Int. J. Appl. Math. Comput. Sci., 2017, Vol. 27, No. 3, 563–573
DOI: 10.1515/amcs-2017-0040

A COMPARATIVE STUDY ON INTERVAL ARITHMETIC OPERATIONS WITH INTUITIONISTIC FUZZY NUMBERS FOR SOLVING AN INTUITIONISTIC FUZZY MULTI–OBJECTIVE LINEAR PROGRAMMING PROBLEM

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In a real world situation, whenever ambiguity exists in the modeling of intuitionistic fuzzy numbers (IFNs), interval valued intuitionistic fuzzy numbers (IVIFNs) are often used in order to represent a range of IFNs unstable from the most pessimistic evaluation to the most optimistic one. IVIFNs are a construction which helps us to avoid such a prohibitive complexity. This paper is focused on two types of arithmetic operations on interval valued intuitionistic fuzzy numbers (IVIFNs) to solve the interval valued intuitionistic fuzzy multi-objective linear programming problem with pentagonal intuitionistic fuzzy numbers (PIFNs) by assuming different α and β cut values in a comparative manner. The objective functions involved in the problem are ranked by the ratio ranking method and the problem is solved by the preemptive optimization method. An illustrative example with MATLAB outputs is presented in order to clarify the potential approach.

Keywords: pentagonal intuitionistic fuzzy number, interval valued intuitionistic fuzzy number, interval valued intuitionistic fuzzy arithmetic, modified interval valued intuitionistic fuzzy arithmetic, interval valued intuitionistic fuzzy multi-objective linear programming problem.

1. Introduction

Fuzzy set theory has been used for managing fuzzy decision-making problems for a long extent of time, but many researchers have shown interest in intuitionistic fuzzy set (IFS) theory and applied it to the field of decision making. The concept of an intuitionistic fuzzy set can be viewed as an alternative approach to recognize a fuzzy set in cases where existing information is not adequate for the definition of an imprecise concept by means of a conventional fuzzy set. Atanassov and Gargov (1989) introduced interval-valued intuitionistic fuzzy sets and many researchers have shown interest in interval valued intuitionistic fuzzy set (IVIFS) theory and successfully applied it to the field of multi-criteria decision making.

Linear programming (LP) is considered one of the most viable optimization techniques. It is based on optimization of a linear function while satisfying a set of linear equality and/or inequality constraints or restrictions, and it involves a lot of parameters whose values are assigned by decision makers. However, in most cases the values of those parameters are not known by either experts or decision makers. In the literature, interval arithmetic was first suggested by Dwyer (1951). The same was developed by Moore (1966). This paper is focused

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on the application of interval valued intuitionistic fuzzy numbers in mathematical optimization problems. In the field of interval linear programming, Ishibuchi and Tanaka (1990), Inuiguchi and Sakawa (1995), Chanas and Kuchta (1996), Chinneck and Ramadan (2000) or Sengupta *et al.* (2001) developed different procedures to deal with these problems. Some frameworks were proposed to solve multi-objective problems with interval parameters by Ida (1999) or Wang and Wang (2001). Oliveira and Antunes (2007) presented an overview of some current procedures of the interval valued multi-objective linear programming problem. Smoczek (2013) used interval numbers in evolutionary optimization. Ben Aicha *et al.* (2013) conceptualized the multi-variable multi-objective predictive controller. Dębski (2016) employed an adaptive multi-spline refinement algorithm in simulation based sailboat trajectory optimization using onboard multi-core computer systems.

With the development of intuitionistic fuzzy set theory, ranking has become a topic that has been studied by many researchers. Deng-Feng (2010) proposed a ratio ranking between triangular intuitionistic fuzzy numbers. We also extended the ratio ranking method to interval valued intuitionistic fuzzy numbers (IVIFNs). Various operations on fuzzy numbers were also available in the past decades which included the new arithmetic operations on interval valued fuzzy numbers by Ganesan and Veeramani (2005). Here, in this work, we modify the same operations to interval valued intuitionistic fuzzy numbers (IVIFNs) to get the preferred maximum conclusion. This paper is focused on the extension of the existing interval valued fuzzy arithmetic operations (Irene Hepzibah and Vidhya, 2015) and modified interval valued intuitionistic fuzzy arithmetic operations based on the results of Ganesan and Veeramani (2005) in a comparative manner.

The paper is focused on pentagonal intuitionistic fuzzy numbers (PIFNs) and interval valued intuitionistic fuzzy numbers by assuming various α and β cut values from them. The pentagonal intuitionistic fuzzy number is represented by five parameters such as a, b, c, d and e, where a and b denote the smallest possible values (decreasing order), c by the most promising value and d, e the largest possible values (increasing order). Each number in the pairwise comparison represents vague judgements.

The paper is organized as follows. Section 2 introduces some preliminaries of pentagonal intuitionistic fuzzy numbers (PIFNs), interval valued intuitionistic fuzzy numbers, extension of interval arithmetic of modified arithmetical operations to interval valued intuitionistic numbers (IVIFNs), and the proposed ratio ranking between interval valued intuitionistic numbers (IVIFNs). Section 3 deals with the formulation of the interval valued intuitionistic fuzzy multi-objective linear programming problem (IVIFMOLPP). In Section 4, some important theorems and results on the interval valued intuitionistic fuzzy multi-objective linear programming problem are given. An application of these new operations is discussed through a numerical illustration in Section 5, and some concluding remarks are given in Section 6.

2. Preliminaries

We introduce the necessary notation in the area of intuitionistic fuzzy set theory.

Definition 1. (*Atanassov, 1986*) Given a fixed set $X = \{x_1, x_2, \dots, x_n\}$, an *intuitionistic fuzzy set* (IFS) is defined as $\tilde{A}^I = \{\langle x_i, \mu_{\tilde{A}^I}(x_i), \nu_{\tilde{A}^I}(x_i)\rangle / x_i \in X\}$, which assigns to each element x_i a membership degree $\mu_{\tilde{A}^I}(x_i)$ and a non-membership degree $\nu_{\tilde{A}^I}(x_i)$ under the condition $0 \le \mu_{\tilde{A}^I}(x_i) + \nu_{\tilde{A}^I}(x_i) \le 1$, for all $x_i \in X$.

Definition 2. (*Atanassov, 1986*) Let $D[0,1]$ be the set of all closed subintervals of the interval $[0,1]$ and X be a given set. An *interval valued intuitionistic fuzzy set* (IVIFS) $\tilde{A}^I$ in X is defined as $\tilde{A}^I = \{\langle x_i, \mu_{\tilde{A}^I}(x_i), \nu_{\tilde{A}^I}(x_i)\rangle / x_i \in X\}$, where $\mu_{\tilde{A}^I}(x_i) : X \to D[0,1]$, $\nu_{\tilde{A}^I}(x_i) : X \to D[0,1]$ with the condition $0 \le \sup \mu_{\tilde{A}^I}(x_i) + \sup \nu_{\tilde{A}^I}(x_i) \le 1$ for any $x \in X$.

Definition 3. (*Yun and Lee, 2013*) A *pentagonal intuitionistic* fuzzy number (PIFN) $\tilde{A}^I$ is defined as an intuitionistic fuzzy set in $\mathbb{R}$ with the following membership function $\mu_{\tilde{A}^I}(x)$ and non-membership function $\nu_{\tilde{A}^I}(x)$:

$$\mu_{\tilde{A}^I}(x) = \begin{cases} \dfrac{x-a}{2b-2a}, & a \le x \le b \\ \dfrac{1}{2} + \left(\dfrac{x-b}{2c-2b}\right), & b \le x \le c \\ 1, & x = c \\ 1 - \left(\dfrac{x-c}{2d-2c}\right), & c \le x \le d \\ \dfrac{e-x}{2e-2d}, & d \le x \le e \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_{\tilde{A}^I}(x) = \begin{cases} \dfrac{2b-a-x}{2b-2a}, & a \le x \le b \\ \dfrac{1}{2} - \left(\dfrac{x-b}{2c-2b}\right), & b \le x \le c \\ 0, & x = c \\ \dfrac{x-c}{2d-2c}, & c \le x \le d \\ \dfrac{x-2d+e}{2e-2d}, & d \le x \le e \\ 1, & \text{otherwise,} \end{cases}$$

where $a \le b \le c \le d \le e$ and $0 \le \mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x) \le 1$ or $\mu_{\tilde{A}^I}(x) = \nu_{\tilde{A}^I}(x)$, for all $x \in \mathbb{R}$.

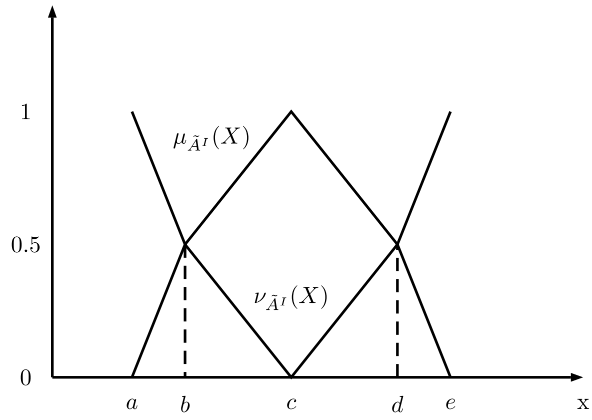

Fig. 1. Pentagonal intuitionistic fuzzy number (PIFN).

Throughout this paper, the PIFN is denoted by $\tilde{A}^I = \{(a,b,c,d,e), 1, 0.5, 0\}$.

Definition 4. An (α,β)-*cut set* of an intuitionistic fuzzy number is defined as $\tilde{A}^I_{\alpha,\beta} = \{x/\mu_{\tilde{A}^I}(x) \geq \alpha, \nu_{\tilde{A}^I}(x) \leq \beta\}$, where $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$ and $0 \leq \alpha+\beta \leq 1$.

Definition 5. If $\tilde{A}^I = \{(a,b,c,d,e), \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)\}$ is a pentagonal intuitionistic fuzzy number, an *interval valued intuitionistic fuzzy number* is defined by

$$\tilde{A}^I_{(\alpha,\beta)} = \{[A^-_\alpha, A^+_\alpha]; [A^-_\beta, A^+_\beta]\},$$

where $[A^-_\alpha, A^+_\alpha] = [a+2\alpha(b-a), e-2\alpha(e-d)]$ is the closed interval which is an α-cut for $\tilde{A}^I$ in $0 \leq \alpha \leq 1/2$ and where $[A^-_\beta, A^+_\beta] = [(2b-a)-2\beta(b-a), (2d-e)+2\beta(e-d]$ is the closed interval which is a β-cut for $\tilde{A}^I$ in $1/2 \leq \beta \leq 1$. An interval valued intuitionistic fuzzy number defined by

$$\tilde{A}^I_{(\alpha,\beta)} = \{[A^-_\alpha, A^+_\alpha]; [A^-_\beta, A^+_\beta]\},$$

where $[A^-_\alpha, A^+_\alpha] = [(2b-c)+2\alpha(c-b), (2d-c)-2\alpha(d-c)]$ is the closed interval which is an α-cut for $\tilde{A}^I$ in $1/2 \leq \alpha \leq 1$ and where $[A^-_\beta, A^+_\beta] = [c-2\beta(c-b), c+2\beta(d-c)]$ is the closed interval which is a β-cut for $\tilde{A}^I$ in $0 \leq \beta \leq 1/2$. A *positive interval valued intuitionistic fuzzy number* is denoted as $\{[a_1,a_2];[a'_1,a'_2]\}$, where all a_i's and a'_i's > 0 for all $i = 1,2$. A *negative interval valued intuitionistic fuzzy number* is denoted as $\{[a_1,a_2];[a'_1,a'_2]\}$, where all a_i's and a'_i 's < 0 for all $i = 1,2$.

2.1. Interval arithmetic operations on interval valued intuitionistic fuzzy numbers.

The existing interval arithmetic operations on fuzzy numbers (Timothy, 2010) are extended to intuitionistic fuzzy numbers (Irene Hepzibah and Vidhya, 2015) and are given below.

Let
$$\tilde{A}^I = \{[a_1,a_2];[a'_1,a'_2]\}$$
and
$$\tilde{B}^I = \{[b_1,b_2];[b'_1,b'_2]\}$$
be two interval valued intuitionistic fuzzy numbers. Then the various arithmetic operations are as follows:

(i) *Addition:*

$$\tilde{A}^I + \tilde{B}^I = \{[a_1+b_1, a_2+b_2]; (a'_1+b'_1, a'_2+b'_2)\}.$$

(ii) *Subtraction:*

$$\tilde{A}^I - \tilde{B}^I = \{[a_1-b_2, a_2-b_1]; [a'_1-b'_2, a'_2-b'_1]\}$$
or
$$\{[a_1-b_1, a_2-b_2]; [a'_1-b'_1, a'_2-b'_2]\},$$
provided that
$$D(\tilde{A}^I) \geq D(\tilde{B}^I)$$
and
$$D(\tilde{A}^{I'}) \geq D(\tilde{B}^{I'}),$$
where
$$D(\tilde{A}^I) = \frac{a_2-a_1}{2}, \quad D(\tilde{B}^I) = \frac{b_2-b_1}{2},$$
$$D(\tilde{A}^{I'}) = \frac{a'_2-a'_1}{2}, \quad D(\tilde{B}^{I'}) = \frac{b'_2-b'_1}{2}.$$
Here D denotes the difference point of a interval valued intuitionistic fuzzy number.

(iii) *Multiplication:*

$$\begin{aligned}\tilde{A}^I \times \tilde{B}^I = \{&[\min(a_1b_1, a_1b_2, a_2b_1, a_2b_2),\\ &\max(a_1b_1, a_1b_2, a_2b_1, a_2b_2)];\\ &[\min(a'_1b'_1, a'_1b'_2, a'_2b'_1, a'_2b'_2),\\ &\max(a'_1b'_1, a'_1b'_2, a'_2b'_1, a'_2b'_2)]\}.\end{aligned}$$

(iv) *Division:*

$$\begin{aligned}&\tilde{A}^I/\tilde{B}^I\\ &= \left\{\left[\min\left(\frac{a_1}{b_1},\frac{a_1}{b_2},\frac{a_2}{b_1},\frac{a_2}{b_2}\right), \max\left(\frac{a_1}{b_1},\frac{a_1}{b_2},\frac{a_2}{b_1},\frac{a_2}{b_2}\right)\right];\right.\\ &\left.\left[\min\left(\frac{a'_1}{b'_1},\frac{a'_1}{b'_2},\frac{a'_2}{b'_1},\frac{a'_2}{b'_2}\right), \max\left(\frac{a'_1}{b'_1},\frac{a'_1}{b'_2},\frac{a'_2}{b'_1},\frac{a'_2}{b'_2}\right)\right]\right\}\end{aligned}$$
or
$$\left\{\left[\frac{a_1}{b_1},\frac{a_2}{b_2}\right];\left(\frac{a'_1}{b'_1},\frac{a'_2}{b'_2}\right]\right\},$$
provided that
$$\left|\frac{D(\tilde{A}^I)}{M(\tilde{A}^I)}\right| \geq \left|\frac{D(\tilde{B}^I)}{M(\tilde{B}^I)}\right|,$$
$$\left|\frac{D(\tilde{A}^{I'}}{M(\tilde{A}^{I'}}\right| \geq \left|\frac{D(\tilde{B}^{I'})}{M(\tilde{B}^{I'})}\right|,$$

where

$$M(\tilde{A}^I) = \frac{a_2 + a_1}{2}, \qquad M(\tilde{B}^I) = \frac{b_2 + b_1}{2},$$
$$M(\tilde{A}^{I'}) = \frac{a'_2 + a'_1}{2} \qquad M(\tilde{B}^{I'}) = \frac{b'_2 + b'_1}{2}.$$

Here M denotes the mid-value of an interval valued intuitionistic fuzzy number.

(v) *Scalar multiplication:*
Let $\lambda \in \mathbb{R}$. Then

$$\lambda\tilde{A}^I = \begin{cases} \{[\lambda a_1, \lambda a_2]; [\lambda a'_1, \lambda a'_2]\}, & \lambda \geq 0, \\ \{[\lambda a_2, \lambda a_1]; [\lambda a'_2, \lambda a'_1]\}, & \lambda < 0. \end{cases}$$

2.2. Modified interval arithmetic operations on interval valued intuitionistic fuzzy numbers. Ganesan and Veeramani (2005) developed modified arithmetic operations on interval fuzzy numbers. These operations are extended here for interval valued intuitionistic fuzzy numbers.

Any interval valued intuitionistic fuzzy number $\tilde{A}^I = \{[a_1, a_2]; [a'_1, a'_2]\}$ is alternatively represented as $\tilde{A}^I = \{[m_1(\tilde{A}^I), w_1(\tilde{A}^I)]; [m'_1(\tilde{A}^I), w'_1(\tilde{A}^I)]\}$, where $m_1(\tilde{A}^I), m'_1(\tilde{A}^I)$ are the mid-points and $w_1(\tilde{A}^I), w'_1(\tilde{A}^I)$ are the half-widths of an interval valued intuitionistic fuzzy number $\tilde{A}^I$, i.e.,

$$m_1(\tilde{A}^I) = \frac{a_1 + a_2}{2}, \qquad w_1(\tilde{A}^I) = \frac{a_2 - a_1}{2},$$
$$m'_1(\tilde{A}^I) = \frac{a'_1 + a'_2}{2}, \qquad w'_1(\tilde{A}^I) = \frac{a'_2 - a'_1}{2}.$$

Let $\tilde{A}^I = \{[a_1, a_2]; [a'_1, a'_2]\}$ and $\tilde{B}^I = \{[b_1, b_2]; [b'_1, b'_2]\}$ be two interval valued intuitionistic fuzzy numbers. Then

$$m_1 = \frac{a_1 + a_2}{2}, \quad m_2 = \frac{b_1 + b_2}{2},$$
$$m'_1 = \frac{a'_1 + a'_2}{2}, \quad m'_2 = \frac{b'_1 + b'_2}{2}.$$

(i) *Addition:*

$$\tilde{A}^I + \tilde{B}^I = \Big\{[(m_1 + m_2 - k); (m_1 + m_2 + k)]; \\ [(m'_1 + m'_2 - k'), (m'_1 + m'_2 + k')]\Big\},$$

where

$$k = \frac{[b_2 + a_2] - [b_1 + a_1]}{2},$$
$$k' = \frac{[b'_2 + a'_2] - [b'_1 + a'_1]}{2}.$$

(ii) *Subtraction:*

$$\tilde{A}^I - \tilde{B}^I = \{[(m_1 - m_2 - k); (m_1 - m_2 + k)]; \\ [(m'_1 - m'_2 - k'), (m'_1 - m'_2 + k')]\},$$

where

$$k = \frac{[b_2 + a_2] - [b_1 + a_1]}{2},$$
$$k' = \frac{[b'_2 + a'_2] - [b'_1 + a'_1]}{2}.$$

(iii) *Multiplication:*

$$\tilde{A}^I \times \tilde{B}^I = \{[(m_1 m_2 - k); (m_1 m_2 + k)]; \\ [(m'_1 m'_2 - k), (m'_1 m'_2 + k')]\},$$

where

$$k = \min\{(m_1 m_2 - \alpha, \beta - m_1 m_2)\},$$
$$\alpha = \min(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2),$$
$$\beta = \max(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2),$$
$$k' = \min\{(m'_1 m'_2 - \alpha, \beta - m'_1 m'_2)\},$$
$$\alpha' = \min(a'_1 b'_1, a'_1 b'_2, a'_2 b'_1, a'_2 b'_2),$$
$$\beta' = \max(a'_1 b'_1, a'_1 b'_2, a'_2 b'_1, a'_2 b'_2).$$

(iv) *Inverse:*

$$\frac{1}{\tilde{A}^I} = \{[a_1, a_2]; [a'_1, a'_2]\}^{-1}$$
$$= \left\{ \left[\frac{1}{m_1} - k, \frac{1}{m_1} + k \right]; \left[\frac{1}{m'_1} - k', \frac{1}{m'_1} + k' \right] \right\},$$

where

$$k = \min\left\{ \frac{1}{a_2}\left(\frac{a_2 - a_1}{a_1 + a_2}\right), \frac{1}{a_1}\left(\frac{a_2 - a_1}{a_1 + a_2}\right) \right\},$$
$$k' = \min\left\{ \frac{1}{a'_2}\left(\frac{a'_2 - a'_1}{a'_1 + a'_2}\right), \frac{1}{a'_1}\left(\frac{a'_2 - a'_1}{a'_1 + a'_2}\right) \right\}$$

for all positive real numbers a_1, a_2, a'_1, a'_2 and

$$0 \notin \{[a_1, a_2]; [a'_1, a'_2]\}.$$

(v) *Scalar multiplication:*
Let $\lambda \in \mathbb{R}$. Then

$$\lambda\tilde{A}^I = \begin{cases} \{[\lambda a_1, \lambda a_2]; [\lambda a'_1, \lambda a'_2]\}, & \lambda \geq 0, \\ \{[\lambda a_2, \lambda a_1]; [\lambda a'_2, \lambda a'_1]\}, & \lambda < 0. \end{cases}$$

2.3. Ranking algorithm for interval valued intuitionistic fuzzy numbers. Deng-Feng (2010) developed a ratio ranking method to compare triangular intuitionistic fuzzy numbers. Here, an algorithm was developed to rank the interval valued intuitionistic fuzzy numbers (IVIFNs).

Step 1. Enter the interval numbers which are to be ranked, i.e.,

$$\tilde{A}^I = \{[a_1, a_2]; [a'_1, a'_2]\}$$

and

$$\tilde{B}^I = \{[b_1, b_2]; [b'_1, b'_2]\}.$$

Step 2. For $\tilde{A}^I$ and $\tilde{B}^I$, compute

$$\nu_\mu(\tilde{A}^I) = \frac{a_1 + a_2}{2}, \qquad \nu_\gamma(\tilde{A}^I) = \frac{a_1' + a_2'}{2},$$
$$\mathcal{A}_\mu(\tilde{A}^I) = \frac{a_2 - a_1}{3}, \qquad \mathcal{A}_\gamma(\tilde{A}^I) = \frac{a_2' - a_1'}{3},$$
$$\nu_\mu(\tilde{B}^I) = \frac{b_1 + b_2}{2}, \qquad \nu_\gamma(\tilde{B}^I) = \frac{b_1' + b_2'}{2},$$
$$\mathcal{A}_\mu(\tilde{B}^I) = \frac{b_2 - b_1}{3}, \qquad \mathcal{A}_\gamma(\tilde{B}^I) = \frac{b_2' - b_1'}{3}.$$

Step 3. Calculate the value index

$$\nu(\tilde{A}^I, \lambda) = \nu_\mu(\tilde{A}^I) + \lambda(\nu_\gamma(\tilde{A}^I) - \nu_\mu(\tilde{A}^I))$$

and the ambiguity index

$$\mathcal{A}(\tilde{A}^I, \lambda) = \mathcal{A}_\gamma(\tilde{A}^I) - \lambda(\mathcal{A}_\gamma(\tilde{A}^I) - \mathcal{A}_\mu(\tilde{A}^I))$$

for $\tilde{A}^I$, and calculate the same for $\tilde{B}^I$, where $\lambda \in (1/2, 1)$ shows that the decision maker prefers certainty or positive feeling while $\lambda = 1/2$ shows that the decision maker is indifferent between positive and negative feelings.

Step 4. Calculate the ratios

$$\mathcal{R}(\tilde{A}^I, \lambda) = \frac{\nu(\tilde{A}^I, \lambda)}{1 + \mathcal{A}(\tilde{A}^I, \lambda)},$$

$$\mathcal{R}(\tilde{B}^I, \lambda) = \frac{\nu(\tilde{B}^I, \lambda)}{1 + \mathcal{A}(\tilde{B}^I, \lambda)}.$$

Step 5. Compare the ratios $\mathcal{R}(\tilde{A}^I, \lambda)$ and $\mathcal{R}(\tilde{B}^I, \lambda)$:

(i) If $\mathcal{R}(\tilde{A}^I, \lambda) < \mathcal{R}(\tilde{B}^I, \lambda)$, then $\tilde{A}^I < \tilde{B}^I$;

(ii) If $\mathcal{R}(\tilde{A}^I, \lambda) > \mathcal{R}(\tilde{B}^I, \lambda)$, then $\tilde{A}^I > \tilde{B}^I$;

(iii) If $\mathcal{R}(\tilde{A}^I, \lambda) = \mathcal{R}(\tilde{B}^I, \lambda)$, then $\tilde{A}^I = \tilde{B}^I$.

3. Problem formulation

In this section, we intend to solve the conventional multi-objective linear programming problem with intuitionistic fuzzy attributes according to the appearance of the vagueness in the real world situation.

3.1. Formulation of the interval valued intuitionistic fuzzy multi objective linear programming problem (IVIFMOLPP). The general form of the multi-objective optimization problem with k intuitionistic fuzzy objective functions $\tilde{Z}_1^I, \tilde{Z}_2^I, \dots, \tilde{Z}_k^I$ and m intuitionistic fuzzy constraints is given by

$$\left.\begin{aligned} &\text{maximize or minimize } \tilde{Z}_l^I = \sum_{j=1}^n \tilde{c}_j^I \tilde{x}_j^I \\ &\text{subject to} \\ &\qquad \sum_{j=1}^n \tilde{a}_{ij}^I \tilde{x}_j^I \preceq \text{ or } \succeq \text{ or } \approx \tilde{b}_i^I,\ \tilde{x}_j^I \succeq \tilde{0}^I \\ &\quad \text{or } \tilde{A}^I \tilde{X}^I \approx \tilde{b}^I, \end{aligned}\right\} \quad (1)$$

where $l = 1, 2, \dots, k$; $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$. Here $\tilde{A}^I = (\tilde{a}_{ij}^I)$, $\tilde{c}_j^I, \tilde{b}^I$ and $\tilde{x}_j^I$ are given by pentagonal intuitionistic fuzzy numbers (PIFNs).

By assuming prescribed values of α and β, the above problem can be restated as

$$\left.\begin{aligned} &\text{maximize or minimize} \\ &[\tilde{Z}_l^I]_{[\alpha,\beta]} = \sum_{j=1}^n [\tilde{c}_j^I]_{[\alpha,\beta]} [\tilde{x}_j^I]_{[\alpha,\beta]} \\ &\text{subject to} \\ &\sum_{j=1}^n [\tilde{a}_{ij}^I]_{[\alpha,\beta]} [\tilde{x}_j^I]_{[\alpha,\beta]} \preceq \text{ or } \succeq \text{ or } \approx [\tilde{b}_i^I]_{[\alpha,\beta]}, \\ &\qquad [\tilde{x}_j^I]_{[\alpha,\beta]} \succeq \tilde{0}^I, \end{aligned}\right\} \quad (2)$$

where $l = 1, 2, \dots, k$; $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$. Here $[\tilde{a}_{ij}^I]_{[\alpha,\beta]}$, $[\tilde{c}_j^I]_{[\alpha,\beta]}, [\tilde{b}^I]_{[\alpha,\beta]}$ are interval valued intuitionistic fuzzy numbers (IVIFNs) and $[\tilde{x}_i^I]_{[\alpha,\beta]}$, whose states are also given by interval valued intuitionistic fuzzy numbers.

Any interval valued intuitionistic fuzzy multi objective linear programming problem can be converted to its standard form as

$$\left.\begin{aligned} &\text{maximize or minimize} \\ &[\tilde{Z}^I]_{[\alpha,\beta]} = \sum_{j=1}^n [\tilde{c}_j^I]_{[\alpha,\beta]} [\tilde{x}_j^I]_{[\alpha,\beta]}, \\ &\qquad l = 1, 2, \dots, k \\ &\text{subject to} \\ &\sum_{j=1}^n [\tilde{a}_{ij}^I]_{[\alpha,\beta]} [\tilde{x}_j^I]_{[\alpha,\beta]} \pm [\tilde{s}_j^I]_{[\alpha,\beta]} \approx [\tilde{b}_i^I]_{[\alpha,\beta]}, \\ &\qquad i = 1, 2, \dots, m, \quad [\tilde{x}_j^I]_{[\alpha,\beta]} \preceq \tilde{0}^I, \end{aligned}\right\} \quad (3)$$

where $[\tilde{s}_i^I]_{[\alpha,\beta]}$ are called interval slack or surplus variables. Here $[\tilde{1}_i^I]_{[\alpha,\beta]}$ and $[\tilde{0}_i^I]_{[\alpha,\beta]}$ are conveniently taken as

$$[\tilde{1}_i^I]_{[\alpha,\beta]} = \{[1, 1]; [1, 1]\}$$

and

$$[\tilde{0}_i^I]_{[\alpha,\beta]} = \{[0, 0]; [0, 0]\}$$

throughout this paper.

4. Main results

Now we are interested in proving interval valued intuitionistic fuzzy equivalents of some important theorems of multi objective linear programming.

Definition 6. Any $\tilde{X}^I = (\tilde{x}_1^I, \tilde{x}_2^I, \dots, \tilde{x}_n^I) \in \mathbb{N}^n(S)$, where each $\tilde{x}_j^I \in \mathbb{N}(S)$ which satisfies all the constraints and non-negativity restrictions of (3) is said to be an *interval valued intuitionistic fuzzy feasible solution* of (3).

Definition 7. Let $\tilde{X}^I = (\tilde{x}_1^I, \tilde{x}_2^I, \dots, \tilde{x}_n^I)$. Suppose $\tilde{X}^I$ solves $\tilde{A}^I \tilde{X}^I \approx \tilde{b}^I$. If all $\tilde{x}_j^I \approx \tilde{0}^I$ for some j, then $\tilde{x}_j^I$ is said to be an *interval valued intuitionistic fuzzy basic solution*. If all $\tilde{x}_j^I \not\approx \tilde{0}^I$ for some j, then $\tilde{X}^I$ has some non-zero components, say $\tilde{x}_1^I, \tilde{x}_2^I, \dots, \tilde{x}_t^I$, $1 \le t \le n$. Then

$$\begin{aligned}\tilde{a}_1^I \tilde{x}_1^I + \tilde{a}_2^I \tilde{x}_2^I + \ldots + \tilde{a}_t^I \tilde{0}^I + \tilde{a}_{t+1}^I \tilde{0}^I \\ + \tilde{a}_{t+2}^I \tilde{0}^I + \cdots + \tilde{a}_n^I \tilde{0}^I \approx \tilde{b}^I.\end{aligned}$$

If the columns $\tilde{a}_1^I, \tilde{a}_2^I, \tilde{a}_3^I, \dots, \tilde{a}_t^I$ corresponding to these non-zero components $\tilde{x}_1^I, \tilde{x}_2^I, \dots, \tilde{x}_t^I$ are linearly independent, then $\tilde{X}^I$ is said to be an *interval valued intuitionistic fuzzy basic solution*.

Let $\tilde{B}^I = (\tilde{b}_1^I, \tilde{b}_2^I, \dots, \tilde{b}_m^I)$ form a basis for the columns of $\tilde{A}^I$. Let $\tilde{X}_{\tilde{B}^I}^I = \tilde{B}^{-1^I} \tilde{b}^I$ be an interval valued intuitionistic fuzzy basic feasible solution and the value of the objective function $\tilde{z}_l^I (l = 1, 2, \dots, k)$ be given by

$$\tilde{z}_{0l}^I \approx \tilde{c}_{\tilde{B}^I}^I \tilde{X}_{\tilde{B}^I}^I,$$

where

$$\tilde{c}_{\tilde{B}^I}^I = (\tilde{c}_{\tilde{B}1^I}^I, \tilde{c}_{\tilde{B}2^I}^I, \tilde{c}_{\tilde{B}3^I}^I, \dots, \tilde{c}_{\tilde{B}m^I}^I)$$

is the cost vector corresponding to $\tilde{X}_{\tilde{B}^I}^I$. Assume that

$$\tilde{a}_j^I = \sum_{i=1}^m \tilde{y}_{ij}^I \tilde{b}_i^I = \tilde{y}_j^I \tilde{B}^I$$

and the interval valued intuitionistic fuzzy numbers

$$\tilde{z}_j^I = \sum_{i=1}^m \tilde{c}_{\tilde{B}i^I}^I \tilde{y}_{ij}^I = \tilde{c}_{\tilde{B}^I}^I \tilde{y}_j^I$$

are known for every column vector $\tilde{a}_j^I$ in $\tilde{A}^I$, which is not in $\tilde{B}^I$. Now we intended to examine the possibility of finding another interval valued intuitionistic fuzzy basic feasible solution with an improved interval valued intuitionistic fuzzy value of $\tilde{z}_l^I (l = 1, 2, \dots, k)$ by replacing one of the columns of $\tilde{B}^I$ by $\tilde{a}_j^I$.

Theorem 1. *Let $\tilde{X}_{\tilde{B}^I}^I = \tilde{B}^{-1^I} \tilde{b}^I$ be an interval valued intuitionistic fuzzy basic feasible solution of (3). If for any column $\tilde{a}_j^I$ in $\tilde{A}^I$ which is not in $\tilde{B}^I$ the condition $(\tilde{z}_j^I - \tilde{c}_j^I) \prec \tilde{0}^I$ holds and $\tilde{y}_{ij}^I \succ \tilde{0}^I$ for some $i \in \{1, 2, 3, \dots, m\}$, then it is possible to obtain a new interval valued intuitionistic fuzzy basic feasible solution by replacing one of the columns in $\tilde{B}^I$ by $\tilde{a}_j^I$.*

Proof. Suppose that

$$\tilde{X}_{\tilde{B}^I}^I = (\tilde{x}_{\tilde{B}1^I}^I, \tilde{x}_{\tilde{B}2^I}^I, \dots, \tilde{x}_{\tilde{B}m^I}^I)$$

is an interval valued intuitionistic fuzzy basic feasible solution with k positive components such that $\tilde{B}^I \tilde{X}_{\tilde{B}^I}^I \approx \tilde{b}$ or $\tilde{X}_{\tilde{B}^I}^I \approx \tilde{B}^{-1^I} \tilde{b}^I$, where $\tilde{x}_{\tilde{B}_i^I}^I \in \mathbb{N}(S)$, for $i = 1, 2, \dots, m$, and $\tilde{x}_{\tilde{B}i^I}^I > \tilde{0}^I$, for $i = 1, 2, \dots, k, k < m$.

Now the equation

$$\tilde{B}^I \tilde{X}_{\tilde{B}^I}^I \approx \tilde{b}^I$$

becomes

$$\sum_{i=1}^k \tilde{x}_{\tilde{B}i^I}^I \tilde{b}_i^I + \tilde{0}^I \tilde{b}_{k+1}^I + \tilde{0}^I \tilde{b}_{k+2}^I + \ldots + \tilde{0}^I \tilde{b}_m^I \approx \tilde{b}^I,$$

where $\tilde{0}^I \approx \{[0,0]; [0,0]\}$ without loss of generality. That is,

$$\sum_{i=1}^k \tilde{x}_{\tilde{B}i^I}^I \tilde{b}_i^I + \sum_{i=k+1}^m \tilde{0}^I \tilde{b}_i^I \approx \tilde{b}^I. \tag{4}$$

Then for any column $\tilde{a}_j^I$ in $\tilde{A}^I$ which is not in $\tilde{B}^I$ we write

$$\tilde{a}_j^I = \sum_{i=1}^m \tilde{y}_{ij}^I \tilde{b}_i^I = \tilde{y}_j^I \tilde{B}^I.$$

We know that if the basis vector $\tilde{b}_r^I$ for which $\tilde{y}_{rj}^I \not\approx \tilde{0}^I$ is replaced by $\tilde{a}_j^I$ in $\tilde{A}^I$, then the new set of vectors $\tilde{b}_i^I (i = 1, 2, \dots, m)$ forms a basis.

Now for $\tilde{y}_{rj}^I \not\approx \tilde{0}^I$ and $r \le k$ we can write

$$\begin{aligned}\tilde{b}_r^I &= \frac{\tilde{a}_j^I}{\tilde{y}_{rj}^I} - \sum_{i=1, i\ne r}^m \frac{\tilde{y}_{ij}^I}{\tilde{y}_{rj}^I} \tilde{b}_i^I \\ &= \frac{\tilde{a}_j^I}{\tilde{y}_{rj}^I} - \sum_{i=1, i\ne r}^m \frac{\tilde{y}_{ij}^I}{\tilde{y}_{rj}^I} \tilde{b}_i^I - \sum_{i=k+1}^m \frac{\tilde{y}_{ij}^I}{\tilde{y}_{rj}^I} \tilde{b}_i^I.\end{aligned}$$

Equation (4) becomes

$$\sum_{i=1, i\ne r}^m \tilde{x}_{\tilde{B}i^I}^I \tilde{b}_i^I + \tilde{x}_{\tilde{B}r^I}^I \tilde{b}_r^I + \sum_{i=k+1}^m \tilde{0}^I \tilde{b}_i^I \approx \tilde{b}^I.$$

This yields

$$\begin{aligned}\sum_{i=1, i\ne r}^m \tilde{x}_{\tilde{B}i^I}^I \tilde{b}_i^I + \frac{\tilde{x}_{\tilde{B}r^I}^I}{\tilde{y}_{rj}^I} \tilde{a}_j^I - \frac{\tilde{x}_{\tilde{B}r^I}^I}{\tilde{y}_{rj}^I} \sum_{i=1, i\ne r}^k \tilde{y}_{ij}^I \tilde{b}_i^I \\ - \frac{\tilde{x}_{\tilde{B}r^I}^I}{\tilde{y}_{rj}^I} \sum_{i=k+1}^m \tilde{y}_{ij}^I \tilde{b}_i^I + \sum_{i=k+1}^m \tilde{0}^I \tilde{b}_i^I \approx \tilde{b}^I.\end{aligned}$$

Consequently,

$$\sum_{i=1,i\neq r}^{k}\left(\tilde{x}^I_{\tilde{B}^I_i}-\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\tilde{y}^I_{ij}\right)\tilde{b}^I_i+\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\tilde{a}^I_j + \sum_{i=k+1}^{m}(\tilde{0}^I-\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\tilde{y}^I_{ij})\tilde{b}^I_i\approx\tilde{b}^I.$$

Since $\tilde{x}^I_{\tilde{B}i^I}\approx\tilde{0}^I$, for $i=k+1,k+2,\dots,m$, we have

$$\sum_{i=1,i\neq r}^{k}\left(\tilde{x}^I_{\tilde{B}i^I}-\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\tilde{y}^I_{ij}\right)\tilde{b}^I_i+\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\tilde{a}^I_j\approx\tilde{b}^I$$
$$\Rightarrow\sum_{i=1,i\neq r}^{m}\hat{\tilde{x}}^I_{\tilde{B}i^I}\tilde{b}^I_i-\hat{\tilde{x}}^I_{\tilde{B}r^I}\tilde{a}^I_j\approx\tilde{b}^I,$$

where

$$\hat{\tilde{x}}^I_{\tilde{B}i^I}=(\tilde{x}^I_{\tilde{B}i^I}-\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\tilde{y}^I_{ij},\quad i\neq r,$$
$$\hat{\tilde{x}}^I_{\tilde{B}r^I}=\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}},$$

which gives a new interval valued intuitionistic fuzzy basic solution to $\tilde{A}^I\tilde{X}^I\approx\tilde{b}^I$.

We shall prove that this new interval valued intuitionistic fuzzy basic solution is also feasible. Choose $\tilde{y}^I_{rj}\succ\tilde{0}^I$ such that

$$\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\approx\min_i\left\{\frac{\tilde{x}^I_{\tilde{B}i^I}}{\tilde{y}^I_{ij}}:\tilde{y}^I_{ij}\succ\tilde{0}^I\right\}.$$

Then

$$\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\preceq\frac{\tilde{x}^I_{\tilde{B}i^I}}{\tilde{y}^I_{ij}}.$$

This implies

$$\frac{\tilde{x}^I_{\tilde{B}i^I}}{\tilde{y}^I_{ij}}-\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\succeq\tilde{0}^I. \tag{5}$$

Hence the new interval valued intuitionistic fuzzy basic solution is an interval valued intuitionistic fuzzy basic feasible solution. ■

Theorem 2. *Assume that $\tilde{X}^I_{\tilde{B}^I}=\tilde{B}^{-1^I}\tilde{b}^I$ is a new interval valued intuitionistic fuzzy basic feasible solution of (3) with $\tilde{z}^I_0\approx\tilde{c}^I_{\tilde{B}^I}\tilde{X}^I_{\tilde{B}^I}$ as the interval valued intuitionistic fuzzy value of the objective function. Let $\hat{\tilde{x}}^I_{\tilde{B}i^I}$ be another interval valued intuitionistic fuzzy basic feasible solution with $\widehat{\tilde{z}_0}^I\approx\widehat{\tilde{c}^I_{\tilde{B}^I}}\widehat{\tilde{X}^I_{\tilde{B}^I}}$ obtained by admitting a non-basic column vector $\tilde{a}^I_j$ in the basis for which $(\tilde{z}^I_j-\tilde{c}^I_j)\prec\tilde{0}^I$ and $\tilde{y}^I_{ij}\succ\tilde{0}^I$ for some $i\in\{1,2,3,\dots,m\}$. Then $\hat{\tilde{z}}^I\succeq\tilde{z}^I_0$.*

Proof. Let $\tilde{X}^I_{\tilde{B}^I}$ be an interval valued intuitionistic fuzzy basic feasible solution and $\tilde{z}^I_0\approx\tilde{c}^I_{\tilde{B}^I}\tilde{X}^I_{\tilde{B}^I}$. Let $\tilde{a}^I_j$ be the column vector introduced in the basis for which $(\tilde{z}^I_j-\tilde{c}^I_j)\prec\tilde{0}^I$. Let $\tilde{b}^I_r$ be the column vector removed from the basis and $\widehat{\tilde{X}^I_{\tilde{B}^I}}$ be a new interval valued intuitionistic fuzzy basic feasible solution. Then

$$\hat{\tilde{x}}^I_{\tilde{B}i^I}=\tilde{x}^I_{\tilde{B}i^I}-\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\tilde{y}^I_{ij},\quad i\neq r$$

and

$$\hat{\tilde{x}}^I_{\tilde{B}r^I}=\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}.$$

Since $\widehat{\tilde{c}^I_{\tilde{B}i^I}}\approx\tilde{c}^I_{\tilde{B}i^I}$, $i\neq r$ and $\widehat{\tilde{c}^I_{\tilde{B}r^I}}\approx\tilde{c}^I_j$, the new interval valued intuitionistic fuzzy value of the objective function is

$$\begin{aligned}
\hat{\tilde{z}}^I&\approx\widehat{\tilde{c}^I_{\tilde{B}^I}}\widehat{\tilde{x}^I}_{\tilde{B}^I}\approx\sum_{i=1}^{m}\widehat{\tilde{c}^I_{\tilde{B}i^I}}\widehat{\tilde{x}^I}_{\tilde{B}i^I}\\
&\approx\sum_{i=1,i\neq r}^{m}\widehat{\tilde{c}^I_{\tilde{B}i^I}}\widehat{\tilde{x}^I}_{\tilde{B}i^I}+\widehat{\tilde{c}^I_{\tilde{B}r^I}}\widehat{\tilde{x}^I}_{\tilde{B}r^I}\\
&\approx\sum_{i=1,i\neq r}^{m}\tilde{c}^I_{\tilde{B}i^I}\left(\tilde{x}^I_{\tilde{B}i^I}-\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\tilde{y}^I_{ij}\right)\\
&\quad+\tilde{c}^I_{\tilde{B}r^I}\left(\tilde{x}^I_{\tilde{B}r^I}-\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\tilde{y}^I_{rj}\right)+\tilde{c}^I_j\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\\
&\approx\sum_{i=1}^{m}\tilde{c}^I_{\tilde{B}i^I}\left(\tilde{x}^I_{\tilde{B}i^I}-\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\tilde{y}^I_{ij}\right)+\tilde{c}^I_j\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\\
&\approx\sum_{i=1}^{m}\tilde{c}^I_{\tilde{B}i^I}\left(\tilde{x}^I_{\tilde{B}i^I}-\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\sum_{i=1}^{m}\tilde{c}^I_{\tilde{B}i^I}\tilde{y}^I_{ij}\right)+\tilde{c}^I_j\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\\
&\approx\tilde{z}^I_0-\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\tilde{z}^I_j+\tilde{c}^I_j\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\approx\tilde{z}^I_0-\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}(\tilde{z}^I_j-\tilde{c}^I_j)
\end{aligned}$$

Since $\tilde{y}^I_{rj}\succ\tilde{0}^I,(\tilde{z}^I_j-\tilde{c}^I_j)\prec\tilde{0}^I$ and

$$\frac{\tilde{x}^I_{\tilde{B}r^I}}{\tilde{y}^I_{rj}}\succeq\tilde{0}^I,$$

we get

$$\hat{\tilde{z}}^I\succeq\tilde{z}^I_0. \tag{6}$$

Hence the new interval valued intuitionistic fuzzy basic feasible solution yields the improved interval valued intuitionistic fuzzy value of the objective function. ■

Remark 1. In the classical linear programming problems, the process of inserting and removing vectors from the basis matrix is conducted for any of the following situations:

(i) there exists j such that $(\tilde{z}^I_j-\tilde{c}^I_j)\prec\tilde{0}^I,\tilde{y}^I_{ij}\preceq\tilde{0}^I,i=1,2,3,\dots,m$, or

(ii) for all j, $(\tilde{z}_j^I - \tilde{c}_j^I) \succeq \tilde{0}^I$.

In the first case, we obtain an unbounded solution, and in the second it is easy to show that the interval valued intuitionistic fuzzy linear programming problem has an interval valued intuitionistic fuzzy optimal solution.

Theorem 3. *If $\tilde{X}^I_{\tilde{B}^I} = \tilde{B}^{-1^I}\tilde{b}^I$ is an interval valued intuitionistic fuzzy feasible solution of (2) and if $(\tilde{z}_j^I - \tilde{c}_j^I) \succeq \tilde{0}^I$ for every column $\tilde{a}_j^I$ of $\tilde{A}^I$, then $\tilde{X}^I_{\tilde{B}^I}$ is an interval valued intuitionistic fuzzy optimal solution.*

Remark 2. Preemptive optimization (Rardin, 2003) treats multi-objective optimization by considering objectives one at a time. The most important one is optimized first, and then the second most important one is optimized subject to the requirement that the first achieved its optimal value in preemptive optimization, and so on. If each stage of preemptive optimization yields a single objective optimum, the final solution is concluded as an efficient point of the full multi-objective model.

5. Illustrative example

Consider a diet problem that makes use of two kinds of nutrients called starch and protein as a group. The two types of food containing this group are Food 1 and Food 2. The minimum demands for starch and protein are 5 units and 6 units per kg of food, respectively. The activities and their levels in the model are given as follows: activity j to include 1 kg of food type j in the diet, with the associated level x_j, for $j = 1, 2$. Restrictions are induced by the two nutrients in the model, each of which leads to a constraint. For example, the amount of starch contained in the diet is $5x_1 + 2x_2$, which must be no less than 5 for feasibility. Similarly, the amount of protein contained in the diet is $x_1 + 2x_2$, which must be no less than 6 for feasibility.

At the present rates of operation the costs to produce and deliver Food 1 and Food 2 are estimated as \$3/kg and \$1/kg, respectively. Also, all other procurement costs (for labor, power, water, maintenance, depreciation of plant and equipment, floor space, insurance, shipping to the wholesaler, etc.) come to \$2/kg and \$3/kg for Food 1 and Food 2, respectively. These prices are utilized to construct the mathematical model.

There are clearly two decision variables in this problem x_1 and x_2. Associated with each variable in the problem is the activity the decision maker can perform. The activities in this example are: Activity 1—to make 1 kg of Food 1, Activity 2—to make 1 kg of Food 2. The variables in the problem just define the levels at which these activities are carried out. As all the data are given on a per kg basis, they provide an indication that the linearity assumptions are quite reasonable in this problem. Here, the total cost and the procurement cost of food should be minimized. Also, the amount of each food manufactured can vary continuously within its present range. In consequence, the multi-objective linear programming problem (MOLPP) is considered.

Since the cost coefficients and all other coefficients are indecisive, the units of food to be produced on each product will be considered uncertain quantities. Therefore, the problem is modeled as a bi-level multi objective intuitionistic fuzzy linear programming problem and solved by preemptive optimization in a comparative manner on two types of interval valued intuitionistic fuzzy arithmetic operations. This is because the two lowest possible values, one most promising value and the largest possible values of all coefficients are treated as pentagonal intuitionisitic fuzzy numbers from a theoretical or practical point of view.

We would like to find $\tilde{X}^I(\tilde{x}_1^I, \tilde{x}_2^I)$ so as to minimize

$$\widetilde{Z_1}^I(\tilde{x}_1^I, \tilde{x}_2^I) \approx \{(2, 2.5, 3, 3.5, 4); 1, 0.5, 0\}\tilde{x}_1^I + \{(0.5, 0.75, 1, 1.25, 1.5); 1, 0.5, 0\}\tilde{x}_2^I$$

(first DM problem) and for given $\tilde{x}_1^I, \tilde{x}_2^I$ minimize

$$\widetilde{Z_2}^I(\tilde{x}_1^I, \tilde{x}_2^I) \approx \{(1, 1.5, 2, 2.5, 3); 1, 0.5, 0\}\tilde{x}_1^I + \{(2, 2.5, 3, 3.5, 4); 1, 0.5, 0\}\tilde{x}_2^I$$

(second DM problem) subject to the constraints

$$\left.\begin{aligned} &\{(4, 4.5, 5, 5.5, 6); 1, 0.5, 0\}\tilde{x}_1^I \\ &\quad + \{(1, 1.5, 2, 2.5, 3); 1, 0.5, 0\}\tilde{x}_2^I \\ &\qquad \succeq \{(4, 4.5, 5, 5.5, 6); 1, 0.5, 0\}, \\ &\{(0.5, 0.75, 1, 1.25, 1.5); 1, 0.5, 0\}\tilde{x}_1^I \\ &\quad + \{(1, 1.5, 2, 2.5, 3); 1, 0.5, 0\}\tilde{x}_2^I \\ &\qquad \succeq \{(5, 5.5, 6, 6.5, 7); 1, 0.5, 0\}, \quad \tilde{x}_1^I, \tilde{x}_2^I \succeq \tilde{0}^I. \end{aligned}\right\} \tag{7}$$

When $\alpha = 0, \beta = 1$ and assuming that the first objective is the single most important one as per ratio ranking, preemptive optimization is used to begin by minimizing the single objective LPP. Preemptive optimization is used to begin by maximizing the single objective LPP. It can be written as follows:

maximize

$$\widetilde{Z_1}^{I*} = \{[4, 6]; [4, 6]\}\tilde{y}_1^I + \{[5, 7]; [5, 7]\}\tilde{y}_2^I + \{[0, 0]; [0, 0]\}\tilde{s}_1^I + \{[0, 0]; [0, 0]\}\tilde{s}_2^I$$

subject to the constraints

$$\begin{aligned} &\{[4, 6]; [4, 6]\}\tilde{y}_1^I + \{[0.5, 1.5]; [0.5, 1.5]\}\tilde{y}_2^I \\ &\quad + \{[1, 1]; [1, 1]\}\tilde{s}_1^I \preceq \{[2, 4]; [2, 4]\}, \\ &\{[1, 3]; [1, 3]\}\tilde{y}_1^I + \{[1, 3]; [1, 3]\}\tilde{y}_2^I \\ &\quad + \{[1, 1]; [1, 1]\}\tilde{s}_2^I \preceq \{[3, 3]; [3, 3]\}, \quad \tilde{y}_1^I, \tilde{y}_2^I \succeq \tilde{0}^I. \end{aligned}$$

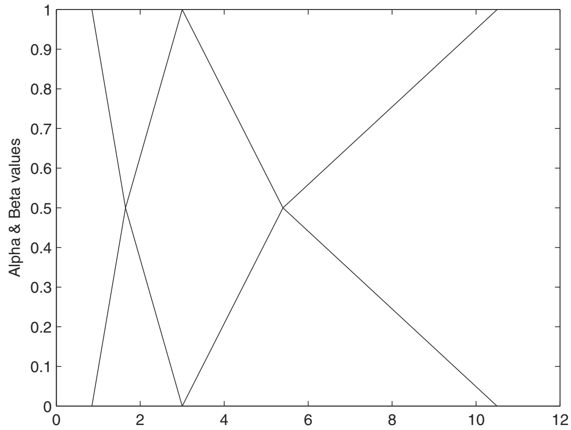

Fig. 2. Optimum $\tilde{z}_1^I$ values.

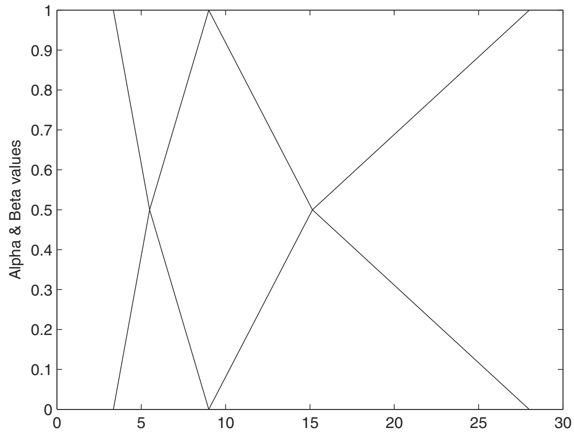

Fig. 4. Optimum $\tilde{z}_2^I$ values.

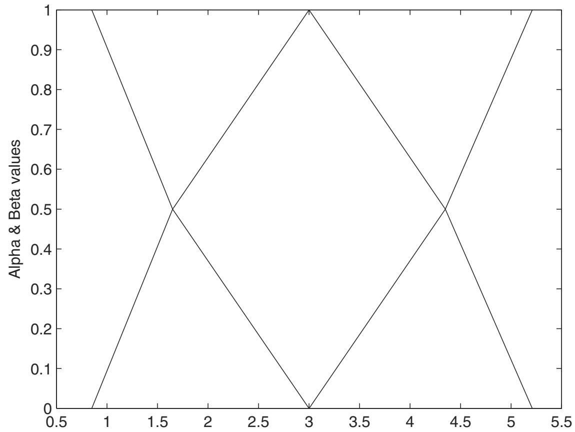

Fig. 3. Optimum $\tilde{z}_1^I$ values.

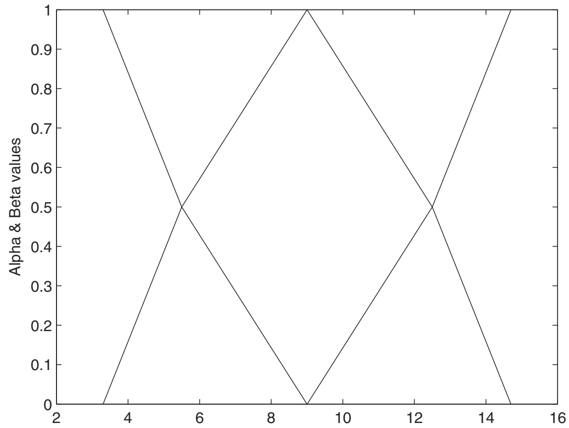

Fig. 5. Optimum $\tilde{z}_2^I$ values.

By Theorems 1 and 2, the optimal solutions of the interval valued intuitionistic fuzzy linear programming problem $\widetilde{Z_1}^I$ for different α and β cut values are displayed in Table 1.

MATLAB outputs of the optimum $\widetilde{Z_1}^I$ values by interval arithmetic (Fig. 2) and modified interval arithmetic operations (Fig. 3) are shown below.

Next, an extra constraint

$$\begin{aligned}\{[2,4];[2,4]\}\tilde{x}_1^I + \{[0.5,1.5];[0.5,1.5]\}\tilde{x}_2^I \\ \succeq \{[0.85,5.21];[0.85,5.21]\}\end{aligned}$$

is imposed. The second objective with the constraints can be written as follows:

maximize

$$\begin{aligned}\widetilde{Z_2}^{I*} &= \{[-5.21,-0.85];[-5.21,-0.85]\}\tilde{y}_1^I \\ &+ \{[4,6];[4,6]\}\tilde{y}_2^I + [5,7];[5,7]\}\tilde{y}_3^I \\ &+ \{[0,0];[0,0]\}\tilde{s}_1^I + \{[0,0];[0,0]\}\tilde{s}_2^I\end{aligned}$$

subject to the constraints

$$\begin{aligned}&\{[-4,-2];[-4,-2]\}\tilde{y}_1^I + \{[4,6];[4,6]\}\tilde{y}_2^I \\ &+ \{[0.5,1.5];[0.5,1.5]\}\tilde{y}_3^I + \{[1,1];[1,1]\}\tilde{s}_1^I \\ &\qquad \preceq \{[1,3];[1,3]\}, \\ &\{[-1.5,-0.5];[-1.5,-0.5]\}\tilde{y}_1^I + \{[1,3];[1,3]\}\tilde{y}_2^I \\ &+ \{[1,3];[1,3]\}\tilde{y}_3^I + \{[1,1];[1,1]\}\tilde{s}_2^I \\ &\qquad \preceq \{[2,4];[2,4]\}, \qquad \tilde{y}_1^I, \tilde{y}_2^I, \tilde{y}_3^I \succeq \tilde{0}^I.\end{aligned}$$

By Theorems 1 and 2, the optimal solutions to the interval valued intuitionistic fuzzy linear programming problem $\widetilde{Z_2}^I$ for using different α and β cut values are presented in Table 2.

MATLAB output of the optimum $\widetilde{Z_2}^I$ values by interval arithmetic and modified interval arithmetic operations are shown in Figs. 4 and 5, respectively.

Therefore, according to the MATLAB, outputs the extension of the modified interval arithmetic operations turns out to be better for obtaining the preferred solution than interval arithmetic operations.

Table 1. Optimum $\widetilde{Z_1}^I$ values for prescribed values of α and β.

$\alpha-$values	β-values	Optimum $\widetilde{Z_1}^I$ values by interval arithmetic operations	Optimum $\widetilde{Z_1}^I$ values by modified interval arithmetic operations
$\alpha = 0$	$\beta = 1$	$\widetilde{Z_1}^I = \{[0.85, 10.5]; [0.85, 10.5]\}$	$\widetilde{Z_1}^I = \{[0.85, 5.21]; [0.85, 5.21]\}$
$\alpha = 0.1$	$\beta = 0.9$	$\widetilde{Z_1}^I = \{[0.97, 9.31]; [0.97, 9.31]\}$	$\widetilde{Z_1}^I = \{[0.97, 5.09]; [0.97, 5.09]\}$
$\alpha = 0.2$	$\beta = 0.8$	$\widetilde{Z_1}^I = \{[1.09, 7.96]; [1.09, 7.96]\}$	$\widetilde{Z_1}^I = \{[1.14, 4.86]; [1.14, 4.86]\}$
$\alpha = 0.3$	$\beta = 0.7$	$\widetilde{Z_1}^I = \{[1.27, 6.97]; [1.27, 6.97]\}$	$\widetilde{Z_1}^I = \{[1.27, 4.73]; [1.27, 4.73]\}$
$\alpha = 0.4$	$\beta = 0.6$	$\widetilde{Z_1}^I = \{[1.46, 6.14]; [1.46, 6.14]\}$	$\widetilde{Z_1}^I = \{[1.46, 4.60]; [1.46, 4.60]\}$
$\alpha = 0.5$	$\beta = 0.5$	$\widetilde{Z_1}^I = \{[1.65, 5.40]; [1.46, 5.40]\}$	$\widetilde{Z_1}^I = \{[1.65, 4.35]; [1.65, 4.35]\}$
$\alpha = 0.6$	$\beta = 0.4$	$\widetilde{Z_1}^I = \{[1.85, 4.8]; [1.46, 4.8]\}$	$\widetilde{Z_1}^I = \{[1.90, 4.10]; [1.90, 4.10]\}$
$\alpha = 0.7$	$\beta = 0.3$	$\widetilde{Z_1}^I = \{[2.11, 4.28]; [1.46, 4.28]\}$	$\widetilde{Z_1}^I = \{[2.11, 3.95]; [2.11, 3.95]\}$
$\alpha = 0.8$	$\beta = 0.2$	$\widetilde{Z_1}^I = \{[2.38, 3.78]; [1.46, 3.78]\}$	$\widetilde{Z_1}^I = \{[2.38, 3.68]; [2.38, 3.68]\}$
$\alpha = 0.9$	$\beta = 0.1$	$\widetilde{Z_1}^I = \{[2.66, 3.36]; [1.46, 3.36]\}$	$\widetilde{Z_1}^I = \{[2.71, 3.29]; [2.71, 3.29]\}$
$\alpha = 1$	$\beta = 0$	$\widetilde{Z_1}^I = \{[3, 3]; [3, 3]\}$	$\widetilde{Z_1}^I = \{[3, 3]; [3, 3]\}$

Table 2. Optimum $\widetilde{Z_2}^I$ values for prescribed values of α and β.

α-values	β-values	Optimum $\widetilde{Z_2}^I$ values by interval arithmetic operations	Optimum $\widetilde{Z_2}^I$ values by modified interval arithmetic operations
$\alpha = 0$	$\beta = 1$	$\widetilde{Z_2}^I = \{[3.35, 28.00]; [3.35, 28.00]\}$	$\widetilde{Z_2}^I = \{[3.30, 14.70]; [3.30, 14.70]\}$
$\alpha = 0.1$	$\beta = 0.9$	$\widetilde{Z_2}^I = \{[3.67, 24.50]; [3.67, 24.50]\}$	$\widetilde{Z_2}^I = \{[3.63, 14.37]; [3.63, 14.37]\}$
$\alpha = 0.2$	$\beta = 0.8$	$\widetilde{Z_2}^I = \{[4.11, 21.56]; [4.11, 21.56]\}$	$\widetilde{Z_2}^I = \{[4.11, 13.89]; [4.11, 13.89]\}$
$\alpha = 0.3$	$\beta = 0.7$	$\widetilde{Z_2}^I = \{[4.51, 19.10]; [4.51, 19.10]\}$	$\widetilde{Z_2}^I = \{[4.51, 13.49]; [4.51, 13.49]\}$
$\alpha = 0.4$	$\beta = 0.6$	$\widetilde{Z_2}^I = \{[4.97, 16.96]; [4.97, 16.96]\}$	$\widetilde{Z_2}^I = \{[4.92, 13.08]; [4.92, 13.08]\}$
$\alpha = 0.5$	$\beta = 0.5$	$\widetilde{Z_2}^I = \{[5.50, 15.15]; [6.05, 13.63]\}$	$\widetilde{Z_2}^I = \{[5.50, 12.50]; [5.50, 12.50]\}$
$\alpha = 0.6$	$\beta = 0.4$	$\widetilde{Z_2}^I = \{[6.05, 13.63]; [6.05, 13.63]\}$	$\widetilde{Z_2}^I = \{[6.11, 11.89]; [6.11, 11.89]\}$
$\alpha = 0.7$	$\beta = 0.3$	$\widetilde{Z_2}^I = \{[6.67, 12.22]; [6.67, 12.22]\}$	$\widetilde{Z_2}^I = \{[6.62, 11.38]; [6.62, 11.38]\}$
$\alpha = 0.8$	$\beta = 0.2$	$\widetilde{Z_2}^I = \{[7.37, 11.04]; [7.37, 11.04]\}$	$\widetilde{Z_2}^I = \{[7.31, 10.69]; [7.31, 10.69]\}$
$\alpha = 0.9$	$\beta = 0.1$	$\widetilde{Z_2}^I = \{[8.14, 9.94]; [8.14, 9.94]\}$	$\widetilde{Z_2}^I = \{[8.21, 9.79]; [8.21, 9.79]\}$
$\alpha = 1$	$\beta = 0$	$\widetilde{Z_2}^I = \{[9, 9]; [9, 9]\}$	$\widetilde{Z_2}^I = \{[9, 9]; [9, 9]\}$

6. Conclusion

This paper presented how extended interval arithmetic operations (Irene Hepzibah and Vidhya, 2015) and modified interval arithmetic operations (Ganesan and Veeramani, 2005) can be efficiently used for solving the multi-objective linear programming problem with pentagonal intuitionistic fuzzy numbers in a comparative manner. Based on MATLAB simulations, extended modified interval arithmetic operations are better than extended interval arithmetic operations. Although here we were considering the linear case, we can extend these operations to multi-objective non-linear programming with interval valued intuitionistic fuzzy coefficients and other intuitionistic fuzzy numbers like triangular, trapezoidal, exponential, piece-wise quadratic, etc., with real life applications.

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Received: 11 August 2016
Revised: 8 January 2017
Re-revised: 24 April 2017
Accepted: 2 May 2017