.g. periodic orbits, invariant tori (Bruno, 1989).

In this paper we demonstrate that our method gives approximately general solutions of equations of motion in the neighborhood of an equilibrium (with prescribed accuracy). Thus, for initial conditions that lie in the neighborhood, we have analytic expressions with numerical coefficients which represent the solution of the initial value problem. We also present applications of our method for finding different classes of solutions.

2. Theory

This section gives a short overview of the theory implemented in our software system LIE (Goździewski and Maciejewski, 1990). We deal with an autonomous Hamiltonian system with N degrees of freedom. As it is well known, such a system is given if we know its Hamiltonian function:

$$H = H(\boldsymbol{x}),$$

where:

$$\boldsymbol{x} = (q_1, \ldots, q_N, p_1, \ldots, p_N)$$

denotes standard canonical coordinates. We also introduce the following matrix:

$$\boldsymbol{I} = \begin{bmatrix} \boldsymbol{O}_N & \boldsymbol{E}_N \\ -\boldsymbol{E}_N & \boldsymbol{O}_N \end{bmatrix},$$

where $\boldsymbol{O}_N, \boldsymbol{E}_N$ are $N \times N$ zero and unit matrices, respectively. Hamilton's equations of motion can be written in the following form:

$$\dot{\boldsymbol{x}} = \boldsymbol{I}\frac{\partial H}{\partial \boldsymbol{x}}. \tag{1}$$

We assume that $\boldsymbol{x}^0$ is an equilibrium point for these equations, i.e.:

$$\frac{\partial H}{\partial \boldsymbol{x}}(\boldsymbol{x}^0) = \mathbf{0}. \tag{2}$$

Without any loss of generality we can assume that $\boldsymbol{x}^0 = \mathbf{0}$. If it is not the case, we can make the canonical transformation:

$$\boldsymbol{x} = \tilde{\boldsymbol{x}} + \boldsymbol{x}^0$$

moving the equilibrium point to zero.

Assuming that H is the analytical function in the neighborhood of zero we can represent it as the power series:

$$H = \sum_{l=2}^{\infty} H_l \tag{3}$$

where H_l are homogeneous polynomials of order l with respect to the $\boldsymbol{x}$ components:

$$H_l = \sum_{|k|=l} h_k \boldsymbol{x}^k,$$

where:

$$k \stackrel{\text{def}}{=} (k_1, \ldots, k_{2N}), \quad |k| \stackrel{\text{def}}{=} \sum_{i=1}^{2N} k_i, \quad \boldsymbol{x}^k \stackrel{\text{def}}{=} x_1^{k_1} \cdots x_{2N}^{k_{2N}}.$$

The right hand sides of Hamilton's equations of motion are determined by one function, namely by the Hamiltonian H. Thus, a simplyfying of them is equivalent to a simplification of the function H. We also want to preserve the Hamiltonian character of the system during every step of the simplification. This requirement restricts a class of simplificating transformations to the canonical ones. The procedure that simplifies a Hamiltonian function is called normalization and it can be presented schematically as follows:

Original problem	Canonical map	Normalized problem
$\dot{\boldsymbol{x}} = I\frac{\partial H}{\partial \boldsymbol{x}}$	$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{y})$	$\dot{\boldsymbol{y}} = I\frac{\partial \tilde{H}}{\partial \boldsymbol{y}}$

In the above table $\tilde{H} = H(\boldsymbol{x}(\boldsymbol{y}))$ is the Hamiltonian function expressed in the new variables.

The normalization is divided into two major stages. Since the procedure depends strongly on the form of the H_2 term in (3), at first we should simplify it as much as possible. This can by obtained by the application of the linear canonical transformation of variables

LINEAR NORMALIZATION		
INPUT	MAP	OUTPUT
$H = \sum_{i=2}^{\infty} H_i,$ $H_2 = \frac{1}{2}\boldsymbol{x}^T \boldsymbol{H} \boldsymbol{x}$	$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{y}$	$\tilde{H} = \sum_{i=2}^{N} \tilde{H}_i,$ $\tilde{H}_2 = \sum_{k=1}^{N} \lambda_k y_k y_{k+N}$

In the above table λ_k, $k = 1, \ldots, N$ denote eigenvalues of the matrix $\boldsymbol{A} = \boldsymbol{IH}$ (it is the coefficients matrix of the linearized system). We also assumed that this matrix is diagonalisable. The new coordinates $\boldsymbol{y}$ are generally complex.

After the linear normalization one has to simplify the terms H_k with $k > 2$. The method of non-linear normalization is based on Lie series transformation and in our settings is purely algebraic. The main advantage of it is that for finding appropriate canonical transformation we need to find coefficients of one function of canonical variables. This function is called the generating Hamiltonian and is denoted by T in the table below.

NONLINEAR NORMALIZATION		
INPUT	MAP	OUTPUT
$H = \sum_{i=2}^{\infty} H_i,$ $H_2 = \sum_{k=1}^{N} \lambda_k x_k x_{k+N}$	$\dot{\boldsymbol{x}} = \boldsymbol{I}\frac{\partial T}{\partial \boldsymbol{x}}$ $\boldsymbol{y} = \boldsymbol{x}(0)$ $\boldsymbol{x} = \exp(D_T)\boldsymbol{y}$	$\tilde{H} = \sum_{i=2}^{\infty} \tilde{H}_i,$ $\tilde{H}_2 = \sum_{k=1}^{N} \lambda_k y_k y_{k+N}$

In this table D_T denotes differential operator:

$$D_T f = \{f, T\},$$

where $\{\cdot, \cdot\}$ is the Poisson bracket.

The normalization procedure is performed step by step i.e., the normalization of terms H_k does not change the terms H_i with $i < k$. If we stop the normalization at the terms of order M we can write:

$$\tilde{H} = \sum_{l=2}^{M} \tilde{H}_l + \tilde{H}_U,$$

where $\tilde{H}_U$ denotes all unsimplified terms. If we neglect these terms, then in a generic case, the system with a Hamiltonian obtained in this way is integrable or simpler for investigations than the original one. The form of the normalized Hamiltonian depends on the so called resonance relations between values of λ_i, $i = 1, \ldots N$.

The overview of normalization algorithms which we have developed is described in (Maciejewski and Goździewski, 1991). We also discussed some aspects of the practical realization of the algorithms, as well as some of their applications (Goździewski and Maciejewski, 1990). The reader can find the full and compact theoretical background in the paper of Bruno (1988). An interesting introduction to the subject is given by Grebiennikov (1986) and Holshevnikov (1985). The paper of Mersman (1970) describes a very efficient and useful algorithm of Lie series transformation.

3. Examples

In order to illustrate the possible applications of the normalization procedure we chose a real, very simple mechanical model: the double pendulum (see Figure 1). The Hamiltonian function of the model is:

$$H(q_1, q_2, p_1, p_2) = -\cos q_1 + l\,(M-1)\cos q_2 + \frac{l^2\,(1-M)\,{p_1}^2 + {p_2}^2 - 2\,l\,(1-M)\,p_1 p_2 \cos(q_1 - q_2)}{l^2\,(1-M)\,(1+M+(M-1)\cos 2\,(q_1-q_2))} \tag{4}$$

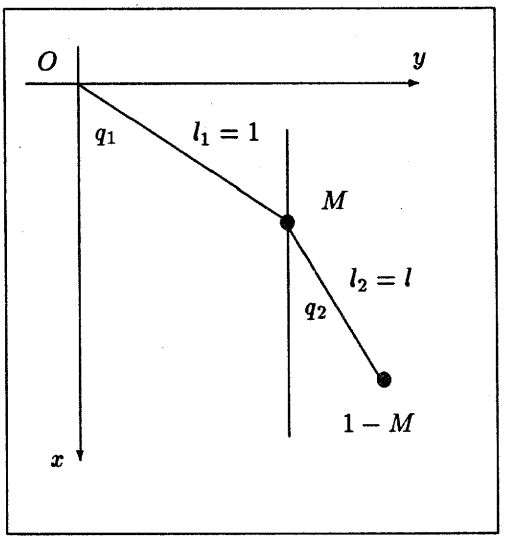

Fig.1. The double pendulum.

It has two parameters: l and M. The physical units are chosen in such a way that the sum of two masses equals to 1; the length of the pendulum of mass M and the gravitational constant are equal 1; the length of the second pendulum is l.

The Hamiltonian system (4) has four equilibrium points:

$$q_1 = q_2 = p_1 = p_2 = 0 \tag{5}$$

$$q_1 = q_2 = \pi,\ p_1 = p_2 = 0 \tag{6}$$

$$q_1 = \pi,\ q_2 = p_1 = p_2 = 0 \tag{7}$$

$$q_1 = 0,\ q_2 = \pi,\ p_1 = p_2 = 0 \tag{8}$$

Equilibrium (5) is of the center–center type (the matrix of linerized equations of motions has two pairs of imaginary eigenvalues), fixed point (6) is of the saddle–saddle type (the matrix has two pairs of real eigenvalues). The last two equilibria (7) and (8) are of center–focus type (the matrix of linearized system has one pair of real and one pair of imaginary eigenvalues).

In the neighborhood of all these equilibria the normalization of the Hamiltonian (3) was performed. We investigated the system for non resonant values of parameters. After the linear and nonlinear normalization up to the order of $2M$, the normalized part of the original system (4) has the following form:

$$\tilde{H} = \sum_{i=1}^{M} \tilde{H}_{2i} \qquad \tilde{H}_{2i} = \sum_{|n|=i} \tilde{h}_n \rho^n \tag{9}$$

where n denotes now a N-multi-index,

$$\rho = (u_1 v_1, \ldots, u_N v_N),$$

and

$$z = (u_1, \ldots, u_N, v_1, \ldots v_N)$$

are normal, complex canonical coordinates. Note that the Hamiltonian (9) depends only on N variables, namely products $\rho_k = u_k v_k$, $k = 1, \ldots, N$. Equations of motion written in normal variables

$$\frac{\mathrm{d}u_k}{\mathrm{d}t} = \frac{\partial \tilde{H}(\rho)}{\partial v_k}, \qquad \frac{\mathrm{d}v_k}{\mathrm{d}t} = -\frac{\partial \tilde{H}(\rho)}{\partial u_k}, \qquad k = 1, \ldots, N \tag{10}$$

are easily integrable:

$$u_k = u_k^0 \exp[\Omega_k(\rho^0)(t - t_0)], \tag{11}$$

$$v_k = v_k^0 \exp[-\Omega_k(\rho^0)(t - t_0)] \tag{12}$$

$$\text{with} \qquad \Omega_k(\rho) = \frac{\partial \tilde{H}(\rho)}{\partial \rho_k}, \qquad k = 1, \ldots, N$$

where a variable with the index '0' denotes its values at the time $t = t_0$. The Cartesian normal variables z can be transformed into the complex 'action-angle' variables (Bruno, 1988):

$$\rho_k = u_k v_k, \qquad \phi_k = \frac{1}{2}(\ln u_k - \ln v_k), \qquad k = 1, \ldots, N \tag{13}$$

Variables ϕ_k are cyclic; thus, ρ_k for $k = 1, \ldots, N$ are integrals of motion and solution (11) can be written as:

$$\rho = \rho^0 = const., \qquad \phi = -\Omega(t - t_0) + \phi^0, \qquad \Omega = \frac{\partial \tilde{H}}{\partial \rho}(\rho^0) \tag{14}$$

The normalization procedure diverges in many cases, neverthless the solutions obtained approximate exact ones very well in a finite period of time. Neglected terms H_U are also the source of errors. Tests show that the order of these errors is higher than the order of the last normalized term. Thus, the accuracy needed during computations can be achieved by choosing a high enough order of normalization. It should also be noted that solution (14) gives good approximation of this part of exact solution that lies in a close neighbourhood of the equilibrium, i.e. when the norm of ρ is small. For prescribed accuracy one can easily determine the maximal allowed values of the ρ norm.

3.1. Quasi–Periodic Solutions

In a close vicinity of the stable equilibrium one can find quasi-periodic solutions. For the normalized Hamiltonian, when the equilibrium is stable, every solution (14) with $\rho \neq 0$ is quasi-periodic. Figure 2 shows the comparison of the numerical integration of the original equations of motion with results derived from (14). The next figure shows differences between these solutions. Let us note that, although the solution lies in the relatively large neighbourhood of the equilibrium, the precission of our method is very good.

3.2. Periodic Solutions

In the neighbourhood of the center–center or the focus–center equilibrium one can prove the existence of families of periodic solutions. We illustrate how to find them for the case of stable equilibrium (5). The basis here is once again, the normal form (9) and explicit transformation between the original and the normal variables which is described by the generating Hamiltonian. In our case system (9) has two degrees of freedom. From general solution (14) we can determine periodic solutions as those fulfilling the condition:

$$\frac{\Omega_1(\rho^0)}{\Omega_2(\rho^0)} = \frac{n}{m} \tag{15}$$

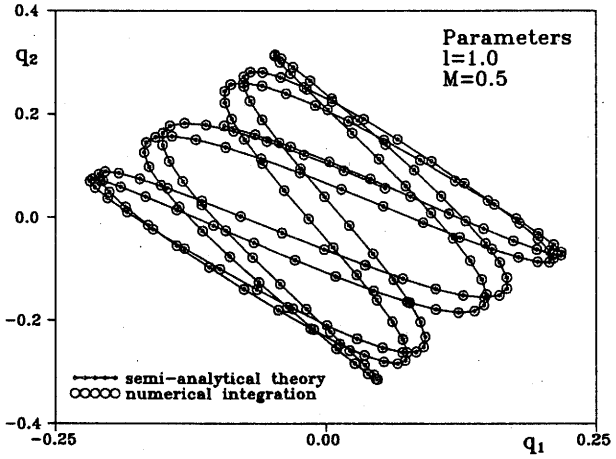

Fig. 2. Quasi periodic solutions on the (q_1, q_2) plane. Initial condition for normal variables was chosen as: $\rho_1^0 = \rho_2^0 = 0.01$, $\phi_1^0 = \phi_2^0 = 0$. It corresponds to the following initial condition for the orginal variables: $q_1 \simeq -0.093\ldots$, $q_2 \simeq 0.177\ldots$, $p_1 \simeq 0.156\ldots$, $p_2 \simeq 0.048\ldots$. The normalization was performed up to the 9-th degree.

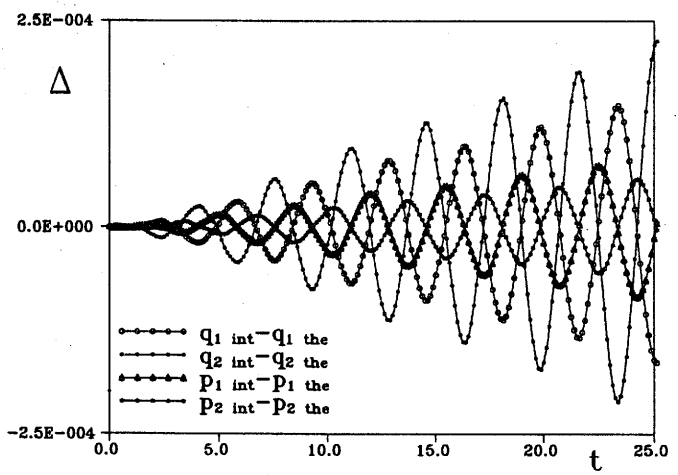

Fig. 3. Differences between numerical and semi-analytical integration for the orbit shown in the previous figure.

where n and m are relative prime integers. We can look for periodic solutions by finding zeros of the function

$$f(\rho_1, \rho_2) = m\,\Omega_1(\rho_1, \rho_2) - n\,\Omega_2(\rho_1, \rho_2) \tag{16}$$

in the neighborhood of the equilibrium position. The search area has to be limited by prescribed accuracy of the transformation between the original and the normal variables. The general solution of (16) (for fixed n, m) forms a curve on the two dimensional plane (ρ_1, ρ_2). Coordinates of every point of this curve, transformed into the original variables give an initial condition of periodic solution (15). The curve represents a family of periodic solutions. Solutions of the family have the same period

$$T = m\frac{2\pi}{|\Omega_2|} = n\frac{2\pi}{|\Omega_1|}.$$

Figure 4 shows an example of application of the algorithm.

3.3. Asymptotic Solutions

The starting point for finding solutions asymptotic to unstable equilibrium (6) are equations (11). In the neighborhood of the equilibrium the solutions can be explicitly classified. They form two manifolds. Without any loss of generality we can assume that $Re(\Omega_k) > 0,\ k = 1, \ldots, N$. Setting now

$$u_k^0 \neq 0, \quad v_k^0 = 0, \qquad k = 1, \ldots, N$$

we define the set of trajectories tending to the equilibrium as $t \to -\infty$. It is the unstable manifold of asymptotic solutions. The stable manifold is determined by

$$u_k^0 = 0, \quad v_k^0 \neq 0, \qquad k = 1, \ldots, N$$

and it is formed by orbits approaching the fixed point as $t \to \infty$. As we can assume that after normalization the transformation between normal and original variables is known, the initial conditions of the asymptotic solutions can be obtained in the original variables. By integrating numerically equations of motion of the original system, we can now investigate the global structure of asymptotic manifolds. A well–known Poincaré cross–section is a very useful tool for this purpose. In the case of two degrees of freedom (as in our example) it has a simple geometrical meaning. The phase space is four–dimensional. The manifold of a fixed energy is three–dimensional. Solutions asymptotic to the equilibrium have the same total energy. Now, instead of considering a whole orbit in the phase space we can consider the sequence of points obtained by intersection of the orbit with a suitable two dimensional submanifold (surface of section). By applying the special algorithm, we are able to reconstruct globally intersections of asymptotic manifolds with the surface of section (Maciejewski and Goździewski, 1992). Generally, they form a collection of smooth pieces of an extremely sophisticated curve on the cross–section plane.

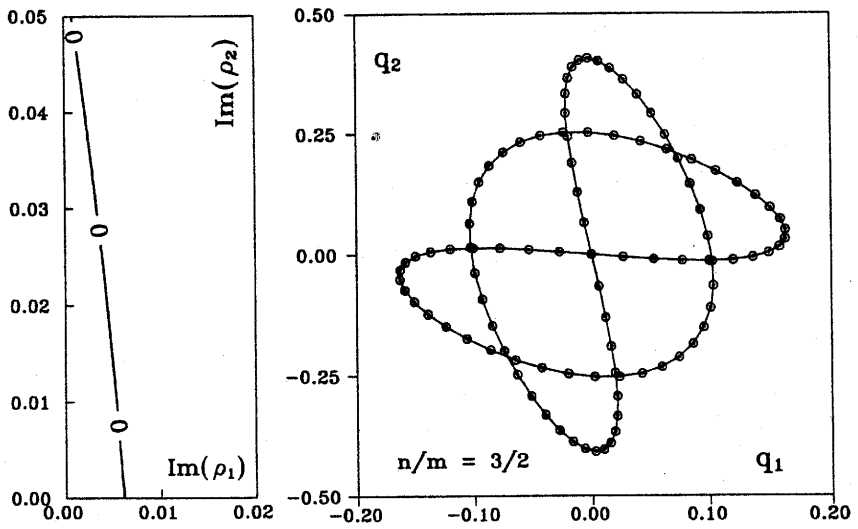

Fig. 4. A family of periodic solutions with nonlinear frequencies ratio $n/m = 3/2$ for parameters $l = 1.1, M = 0.85$ (on the left), shown on the plane (ρ_1, ρ_2). The second plot shows one of the periodic solutions of the family — results of numerical integration (open circles) are compared with semi-analytical computation of the orbit (small dots). The orbit computed covers the time span of one period. Notice the range of angular variables!

This reconstruction can give a very important result in the case of a Hamiltonian system with two degrees of freedom, which has an equilibrium of saddle–focus or saddle–saddle type. If we find that the stable and unstable manifolds intersect transversely, we can proof numerically the nonintegrability of the system investigated. This statement has solid theoretical background in the theorems of Devaney (1976) and Bolotin (1990). Figure 5 shows the intersection of asymptotic manifolds with the cross-section plane $q_1 = \pi$ (the parameters are taken as $l = 2, M = 0.7$, the origin of angular coordinates is in the saddle–saddle equilibrium (6)). As it can be seen, they intersect transversely. The next figure shows orbit whose initial conditions are determined by a point denoted as X. This orbit has a special feature—it is double asymptotic (homoclinic). With the time tending to infinity, as well as to minus infinity, the trajectory tends to the equilibrium. The existence of such an orbit in our system suggests that it is nonintegrable. To finish the proof we should check whether some additional assumptions of the theorem of Bolotin are fulfilled (1990). We do not present them here. More details can be found in our paper (Maciejewski and Goździewski, 1992).

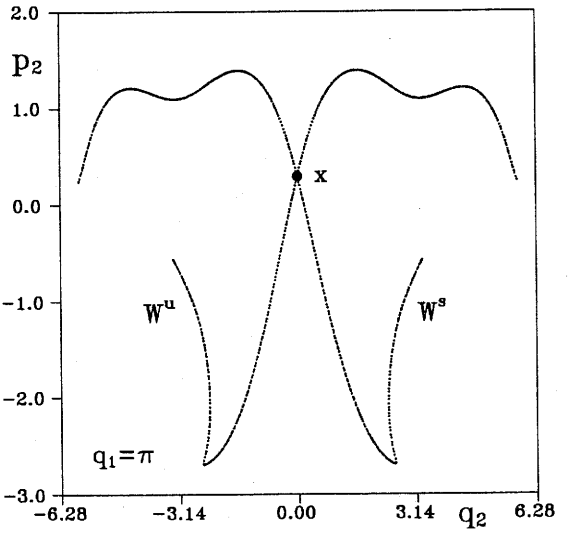

Fig. 5. Intersection of manifolds asymptotic to unstable equilibrium for parameters $l = 2, M = 0.75$ with cross–section plane $q_1 = \pi$. Angular coordinates are relative to the equilibrium.

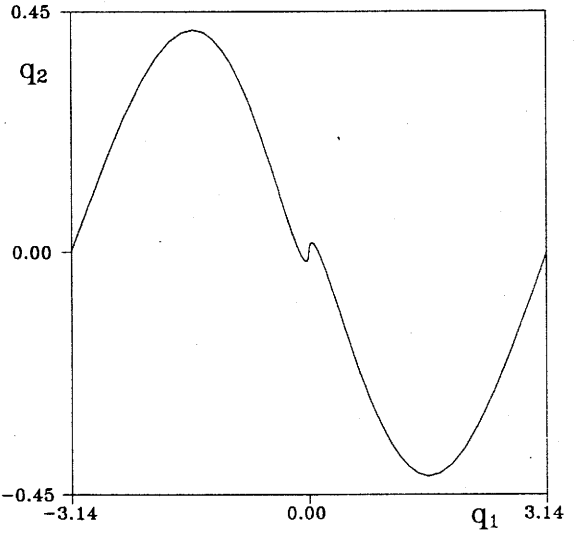

Fig. 6. Orbit homoclinic to unstable equilibrium (2). The coordinates' origin is in the equilibrium (the coordinates are taken mod 2π). Initial conditions of the orbit are denoted X in the previous figure.

3.4. Nonlinear Stability of Equilibria

In the case of a Hamiltonian system with two degrees of freedom normalization can be used as a tool for determining nonlinear stability of an equilibrium with purely imaginary eigenvalues. If the Hamiltonian is positively (or negatively) defined, then Liapunov's direct method cannot be applied. Instead, we can use a famous theorem of Arnold and Moser (see e.g. Meyer and Schmidt, 1986). Assuming the lack of resonances up to the order of $2M$ we can present the Hamiltonian in terms of the real action–angle variables $(\rho_1, \rho_2, \phi_1, \phi_2)$

$$\tilde{H} = \tilde{H}_2 + \tilde{H}_4 + \dots$$

where $H_{2k}(\rho_1, \rho_2)$ are homogenous polynomials of degree k in ρ_1 and ρ_2, $\tilde{H}_2 = \omega_1\rho_1 - \omega_2\rho_2$. The equilibrium is stable if for some k

$$D_{2k} = \tilde{H}_{2k}(\omega_2, \omega_1) \neq 0.$$

The theorem gives a constructive algorithm for investigating the stability. We implemented it in the system LIE (Goździewski and Maciejewski, 1990). To illustrate the above, we show here the answer given by LIE, when asked if equilibrium (5) is stable for the parameters $l = 1, M = 0.5$ (the example demonstrates only the result of calculations, obviously the Hamiltonian is positively defined):

```
o1 = 1.84776
o2 = 0.76537
D2 = 2.828 + I 3.16E-16
D4 =-1.813 + I 7.82E-16
D6 = 7.769 + I 5.87E-16
```

Here `o1,o2` denote frequencies of the linearized system, `D2`, `D4`, `D6` are the determinants defined above for $k = 1, 2, 3$ and `I` is the imaginary unit. We used the method to solve a much more sophisticated real problem (Goździewski, Maciejewski and Niedzielska, 1991).

4. Remarks

It should be emphasized that:

1. The method described seems to be rather complicated in comparison with a direct numerical integration of equations of motion. It gives however *the general* solution of an initial value problem—for *arbitrary* initial conditions taken from a close neighborhood of an equilibrium our method gives the analytical formula with numerical coefficients that represents this solution. Thus, we can investigate general classes of specific solutions (as we have demonstrated in the examples). The accuracy of the solution depends on the maximal order of normalization procedure applied and the distance of the chosen initial condition from the equilibrium.

2. The method is general. It can be applied for an arbitrary system of ordinary differential equations satisfying the restrictions described above.

3. The process of normalization is time consuming (especially in the case of high maximal order of normalization). However, for a given system this step has to be performed *only once* and additional calculations needed for finding the specific solutions are very fast.

The universality of the method could be find interesting for solving different kinds of problems in applied sciences.

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